On the derivative of the Lebesgue area of continuous surfaces.

By

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§ 1. Let the continuous surface S be given by the equations

(1) $$x=x(u,v), \qquad y=y(u,v), \qquad z=z(u,v),$$

where $x(u,v)$, $y(u,v)$, $z(u,v)$ are continuous in the closed square

(2) $$Q_0: \qquad 0\leqslant u\leqslant 1, \quad 0\leqslant v\leqslant 1.$$

If q is a closed square comprised in Q_0, then $L(q)$ will denote the area, in the sense of Lebesgue[1]), of the portion of S which corresponds to q by means of the equations (1). Thus $L(Q_0)$ stands for the area of S itself. We assume in the sequel that $L(Q_0)<+\infty$.

§ 2. The *function of squares* $L(q)$ possesses the following properties:

1) $L(q)\geqslant 0$;

2) For every square q comprised in Q_0 and for every non-overlapping system of squares $q_1,\ldots,q_j$ comprised in q, we have the inequality $L(q_1)+\ldots+L(q_j)\leqslant L(q)$ [2]).

§ 3. Let (u,v) be an interior point of Q_0. Take a sequence q_n of squares in Q_0 which contain (u,v) and are such that $mq_n\to 0$, where mq_n denotes the area of q_n. Consider the quotient $L(q_n)/mq_n$. If, for every such sequence q_n, this quotient converges to a finite limit (which is then independent of the choice of the sequence), then this limit is called the *derivative* of the function of squares $L(q)$ at the point (u,v), and is denoted by $L'(u,v)$.

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[1]) For the simple facts concerning the Lebesgue area, used in the sequel, the reader may consult the author's paper *Über das Flächenmass rektifizierbarer Flächen*, Math. Annalen **100** (1928), pp. 445-479.

[2]) See loc. cit. sub [1]).

§ 4. As a consequence of the properties 1) and 2), stated in § 2, the derivative $L'(u,v)$ exists almost everywhere in Q_0, is summable there and satisfies the inequality

$$\iint_{Q_0} L'(u,v)\, du\, dv \leqslant L(Q_0).$$

While this fact is not explicitly stated in the literature, it was proved implicitly by Banach[3]).

§ 5. We assume now that the functions $x(u,v)$, $y(u,v)$, $z(u,v)$ in (1) have partial derivatives of the first order almost everywhere in Q_0, and we put:

$$X=\partial(y,z)/\partial(u,v), \qquad Y=\partial(z,x)/\partial(u,v), \qquad Z=\partial(x,y)/\partial(u,v),$$
$$W=(X^2+Y^2+Z^2)^{1/2}.$$

§ 6. If, for every square q in Q_0, we happen to have

(3) $$L(q)=\iint_q W\, du\, dv,$$

then it would follow, by a well-known theorem of Lebesgue, that

(4) $$W(u,v)=L'(u,v)$$

almost everywhere in Q_0. But (4) might hold even if (3) does not hold. Indeed, for the special case $x=u$, $y=v$ (that is, for surfaces which are given by an equation of the form $z=f(x,y)$) Saks proved that (4) is true under the only assumption that $L(Q_0)$ is finite[4]). Earlier results of Lampariello[5]) implied only that $W(u,v)\leqslant L'(u,v)$.

§ 7. The facts referred to in § 6 suggest various questions which were not satisfactorily answered as yet, as far as the author is aware. The following result might therefore be of some interest, particularly on account of the light it throws upon certain recent results in the theory of the Lebesgue area of surfaces (cf. § 18).

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[3]) S. Banach, *Sur une classe de fonctions d'ensemble*, Fund. Math. **6** (1924), pp. 170-188, in particular the proof of th. 7 on pp. 177-178.

[4]) S. Saks, *Sur l'aire des surfaces $z=f(x,y)$*, Acta Szeged **3** (1927), pp. 170-176.

[5]) G. Lampariello, *Sulle superficie continue che ammettono area finita*, Rendiconti Lincei, Ser. 6, Vol. **3** (1926), pp. 294-298.

§ 8. ***Theorem.*** *Suppose that the surface S, given by* (1), *has a finite area* $L(Q_0)$, *and that the first partial derivatives of the functions* $x(u,v)$, $y(u,v)$, $z(u,v)$ *exist almost everywhere in* Q_0. *Then*

$$W(u,v) \leqslant L'(u,v) \tag{5}$$

almost everywhere in Q_0.

