Désignons par p_i le point d'intersection de a_i^k et b_i^{n-k} (c. à d. le centre de gravité de a_i^k) et par $\hat{P}^0$ le polyèdre 0-dimensionnel composé de points $p_1, p_2, \dots, p_r$. On a alors

$$(25.2) \qquad |s^{n-k}| - \hat{P}^0 \subset Q - P^k.$$

Le polyèdre $|s^{n-k}|$ est homéomorphe à S^{n-k} et la première subdivision barycentrique de Q^n fournit une division simpliciale $\hat{Q}^{n-k}$ de $|s^{n-k}|$. Le point p_i devient alors un sommet $\hat{a}_i^0$ et $\hat{P}^0$ devient un sous-polyèdre de $\hat{Q}^{n-k}$.

Toute transformation $f(X+Q^n-P^k)\subset S^m$ induit en vertu de (25.2) une transformation $f(\hat{Q}^{n-k}-\hat{P}^0)\subset S^m$; celle-ci nous fournit la chaîne $\gamma^0(f)$ et les éléments $\hat{a}_i(f)\,\epsilon\,(m^{n-k-1})$ pour $i=1,2,\dots,r$.

D'après **24**, on a donc

$$(25.3) \qquad \sum_{i=1}^{r} \hat{a}_i(f) = 0.$$

La cellule $\hat{b}_i^{n-k}$, duale à $\hat{a}_i^0$ dans $\hat{Q}^{n-k}$, est contenue dans la cellule b_i^{n-k} et, en vertu de (25.1), elle est orientée comme $\eta_i b_i^{n-k}$. Par conséquent $\hat{a}_i(f)=\eta_i a_i(f)$ et d'après (25.3) $\sum_{i=1}^{r}\eta_i a_i(f)=0$.

Comme c'était le coefficient de a^{k-1} dans la chaîne infinie $\partial[\gamma^k(f)]$, la chaîne $\gamma^k(f)$ est un cycle infini. Il résulte de (5.5) qu'elle est un cycle infini mod Y.

Supposons maintenant que $\gamma^k(f)\sim 0$ mod Y. Soit donc A^{k+1} la chaîne infinie $(k+1)$-dimensionnelle mod Y à coefficients de (m^{n-k-1}), telle que

$$\gamma^k(f) + \partial A^{k+1} = 0.$$

En vertu de **23**, il existe un sous-polyèdre k-dimensionnel P_1^k de Q tel que $\overline{P_1^k}\cdot X=0$, et une transformation $f_1(X+Q-P_1^k)\subset S^m$ telle que $\gamma^k(f_1)=0$ et $f_1(x)=f(x)$ pour $x\,\epsilon\,X$. Comme $|\gamma^k(f_1)|=0$, on trouve, en appliquant **20**, un polyèdre $(k-1)$-dimensionnel $P^{k-1}\subset P_1^k$ et une transformation

$$f^*(X+Q-P^{k-1})\subset S^m$$

qui coïncide sur X avec f_1, donc aussi avec f.

On the equivalence of any set of first category to a set of measure zero.

By

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In n-dimensional euclidean space $R^{(n)}$ it is an easy matter to define sets of first category which have positive measure, and even, ones whose complements are of measure zero. For instance, the complement of the intersection of any sequence of dense open sets whose measures tend to zero defines such a set. Nevertheless one may ask whether such a set is equivalent to a set of measure zero under the group of homeomorphisms of the space onto itself. In other words, does there exist an automorphism (that is, a homeomorphism of the space onto itself) which carries the given set into one of measure zero. The object of the present note is to show that such a transformation always exists, indeed that the automorphisms which carry a given set of first category into one of measure zero form a residual set in the space of antomorphisms, provided the latter is suitably metrised.

To begin with, consider the unit cube $\mathcal{I}^{(n)}$ in n-dimensional euclidean space, $(n\geqslant 1)$. Let $[H]$ denote the space of all automorphisms of $\mathcal{I}^{(n)}$ metrise by the formula [1])

$$\varrho(g,h) = \max_{x\,\epsilon\,\mathcal{I}^{(n)}} (|gx-hx|, |g^{-1}x-h^{-1}x|),$$

where $|x-y|$ denotes the euclidean distance between x and y. The space $[H]$ is complete and the group operations are continuous in the metric, so that it forms a metric group.

