Remark on a paper by Śpiewa-Neyman.

By


In Fund. Math. 5, p. 330, J. Śpiewa-Neyman refers to the following problem which W. Sierpiński suggested to him:

\[ E \text{ étant un ensemble linéaire de mesure nulle et } \mathcal{F} \text{ une famille d'intervalles, telle qu'il existe pour tout point } p \text{ de } E \text{ et tout } \varepsilon > 0 \text{ un intervalle } \delta \text{ de } \mathcal{F} \text{ de longueur } < \varepsilon \text{ contenant à son intérieur } p, \text{ existe-t-il pour tout } \varepsilon > 0 \text{ une suite (finie ou infinie) d'intervalles de } \mathcal{F} \text{ recouvrant } E \text{ et dont la somme de longueurs est } < \varepsilon? \]

I shall show by an example that the answer is in the negative.

Let \( G_n \) be the open interval \( 0 < x < 1 \). Introduce auxiliary variables

\[ x_k = 2^{-2^k} - 1, \quad x_{k,l} = 2^{-2^l} x_k - 1, \quad \ldots \]

\( k, l, \ldots \) being positive integers. Let \( I_{k,l}, \ldots, p \) be the set of values of \( x \) for which

\[ 0 < x_{k,l}, \ldots, p < 1. \]

Thus

\[ |I_{k,l}, \ldots, p| = 2^{-2^k - 2^{l-1} - \ldots - 2^p}. \]

Write

\[ G_1 = \sum I_k, \quad G_2 = \sum \sum I_{k,l}, \quad \ldots, \quad E = \bigcup G_n. \]

Then

\[ |G_{n+1}|/|G_n| = |G_1|/|G_0| = \sum 2^{-2^k} = 1/3, \quad |E| = 0. \]

Each sequence \( (k, l, \ldots) \) of positive integers defines a point of \( E \), namely

\[ x = 2^{-2^k} + 2^{-2^{k-1}} + \ldots \]

Let \( f \) be the family of intervals

\[ \delta_1; \quad 2^{-2^k} < x < 1 \]

where \( k \) is a positive integer. They cover every point of \( G_0 \).

Let \( f'_n \) be the family of intervals (of values of \( x \))

\[ \delta_{k,l}; \quad 2^{-2^l} < x < 1. \]

They cover every point of \( I_k \), so that the intervals of \( \Sigma f'_n \) cover every point of \( G_1 \). Define similarly \( f''_{k,l}, f'''_{k,l,m}, \ldots \) and take

\[ \mathcal{F} = f + \sum f'_n + \sum \sum f''_{k,l} + \ldots \]

Then each point of \( E \) is covered by an interval \( \delta \) of \( f \), an interval \( \delta_{k,l} \) of \( f'_n \), and these intervals have lengths at most \( 1, 2^{-2^k}, 2^{-2^{k-1}} \),

or at most

\[ 1, 2^{-2}, 2^{-4}, \ldots \]

Thus every point of \( E \) is covered by arbitrarily small intervals of the family \( \mathcal{F} \).

Now let \( \{A_n\} \) be a sequence of intervals of the family subject to the condition

\[ \sum |A_n| < \infty. \]

We shall show that they do not cover all points of \( E \).

Since \( |A_0| > \frac{1}{3} \), only a finite number of intervals \( \delta_k \) are included in the sequence \( \{A_n\} \). If any at all are included, let \( \delta_{K-1} \) be the largest which is included; if none are included, take \( K = 1 \). Then no point of \( I_K \) is covered by any interval \( A_n \) belonging to \( f \).

Since \( |\delta_{K,L}| > 2^{-2^{K-1}} \), only a finite number of intervals \( \delta_{K,L} \) are included in the sequence \( \{A_n\} \). If any at all are included let \( \delta_{K,L-1} \) be the largest which is included; if none are included, take \( L = 1 \). Then no point of \( I_{K,L} \) is covered by any interval \( A_n \) belonging to \( f \).

Continuing in this way we construct a sequence

\[ I_K, I_{K,L}, I_{K,L,M}, \ldots \]

which contract on to a point of \( E \) not covered by any \( A_n \).

Thus any sequence of intervals of \( \mathcal{F} \) which cover \( E \) must necessarily have lengths adding up to infinity.