§ 9. The purpose of this paper is to give a simple proof of this theorem. As is so often the case in the theory of the area, the proof is based upon methods which are similar to those developed for the important class of rectifiable surfaces (surfaces which admit of a representation (1) where the functions $x(u,v)$, $y(u,v)$, $z(u,v)$ satisfy a Lipschitz condition). In fact the theorem represents, in a sense, part of what can be salvaged if the Lipschitz condition is replaced by the weaker assumptions of § 8. For this reason, we shall freely refer, for details of the proof, to arguments used in the theory of rectifiable surfaces [6]), whenever the necessary modifications are too obvious for explicit discussion.

§ 10. It will be convenient to use the following definition. Let $f(u,v)$ be continuous in Q_0. Let (u_0,v_0) be an interior point of Q_0 where the partial derivatives f_u, f_v exist. Let q be a square contained in Q_0 and containing (u_0,v_0) in its interior. Put

$$\varrho = [(u-u_0)^2+(v-v_0)^2]^{1/2},$$

$$\Delta = f(u,v) - f(u_0,v_0) - f_u(u_0,v_0)(u-u_0) - f_v(u_0,v_0)(v-v_0),$$

and denote by $\mu_f(q,u_0,v_0)$ the maximum of $|\Delta|/\varrho$ on the perimeter of q. Consider now a family (F) of such squares q. If (F) contains sequences q_n such that $mq_n \to 0$, and if for every such sequence we have the relation $\mu_f(q_n,u_0,v_0) \to 0$, then we shall say that the family (F) is *admissible* for the function $f(u,v)$ at the point (u_0,v_0).

§ 11. Consider now a transformation

$$x = \varphi(u,v), \qquad y = \psi(u,v), \tag{6}$$

where $\varphi(u,v)$, $\psi(u,v)$ are continuous in Q_0. Take in Q_0 a square q, denote by b the boundary of q, and by $\bar{b}$ the image of b under the transformation (6). Define $O_{xy}(q)$ as the set of points, in the xy plane, which are not on $\bar{b}$ and whose topological index, with respect to the closed continuous curve $\bar{b}$, is different from zero. The set

[6]) See, also for further literature, loc. cit. sub [1]).

$O_{xy}(q)$ is clearly open. Let $mO_{xy}(q)$ denote its measure. Consider now a point (u_0,v_0), interior to Q_0, such that the partial derivatives $\varphi_u, \varphi_v, \psi_u, \psi_v$ exist at (u_0,v_0). If we assume, for a moment, that φ and ψ admit of complete differentials [7]) at the point (u_0,v_0), then it follows [8]) that $mO_{xy}(q_n)/mq_n \to |J(u_0,v_0)|$, where $J=\varphi_u\psi_v-\varphi_v\psi_u$, for *every* sequence of squares q_n, containing (u_0,v_0) as an interior point, and such that $mq_n \to 0$. The proof of this fact depends however only upon the circumstance that, as a consequence of the existence of the complete differentials, we have $\mu_\varphi(q_n,u_0,v_0)\to 0$, $\mu_\psi(q_n,u_0,v_0)\to 0$. The existence of the complete differentials is used only to secure these last relations. Hence the same proof yields the following result:

§ 12. *If the partial derivatives* $\varphi_u, \varphi_v, \psi_u, \psi_v$ *exist at the interior point* (u_0,v_0) *of* Q_0, *and if there exists a family* (F) *of squares which is admissible for both* $\varphi(u,v)$ *and* $\psi(u,v)$ *at* (u_0,v_0), *then*

$$mO_{xy}(q_n)/mq_n \to |J(u_0,v_0)|, \tag{7}$$

where $J=\varphi_u\psi_v-\varphi_v\psi_u$, *for every sequence* q_n *of* (F) *such that* $mq_n \to 0$.

§ 13. Consider a function $f(u,v)$ which is continuous in Q_0. Suppose for a moment that $f(u,v)$ satisfies a Lipschitz condition in Q_0. Then $f(u,v)$ admits of a complete differential almost everywhere in Q_0 [9]). In our terminology, this fact implies that for almost every point (u_0,v_0) in Q_0 the family of *all* the squares q, comprised in Q_0 and containing (u_0,v_0) as an interior point, is an admissible family for $f(u,v)$. Suppose now only that the *approximate* partial derivatives of $f(u,v)$ exist almost everywhere in Q_0. According to Stepanoff, $f(u,v)$ admits then of an *approximate* complete differential almost everywhere in Q_0 [10]). While this conclusion would not be sufficient for our purposes, we can obtain a sufficiently strong statement if we make the stronger assumption that the partial derivatives f_u, f_v exist almost everywhere in Q_0 *in the usual sense.* The reasoning which leads to the theorem of Stepanoff [11]) yields then, after modifications which are too trivial to be put down explicitly, the following corollary.