[1]) This metric is equivalent to the usual one. See S. Banach, *Théorie des opérations linéaires*, Monografie Matematyczne **1**, Warszawa 1932, p. 229.

Theorem 1. *Let A be any set of first category in $\mathcal{I}^{(n)}$. The automorphisms of $\mathcal{I}^{(n)}$ which carry A into a set of measure zero form a residual set in $[H]$.*

By hypothesis, $A=\sum_{i=1}^{\infty}F_i$, F_i nowhere dense. We may suppose all the F_i closed, for the union of their closures defines a first category set containing A, and if the theorem is proved for this set it will be all the more true for A. Let E_{ik} $(i,k=1,2,\ldots)$ denote the set of automorphisms h such that $m(hF_i)<1/k$, where $m(X)$ denotes the n-dimensional measure of X. Then the set E of automorphisms such that $m(hA)=0$ is represented by $E=\prod_{i,k=1}^{\infty}E_{ik}$. To prove the theorem it will be sufficient to show that the sets E_{ik} are all open and dense in $[H]$, for the complement of E will then be represented as a union of countably many nowhere dense sets. Let h be any element of E_{ik}. Then $m(hF_i)<1/k$. Since hF_i is closed, there is an $\varepsilon>0$ such that the ε-neighborhood $(hF_i)_\varepsilon$ of hF_i likewise has measure less than $1/k$. Consider any g in $[H]$ such that $\varrho(g,h)<\varepsilon$. We have $gF_i\subset(hF_i)_\varepsilon$, hence $m(gF_i)<1/k$, and so $g\,\epsilon\,E_{ik}$ and E_{ik} is open. To show that E_{ik} is dense, consider any $h\,\epsilon\,[H]$, $\varepsilon>0$. By the continuity of the group composition, there is a $\delta>0$ such that $\varrho(gh,h)<\varepsilon$ if $\varrho(g,I)<\delta$, I denoting the identical transformation. Divide $\mathcal{I}^{(n)}$ into rectangular cells $\sigma_1,\sigma_2,\ldots,\sigma_l$ of diameter less than δ. In the interior of each σ_j select a closed sphere s_j which lies outside hF_i. This can be done since hF_i is nowhere dense. Now define g so as to expand each s_j within σ_j while leaving the boundary points of the cells σ_j fixed, the amount of expansion being such that $m(ghF_i)<1/k$. Such an automorphism g may conveniently be defined by a linear expansion and contraction along each radial line from the center of s_j to the boundary of σ_j. Since g moves no point by more than δ we have $\varrho(g,I)<\delta$. Hence $\varrho(gh,h)<\varepsilon$, $gh\,\epsilon\,E_{ik}$, which completes the proof.

Since $[H]$ is a complete space, it follows that there exists an automorphism of $\mathcal{I}^{(n)}$ which carries A into a set of measure zero. If desired, the automorphism can be chosen so as to leave all boundary points of $\mathcal{I}^{(n)}$ fixed, for the above proof applies without change to the complete sub-space of automorphisms which leave boundary points fixed.

One may ask whether a stronger theorem can be asserted, for instance, whether it is always possible to find a differentiable transformation, or an automorphism fulfilling a Lipschitz or Hölder condition, which will effect the desired transformation. That this is not always possible is shown by the following considerations.

Let $\delta_k(\varepsilon)$ be any sequence of positive monotone increasing functions defined for $\varepsilon>0$ and tending to zero with ε. There is a set A of first category in $\mathcal{I}^{(n)}$ such that no automorphism of $\mathcal{I}^{(n)}$ having a modulus of continuity $\leqslant$ than one of the functions $\delta_k(\varepsilon)$ carries A into a set of measure zero.

