Remark on the symmetrical derivates of additive functions of intervals.

By

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Some time ago, S. Saks[1]) raised the question whether it is possible for the upper and lower limits, as $h, k \to 0$, of

$$\frac{1}{4hk}\int_{x-h}^{x+h}\int_{y-k}^{y+k} f(x,y)\,dx\,dy,$$

to be finite and unequal at the points (x,y) of a set of positive measure. Although the analogous problem concerning

$$\frac{1}{hk}\int_{x}^{x+h}\int_{y}^{y+k} f(x,y)\,dx\,dy$$

was, soon afterwards, solved by A. S. Besicovitch[2]), the question in its original form has never been explicitly answered. It seems worth while to point out that the answer can be easily obtained from the result of Besicovitch, by means of a lemma due in principle to A. Khintchine[3]), but stated by him only for functions of one variable. For the sake of completeness we give a statement of the two-dimensional form of the lemma which is more general than that actually required. It is easily seen that the method of proof may be extended to cover the case of m-dimensional space, $m>2$.

[1]) Fund. Math. **22** (1934), 257-261.

[2]) Fund. Math. **25** (1935), 209-216.

[3]) Fund. Math. **9** (1927), 212-279 (see especially 217).

$F(R)$ denotes an additive function of rectangles[4]). If $x=(x_1,x_2)$ is a fixed point and $\xi=(\xi_1,\xi_2)$ a variable point, and if $f(x,\xi)=$ $=F[x_1,\xi_1;\, x_2,\xi_2]$ is, for any one value of x, a function of ξ measurable (B), then it is so also for any other (fixed) value of x. In this case we say that F is measurable (B)[5]). If h_1, h_2 are any two positive numbers, we denote by $\mathrm{r}(h_1,h_2)$ the parameter of regularity of a rectangle with sides of lengths h_1 and h_2; that is,

$$\mathrm{r}(h_1,h_2)=\min(h_1/h_2,\, h_2/h_1).$$

Lemma. *If $F(R)$ is measurable* (B), *then, almost everywhere,*

(i) $$\overline{F}_s(x) \leqslant \overline{\lim_{h_1,h_2\to+0}} \frac{1}{h_1h_2} F[x_1-h_1,\; x_1+h_1;\; x_2-h_2,\; x_2+h_2];$$

(ii) *if* $0\leqslant\alpha<\beta<1$,

$$\overline{F}_{(\beta)}(x) \leqslant \overline{\lim_{\substack{h_1,h_2\to+0\\ \mathrm{r}(h_1,h_2)\geqslant\alpha}}} \frac{1}{h_1h_2} F[x_1-h_1, x_1+h_1;\quad x_2-h_2,\quad x_2+h_2].$$

We give the proof of part (ii) of the lemma. (The proof of part (i) is the same in essence, but is actually rather simpler.) If our statement were false, we could choose, in succession, fixed α, β, $0\leqslant\alpha<\beta<1$, a rational number s, and $\delta>0$, such that, at each point $x=(x_1,x_2)$ of a set E of positive outer measure, we should have, writing $G(R)=F(R)-s\cdot|R|$,

$$G[x_1-h_1,\; x_1+h_1;\; x_2-h_2,\; x_2+h_2]<0$$

whenever $0<h_1\leqslant\delta$, $0<h_2\leqslant\delta$, $\mathrm{r}(h_1,h_2)\geqslant\alpha$, but

$$\overline{G}_{(\beta)}(x)>0.$$

Let x_0 be a point of outer density of E, belonging to E. Then we can find a rectangle $R=[a_1,b_1;\, a_2,b_2]$ including x_0, such that $G(R)>0$, $\mathrm{r}(R)\geqslant\beta$, $\delta(R)\leqslant\delta$, and

$$|ER|>(1-\tfrac{1}{16}\epsilon^2)\cdot|R|,$$

where

$$\epsilon=\frac{\beta-\alpha}{\beta+\alpha},$$

[4]) For the terminology and notations see S. Saks, *Theory of the Integral*, Monogr. Matem. **7** (Warsaw, 1937), especially pp. 57, 106, 134.

