Remark on a paper by Spława-Neyman.

By

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In Fund. Math. **5**, p. 330, J. Spława-Neyman refers to the following problem which W. Sierpiński suggested to him:

E étant un ensemble linéaire de mesure nulle et $\mathcal{F}$ une famille d'intervalles, telle qu'il existe pour tout point p de E et tout $\epsilon>0$ un intervalle δ de $\mathcal{F}$ de longueur $<\epsilon$ contenant à son intérieur p, existe-t-il pour tout $\epsilon>0$ une suite (finie ou infinie) d'intervalles de $\mathcal{F}$ recouvrant E et dont la somme de longueurs est $<\epsilon$?

I shall show by an example that the answer is *in the negative*.

Let $G_0=I$ be the open interval $0<x<1$. Introduce auxiliary variables

$$x_k=2^{2k}x-1, \quad x_{k,l}=2^{2l}x_k-1, \quad \ldots$$

$k,l,\ldots$ being positive integers. Let $I_{k,l,\ldots,p}$ be the set of values of x for which

$$0<x_{k,l,\ldots,p}<1.$$

Thus

$$|I_{k,l,\ldots,p}|=2^{-2k-2l-\ldots-2p}.$$

Write

$$G_1=\sum I_k, \quad G_2=\sum\sum I_{k,l}, \quad \ldots, \quad E=\prod G_n.$$

Then

$$|G_{n+1}|/|G_n|=|G_1|/|G_0|=\sum 2^{-2k}=1/3, \qquad |E|=0.$$

Each sequence $\{k,l,\ldots\}$ of positive integers defines a point of E, namely

$$x=2^{-2k}+2^{-2k-2l}+\ldots$$

Let f be the family of intervals

$$\delta_k: \qquad 2^{-2k}<x<1$$

where k is a positive integer. They cover every point of G_0.

Let f_k be the family of intervals (of values of x)

$$\delta_{k,l}: \qquad 2^{-2l}<x_k<1.$$

They cover every point of I_k, so that the intervals of Σf_k cover every point of G_1. Define similarly $f_{k,l}, f_{k,l,m}, \ldots$ and take

$$\mathcal{F}=f+\sum f_k+\sum\sum f_{k,l}+\ldots$$

Then each point of E is covered by an interval δ_k of f, an interval $\delta_{k,l}$ of $f_k, \ldots$ and these intervals have lengths at most

$$1,\ 2^{-2k},\ 2^{-2k-2l}, \ldots$$

or at most

$$1,\ 2^{-2},\ 2^{-4}, \ldots$$

Thus every point of E is covered by arbitrarily small intervals of the family $\mathcal{F}$.

Now let $\{\Delta_n\}$ be a sequence of intervals of the family subject to the condition

$$\sum|\Delta_n|<\infty.$$

We shall show that they do not cover all points of E.

Since $|\delta_k|>\frac{1}{2}$, only a finite number of intervals δ_k are included in the sequence $\{\Delta_n\}$. If any at all are included, let δ_{K-1} be the, largest which is included: if none are included, take $K=1$. Then no point of I_K is covered by any interval Δ_k belonging to f.

Since $|\delta_{K,l}|>2^{-2K-1}$, only a finite number of intervals $\delta_{K,l}$ are included in the sequence $\{\Delta_n\}$. If any at all are included let $\delta_{K,L-1}$ be the largest which is included: if none are included, take $L=1$. Then no point of $I_{K,L}$ is covered by any interval Δ_n belonging to f_K.

Continuing in this way we construct a sequence

$$I_K,\ I_{K,L},\ I_{K,L,M}, \ldots$$

which contract on to a point of E not covered by any Δ_n.

Thus any sequence of intervals of $\mathcal{F}$ which cover E must necessarily have lengths adding up to infinity.