Let ε_i $(i=1,2,\ldots)$ be the radius of a sphere of n-dimensional measure $1/2^{i+1}$ and let E_i' be a positive number such that $\delta_k(E_i')\leqslant E_i$. Let $x_1,x_2,\ldots$ be a sequence of points dense in $\mathcal{I}^{(n)}$. Define F_k as the complement of the open set obtained by taking a sphere of radius $\delta_k(\varepsilon_i)$ about each point x_i. Then F_k is nowhere dense and any automorphism having modulus of continuity $\leqslant\delta_k(\varepsilon)$ will take F_k into a set of measure at least $1/2$. The set $A=\sum_{k=1}^{\infty}F_k$ is therefore carried into a set of measure at least $1/2$ by any automorphism which has for modulus of continuity a function $\leqslant$ than one of the functions $\delta_k(\varepsilon)$.

An automorphism is said to fulfill a Hölder condition if there exist constants $K>0$, $0<\alpha\leqslant 1$ such that for all x,y in $\mathcal{I}^{(n)}$ we have $|hx-hy|\leqslant K|x-y|^\alpha$. Such a transformation has a modulus of continuity $\leqslant$ than a function of the form $\delta_k(\varepsilon)=1/K\varepsilon^{1/k}$, whence it follows that not every first category set is equivalent to a set of measure zero under an automorphism fulfilling a Hölder condition.

Returning now to the consideration of unbounded sets we can deduce at once the following theorem.

Theorem 2. *Any set of first category in $R^{(n)}$ is equivalent to a set of n-dimensional measure zero under an automorphism of $R^{(n)}$.*

It is only necessary to divide the space into unit cubes and transform the part in each cube into a set of measure zero. If an automorphism of each is chosen which leaves boundary points fixed, the automorphisms of the separate cubes will join together to form an automorphism of the whole space which carries the given set into one of measure zero.

If one seeks to generalize the result that the transformations effecting the equivalence form a residual set in the space of automorphisms, one must first introduce a suitable metric. The metric previously used is not satisfactory because the domain is no longer compact. However, if one maps $R^{(n)}$ into the surface $S^{(n)}$ of the unit sphere in $R^{(n+1)}$ by projection from the north pole, one sees that automorphisms of $R^{(n)}$ are set in one-to-one correspondence with automorphisms of $S^{(n)}$ which leave the north pole fixed. Since $S^{(n)}$ is compact, the metric previously employed can be used, the absolute value signs now denoting distances in $R^{(n+1)}$. The proof given above for Theorem 1 carries through without change except that in the representation $A=\sum_{i=1}^{\infty}F_i$ the sets F_i should be taken to be bounded. Thus with this convention as to the metric in the space of automorphisms of $R^{(n)}$ one can again assert that the automorphisms which carry a given set of first category into a set of measure zero form a residual set in the space of automorphisms.

A set is said to be of *first category at a point* x if it intersects a neighborhood of x in a set of first category. It is said to have *the property of Baire* (in the weak sense) if it can be represented in the form $G-P_1+P_2$, where G is open, P_1 and P_2 of first category [2]). Applying the above results to the set P_1+P_2 we conclude:

Theorem 3. *Any set in $R^{(n)}$ (of $\mathfrak{S}^{(n)}$) having the property of Baire (in the weak sense) is equivalent under an automorphism of $R^{(n)}$ (or $\mathfrak{S}^{(n)}$) to a measurable set, indeed to a set differing from an open set by a set of measure zero. Points at which the set or its complement are of first category go into points of density 0 and 1 respectively. The automorphisms effecting such a transformation form a residual set in the space of automorphisms of $R^{(n)}$ (or $\mathfrak{S}^{(n)}$).*

Noting that a non-void open set has positive measure, on combining with previous results we obtain.

Theorem 4. *A set in $R^{(n)}$ (or $\mathfrak{S}^{(n)}$) having the property of Baire (in the weak sense) is of first or second category according as the automorphisms which carry it into a set of measure zero form a residual set or a first category set in the space of automorphisms of $R^{(n)}$ (or $\mathfrak{S}^{(n)}$).*

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[2]) C. Kuratowski, *Topologie* I, Monografie matematyczne **3**, Warszawa-Lwów 1933, p. 49.

