Remark on weakly convergent cycles
(from a letter to Karol Borsuk).

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In recent papers *) you have introduced and used the notion of a weakly convergent cycle:

A sequence \( \langle \gamma_n \rangle \) of k-dimensional \( \delta \)-cycles with integer coefficients (\( \lim \delta = 0 \)) is weakly convergent if \( \gamma_n \overset{\delta}{\Rightarrow} \gamma_{n+1}, \lim \varepsilon = 0. \)

It seems to me to be possible that you could have used throughout the usual notion of convergent cycle. My reason for this belief is based on the following proposition.

1. If \( \langle \gamma_n \rangle \) is a weakly convergent cycle in the compact metric space \( K \), there is a subsequence \( \langle \gamma_{n_k} \rangle \) which is convergent.

The proof divides into several parts.

2. If \( H(\varepsilon, \delta) \) (\( \varepsilon > \delta > 0 \)) is the group of \( \delta \)-cycles (of dimension \( p \)) of \( K \) reduced modulo those which are \( \varepsilon \)-homologous to zero, then \( H(\varepsilon, \delta) \) has a finite basis.

Choose \( \nu > 0 \) so that \( 2\nu + \delta < \varepsilon \). Let \( A \) be a finite subset of \( K \) so that each point of \( K \) is within a distance \( \nu \) of \( A \). Let us agree that each subset of \( A \) of diameter \( < \varepsilon \) constitutes a simplex. Then \( A \) is an abstract complex. For each point \( P \) of \( K \), choose one of the nearest points of \( A \) and denote it by \( f(P) \).

*) Quelques relations entre la situation des ensembles et la retraction dans les espaces euclidiens, Fund. Math. 29 (1937), and with S. Eilenberg, Über etliche Abbildungen der Teilmengen euklidischer Räume auf die Kreislinie, Fund. Math. 26 (1936).
If $\sigma$ is a $\delta$-simplex of $K$, diameter $f(\sigma)<2\varepsilon+\delta<\varepsilon$. Hence, for any $\delta$-cycle $\gamma$ of $K$, $f(\gamma)$ is a simplicial map of $\gamma$ into the complex $A$, it is therefore a cycle of $A$. Those cycles of $A$ which are images of $\delta$-cycles of $K$ under $f$ form a subgroup. This group has a finite basis $a_1, a_2, \ldots, a_n$. Let $a_i'$ be a $\delta$-cycle of $K$ such that $f(a_i')=a_i$. Then $a_1', a_2', \ldots, a_n'$ is a basis for the $\delta$-cycles of $K$ relative to $\varepsilon$-homologies.

3. If $\langle \gamma_n' \rangle$ is a weakly convergent cycle and a number $\varepsilon>0$ is given, there is a subsequence $\langle \gamma_n'' \rangle$ such that any two of its terms are $\varepsilon$-homologous.

Choose a $\delta$ so that $\varepsilon>\delta>0$. Then choose an integer $N$ so that $n, m \geq N$ implies that $\gamma_n, \gamma_m$ are $\delta$-cycles and $\gamma_n \equiv \gamma_m$. Let $T(\varepsilon, \delta)$ be the torsion subgroup of $H(\varepsilon, \delta)$. Then $\gamma_n - \gamma_m$ is in $T(\varepsilon, \delta)$. Let $T'(\varepsilon, \delta)$ be the cokernel of $T(\varepsilon, \delta)$ containing $\gamma_n'$. Then each $\gamma_n'$ ($n \geq N$) is in $T'(\varepsilon, \delta)$. From 2 it follows that $T(\varepsilon, \delta)$ is finite; hence also $T'(\varepsilon, \delta)$. Therefore some element of $T'(\varepsilon, \delta)$ must contain infinitely many of the $\gamma_n'$.

To prove 1 we choose a sequence $\varepsilon_i \to 0$. Apply 3 to $\langle \gamma_n'' \rangle$ and $\varepsilon_i$ obtaining a subsequence $\langle \gamma_n''' \rangle$. Then apply 3 to $\langle \gamma_n''' \rangle$ and $\varepsilon_i$ obtaining $\langle \gamma_n^\infty \rangle$, and so on. The diagonal sequence $\langle \gamma_n^\infty \rangle$ is then a convergent cycle.

It follows from 1 that the homology groups based on weakly convergent cycles with weak homologies are identical with the homology groups based on convergent cycles with weak homologies.