[7]) See [6]).

[8]) See [6]).

[9]) See H. Rademacher, *Über partielle und totale Differenzierbarkeit von Funktionen mehrerer Variablen*, part. I, Math. Annal. **79** (1919), pp. 340-359.

[10]) See the presentation in S. Saks, *Theory of the Integral*, Monogr. Matem. VII, Warszawa-Lwów 1937, Chapter IX.

[11]) See loc. cit. sub [10]), pp. 300-303.

§ 14. *If $f(u,v)$ is continuous in Q_0, and if the partial derivatives f_u, f_v exist almost everywhere in Q_0, then at almost every point (u_0,v_0) in Q_0 there exists an admissible family (F) for $f(u,v)$ such that:*

1) *the squares of (F) have sides parallel to the axes and have (u_0,v_0) for center,*

2) *the sets of points in which the perimeters of all the squares of (F) intersect the lines $u=u_0$, $v=v_0$ respectively, are measurable and have the (linear) density 1 at the point (u_0,v_0).*

§ 15. Obviously, it follows that if a finite number of functions $f_1(u,v),\dots,f_j(u,v)$ are given, such that every one satisfies the conditions of § 14, then at almost every point of Q_0 there exists a family (F) which possesses the properties stated in § 14 with respect to all these functions *simultaneously*.

§ 16. The proof of the theorem of § 8 is now immediate. Consider the three transformations

$$y=y(u,v),\ z=z(u,v);\quad z=z(u,v),\ x=x(u,v);\quad x=x(u,v),\ y=y(u,v). \tag{8}$$

Under the assumptions of § 8, it follows by §§ 15 and 12 that for almost every point (u_0,v_0) in Q_0 there exists a sequence of squares q_n such that

1) q_n is comprised in Q_0,
2) (u_0,v_0) is interior to q,
3) $mq_n\to 0$,
4) we have the relations:

$$\frac{mO_{yz}(q_n)}{mq_n}\to|X(u_0,v_0)|,\quad \frac{mO_{zx}(q_n)}{mq_n}\to|Y(u_0,v_0)|,\quad \frac{mO_{xy}(q_n)}{mq_n}\to|Z(u_0,v_0)|, \tag{9}$$

where the meaning of the symbols O_{yz}, O_{zx}, O_{xy} follows from § 11, while $X(u,v)$, $Y(u,v)$, $Z(u,v)$ are the Jacobians of the transformations (8). Since $([mO_{yz}(q)]^2+[mO_{zx}(q)]^2+[mO_{xy}(q)]^2)^{1/2}\leqslant L(q)$ for every square q [12]), it follows from (9) that we have

$$\liminf \frac{L(q_n)}{mq_n}\geqslant W(u_0,v_0). \tag{10}$$

On the other hand, by § 4, the derivative $L'(u,v)$ exists at almost every point (u_0,v_0) of Q_0. Hence (10) implies that $L'(u,v)\geqslant W(u,v)$ almost everywhere in Q_0, and the theorem is proved.

[12]) See loc. cit. sub [1]).

§ 17. We mention an immediate application. To the assumptions of § 8 let us add the following one: There exists a sequence S_n of continuous surfaces, such that S_n converges to S in the sense of Fréchet and

$$L_n\to \iint_{Q_0} W\,du\,dv, \tag{11}$$

where L_n is the Lebesgue area of S_n. *Then we have*

$$L(Q_0)=\iint_{Q_0} W\,du\,dv. \tag{12}$$

Indeed, by the lower semi-continuity of the Lebesgue area, we have $\liminf L_n\geqslant L(Q_0)$, and hence (11) implies that

$$\iint_{Q_0} W\,du\,dv\geqslant L(Q_0),$$

while the theorem of § 8 yields, with regard to § 4, the complementary inequality

$$\iint_{Q_0} W\,du\,dv\leqslant L(Q_0). \tag{13}$$

§ 18. Despite its trivial character, the preceding result permits us to account for and to improve upon several recent results [13]) concerned with the formula (12) and the inequality (13) respectively. Indeed, the various sets of assumptions used loc. cit. [13]) are obviously more than sufficient to justify the application of the result of § 17. We leave it to the reader to formulate the generalized theorems suggested by these remarks.

[13]) E. J. McShane, *Integrals over surfaces in parametric form*, Annals of Math. **34** (1933), pp. 815-838; C. B. Morrey, *A class of representations of manifolds*, part I, Americ. Journ. of Math. **55** (1933), pp. 683-707; T. Radó, *A remark on the area of surfaces*, ibid. **58** (1936), pp. 598-606.

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