[5]) If $f(x,\xi)$ is measurable, but not measurable (B), with respect to ξ for one fixed x, it does not follow that it is so also for any other fixed x.

so that
$$0<\epsilon<1.$$

We write
$$c_i=\frac{a_i+b_i}{2}, \qquad d_i=\frac{b_i-a_i}{2}, \qquad (i=1,2).$$

Suppose for convenience that $d_1\leqslant d_2$; then, since $\mathrm{r}(R)\geqslant\beta$,
$$\beta d_2\leqslant d_1\leqslant d_2.$$

Let $x=(x_1,x_2)$ be any point of R, and define the four intervals
$$R_1(x)=[a_1,x_1;\, a_2,x_2] \qquad R_2(x)=[a_1,x_1;\, x_2,b_2]$$
$$R_3(x)=[x_1,b_1;\, a_2,x_2] \qquad R_4(x)=[x_1,b_1;\, x_2,b_2].$$

Let A_i $(i=1,2,3,4)$ be the set of points x of R such that $G[R_i(x)]>0$. Since, for any x, the $R_i(x)$ together make up R, each point x of R must belong to at least one set A_i. Now since F is measurable (B), it is easily seen that the sets A_i are measurable. Hence, if we denote by S the interval $[c_1-\epsilon d_1, c_1+\epsilon d_1;\, c_2-\epsilon d_2, c_2+\epsilon d_2]$ at least one set A_i satisfies
$$|SA_i|\geqslant\tfrac{1}{4}|S|=\tfrac{1}{4}\epsilon^2|R|.$$

Suppose for example that $|SA_1|\geqslant\frac{1}{4}\epsilon^2|R|$. Let B_1 be the set of points $y=(y_1,y_2)$ such that $(2y_1-a_1, 2y_2-a_2)$ belongs to SA_1. Then B_1 is measurable and $|B_1|\geqslant\frac{1}{16}\epsilon^2|R|$, hence there exists a point y, say, of B_1E. Then, since $(2y_1-a_1, 2y_2-a_2)$ is in S, we obtain
$$\mathrm{r}(y_1-a_1, y_2-a_2)\geqslant\frac{d_1(1-\epsilon)}{d_2(1+\epsilon)}\geqslant\frac{\beta(1-\epsilon)}{1+\epsilon}\geqslant\alpha.$$

Hence, by the definition of E (putting $h_i=y_i-a_i$),
$$G[a_1, 2y_1-a_1;\, a_2, 2y_2-a_2]<0.$$

But $(2y_1-a_1, 2y_2-a_2)$ is in A_1, and so we have a contradiction. Thus the lemma (ii) is proved, and (i) is proved in a similar way.

We write for brevity $\overline{F}_{\text{s-sym}}(x)$ and $\overline{F}_{(\alpha)\,\text{sym}}(x)$ for the upper limits occurring, on the right-hand side, in the statements (i) and (ii), respectively, of the lemma. We define similarly $\underline{F}_{\text{s-sym}}(x)$ and $\underline{F}_{(\alpha)\,\text{sym}}(x)$. It is at once obvious that, for each x,
$$\underline{F}_{\text{s}}(x)\leqslant\underline{F}_{\text{s-sym}}(x)\leqslant\overline{F}_{\text{s-sym}}(x)\leqslant\overline{F}_{\text{s}}(x).$$
and also that for any α, $0<\alpha\leqslant1$,
$$\underline{F}_{(\alpha)}(x)\leqslant\underline{F}_{(\alpha)\,\text{sym}}(x)\leqslant\overline{F}_{(\alpha)\,\text{sym}}(x)\leqslant\overline{F}_{(\alpha)}(x),$$

Hence, combining our lemma with known results [6]) concerning $\overline{F}_s(x)$, $\overline{F}_{(\alpha)}(x)$, we obtain finally the following theorem.

Theorem. *If $F(R)$ is measurable* (B), *in particular if*
$$F(R)=\iint_R f(u,v)\,du\,dv,$$
then, almost everywhere,

(i) *either* $\overline{F}_{\text{s}}(x)=\overline{F}_{\text{s-sym}}=+\infty$, *or* $\overline{F}_{\text{s}}(x)=\overline{F}_{\text{s-sym}}(x)=F'(x)$, *finite*,

and

(ii) *either* $\overline{F}(x)=\overline{F}_{(\alpha)\,\text{sym}}(x)=+\infty$ *and* $\underline{F}(x)=\underline{F}_{(\alpha)\,\text{sym}}(x)=-\infty$, *for all* α, $0<\alpha<1$, *or* $\overline{F}_{(\alpha)\,\text{sym}}(x)=\underline{F}_{(\alpha)\,\text{sym}}(x)=F'(x)$, *finite, for all* α, $0<\alpha<1$.

Corollary. *If, at each point of a set E,*
$$-\infty<\underline{F}_{\text{s-sym}}(x)\leqslant\overline{F}_{\text{s-sym}}(x)<\infty,$$
then $F'_{\text{s}}(x)$ exists [*and therefore* $\overline{F}_{\text{s-sym}}(x)=\underline{F}_{\text{s-sym}}(x)=F'_{\text{s}}(x)$] *almost everywhere in E.*

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[6]) S. Saks, *Theory of the Integral*, l. c., pp. 137, 139; or A. T. Ward, Fund. Math. **28** (1937), 265-279.