To set up an automorphism which will transform a given set of first category into a set of measure zero, one may proceed as follows. Represent the given set in the form $A=\sum_{i=1}^{\infty}N_i$ where the sets N_i are bounded, nowhere dense, and $N_i\subset N_{i+1}$ $(i=1,2,\ldots)$. First compress N_1 to a set of measure less than 1 by a transformation h_1 which expands spheres in the complement, as described in the proof of Theorem 1. Follow this by a smaller transformation h_2 compressing h_1N_2 to measure less than $1/2$. At the i-th step let h_i compress $h_{i-1}\ldots h_2h_1N_i$ to measure less than $1/i$. Proceed in this manner making the transformations h_i grow small so rapidly that the successive products will converge uniformly together with their inverses. The limiting transformation is the desired automorphism. Now observe that at the i-th step the compression can be made continuously. The final automorphism can thus be joined to the identity by a continuous family of automorphisms, and we have in fact an isotopic deformation of the space. Hence we can assert the following more precise theorem.

Theorem 5. *Any set of first category in $R^{(n)}$ (or $\mathfrak{S}^{(n)}$) is equivalen to a set of measure zero under an isotopic deformation of the space. Likewise any set having the property of Baire (in the weak sense) can be deformed into a measurable set by an isotopic deformation of the space.*

The general question of the equivalence of an arbitrary set of first category to one of measure zero under automorphism has meaning in any complete metric space in which a measure is defined. One may ask how far the results here obtained can be generalized. Regarding more general measures in the spaces $R^{(n)}$ and $\mathfrak{S}^{(n)}$ it may be remarked at once that the proofs of Theorems 1, 2 and 3 apply to any Carathéodory measure, provided that each time a division into cells is made one chooses them so that their boundaries have measure zero. On the other hand, the methods of proof are restricted to euclidean or at least to locally euclidean spaces.

An example of a space requiring independent consideration is the Cantor set C with the usual probability measure. That is, C is assigned measure 1 and congruent subsets are assigned equal measure. Here the analog of Theorem 1 holds.

Theorem 6. *The automorphisms of C which carry a given set of first category (with respect to C) into a set of measure zero (in the probability measure) form a residual set in the space of automorphisms of C.*

The proof runs parallel to that of Theorem 1 until the proof that E_{ik} is dense. Here we divide C into disjoint segments $\sigma_1, \dots, \sigma_l$ of length less than δ. In each σ_i select a prefect subset lying outside hF_i. Since any two perfect subsets of C are homeomorphic, g can be chosen so as to leave each σ_j invariant and yet map s_j into a subset of σ_j having an arbitrarily large fraction of the measure of σ_i. Thus ghF_i can be made to have measure less than $1/k$. The proof then proceeds similarly as before.

Isotopic deformation of C is of course impossible, so that no analog of Theorem 5 is to be sought.

The possible extension of the results here obtained to locally compact groups with Haar measure is a question which the authors hope to discuss subsequently.

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The characteristic function of a sequence of sets and some of its applications [1]).

By

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The characteristic function of a sequence of sets, which is the subject of this paper, is a simple generalization of the well known notion of characteristic function of a single set due to de la Vallée-Poussin.

Since the time when the notion of characteristic function of a sequence of sets has been introduced [2]) it was applied in several publications by Knaster, Kuratowski, Sierpiński and the author of this paper [3]). In general, it may be stated that, for each sequence of sets which approximates a space in one sense or another, the application of its characteristic function leads to certain results. This happens e.g. in the case of 1° a basis of a separable space [3.8 (i)], 2° a basis consisting of closed-and-open subsets of a 0-dimensional separable space [3.8 (ii)], 3° the sequence of all closed-and-open subsets of a compact metrical space [4]), 4° a sequence approximating all the measurable sets from the point of view of the measure [5]).

[1]) Presented in part to the 3rd Polish Mathematical Congress in Warsaw, on September 30, 1937 (§§ 2 and 3) and to the Polish Mathematical Society, Warsaw section, on May 20, 1938 (§ 4).

[2]) Kuratowski [1], p. 124 defines in a certain proof a function which is the characteristic function of a basis of a 0-dimensional space. In a general way, the characteristic function of a sequence of sets has been defined in the paper: Szpilrajn [1], which contains several theorems on this notion and with its help solves some problems of Ulam.

[3]) Kuratowski [2], p. 56 and [3], pp. 244 and 245, Sierpiński [2], [3] and [4], Szpilrajn [1] and the preliminary reports [4a, b, c and d].

[4]) Kuratowski [3], p. 244.

[5]) Szpilrajn [4c].