A theorem concerning connected point sets.

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Knaster and Kuratowski[1]) have recently given an example of a connected[2]) point set M which contains a point P such that $M-P$ is a totally disconnected[3]) point set. The purpose of the present note is to show that no connected point set can have more than one point such that when it is removed, the remainder is totally disconnected.

Theorem A. *Suppose M is a connected point set which contains a point P having the property that $M-P$ is totally disconnected. Under these conditions, P is the only point whose omission totally disconnects M.*

Proof: — Let us suppose our theorem false. Then there is in M a point Q such that (1) Q is different from P and (2) $M-Q$ is totally disconnected. By hypothesis as $M-P$ is totally disconnected, $M-P=\overline{M}_1+\overline{M}_2$, two mutually exclusive sets neither of which cointains a limit point of the other one. Let M_1 denote that one of the two sets, $\overline{M}_1$ and $\overline{M}_2$, which contains Q while M_2 denotes the other one. Likewise, $M-Q=\overline{S}_1+\overline{S}_2$, two mutually exclusive sets neither of which contains a limit point of the other one. Let S_1 denote that one of the sets, S_1 and S_2, which contains P while S_2 denotes the other one. Two cases may arise.

[1]) *Fundamenta Mathematicae* t. II, p. 241, 1921.

[2]) A set of points is said to be *connected* if, however it be divided into two mutually exclusive subsets one of them contains a limit point of the other one.

[3]) A set of points is said to be totally disconnected if it contains no connected subset containing more than a single point.

Case I. *$S_2 + Q$ is connected.* But $S_2 + Q$ is a subset of $M - P$. Then $S_2 + Q$ is a connected subset of $M \quad P$ containing more than a single point. This is contrary to the hypothesis that $M - P$ is totally disconnected.

Case II. *$S_2 + Q$ is not a connected point set.* Then $S_2 + Q = \overline{H}_1 + \overline{H}_2$, two mutually exclusive sets neither of which contains a limit point of the other one. Let H_1 denote that one of the sets, $\overline{H}_1$ and $\overline{H}_2$, which contains Q while H_2 denotes the other one. Consider the sets I_1 and I_2, where I_1 is the sum of H_1 and S_1 while I_2 is identical with H_2. Clearly neither of the sets, I_1 and I_2, contains a limit point of the other one. But $M = I_1 + I_2$. Hence M is not connected, contrary to hypothesis.

Thus in both cases we are led to a contradiction. Hence it is not possible that $M - Q$ be totally disconnected. Hence our theorem is established.

It is interesting to note that in our proof we used only the fact that $M - Q$ was not a connected point set. In view of this, we may state the following result.

Theorem B. *Suppose M is a connected point set which contains a point P having the property that $M - P$ is totally disconnected. Then, if Q is any point of M, different from P, $M - Q$ is a connected point set.*

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