Concerning connectedness im kleinen and a related property.

By

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Sierpiński has recently shown[1]) that in order that a closed and connected set of points M should be a continuous curve it is necessary and sufficient that, for every positive number ε, the point-set M should be the sum of a finite number of closed and connected point-sets each of diameter less than ε. It follows that, *as applied to point-sets which are closed, bounded and connected*, this property is equivalent to that of connectedness im kleinen. In the present paper I will make a further study of these two properties (or rather of suitable modifications of these properties) especially as applied to sets which are not necessarily closed.

A point-set M will be said to have property S if and only if, for every positive number ε, M is the sum of a finite number of connected[2]) point-sets $M_{1n}, M_{2n}, \ldots M_{(n-1)n}$ each of diameter less than ε.

The only difference between property S and the above mentioned property of Sierpiński's is that Sierpiński stipulates that the connected sets $M_{1n}, M_{2n}, \ldots M_{(n-1)n}$ should be *continuous* (and therefore closed) while in the statement of property S this stipulation is not made. It follows that every point-set which has Sierpiński's property in its original form possesses also the modified form S. But the converse does not hold true.

For instance every Jordan region[3]) has property S but no Jordan region, indeed no point-set which is not closed, can have Sierpiński's property in its first form.

[1]) W. Sierpiński, Sur une condition pour qu'un continu soit une courbe jordanienne, *Fundamenta Mathematicae* T. I (1920), pp. 44—60.

[2]) A point-set is said to be connected if it is not the sum of two mutually exclusive point-sets neither of which contains a limit point of the other one.

[3]) By a Jordan region is meant the interior of a simple closed curve.

A point-set M is said to be connected im kleinen [1]) at the point P of M if for every positive number ε there exists a positive number $\delta_{P\varepsilon}$ such that if X is a point of M at a distance from P less than $\delta_{P\varepsilon}$ then X and P fie together in a connected subset of M of diameter less than ε.

A point-set M is said to be uniformly [2]) connected im kleinen if for every positive number ε there exists a positive number δ_ε such that if X and Y are two points of M at a distance apart less than δ_ε then X and Y lie together in (can be joined by) a connected subset of M of diameter less than ε.

Theorem 1. *If the point-set M is uniformly connected im kleinen then the point-set M' (composed of M together with its boundary) is connected im kleinen.*

Proof. Let P denote a point belonging to M'. Let K denote a circle with center at P and let $\overline{K}$ denote a second circle, concentric with K and lying within it. Since M is uniformly connected im kleinen there exists a circle K^* with center at P such that if X and Y are two points of M lying within K^* then X and Y lie together in a connected subset of M that lies wholly within $\overline{K}$. Suppose now that A is any point of M' lying within K^*. Then there exist two sequences of points $A_1, A_2, A_3, \ldots$ and $P_1, P_2, P_3, \ldots$ belonging to M and having A and P respectively as sequential limit points. For each n there exists a connected point-set t_n which contains A_n and P_n, is a subset of M and lies wholly within K^*. Let t denote the limiting set of the sequence of connected sets $t_1, t_2, t_3, \ldots$. Then t is a closed, connected subset of M' which contains A and P and lies wholly within K. Thus M' is connected im kleinen at every point P.

It does not follow [3]) that if a set is uniformly connected im kleinen then its boundary is connected im kleinen.

Theorem 2. *Every point-set which satisfies Condition S is connected im kleinen.*

[1]) See Hans Hahn, *Wien. Ber.* 123 (1914), p. 2433. According to Hahn's definition X and P must lie together in a *closed* and connected subset of M of diameter less than ε.

[2]) See Hahn, loc. cit.

[3]) See A. Rosenthal: Über Peanoflächen und ihren Rand, *Mathematische Zeitschrift* 10 Band (1921) p. 102.

Proof[1]). Suppose on the contrary that there exists a point-set M which satisfies Condition S but is not connected im kleinen. Then for some point P and some positive number ε there exists an infinite sequence α of distinct points $P_1, P_2, P_3, \ldots$, all belonging to M, such that P is the sequential limit point of α and such that no point of α lies with P in a connected subset of M of diameter less than ε. But since M satisfies Condition S it is the sum of a finite collection β of connected point-sets $M_1, M_2, M_3, \ldots M_k$ each of diameter less than ε. Since each point of the infinite sequence α belongs to some point-set of the finite collection β therefore there exists a set M_i of the collection β that contains infinitely many distinct points of the sequence α. The point-set $M_i + P$ is a connected subset of M, of diameter less than ε, which contains P and points of α. Thus the supposition that M is not connected im kleinen has led to a contradiction. Theorem 2 is therefore established.

An example of a bounded, connected point-set which is connected im kleinen but does not satisfy Condition S is the set of all those points on the curve $y = \sin \frac{1}{x}$ whose abscissas lie between 0 and 1. From Theorem 2 and the existence of this example it follows that, as a condition applied to bounded connected point-sets, Condition S is stronger than that of connectivity im kleinen.

Theorem 3. *If the bounded point-set M is uniformly connected im kleinen then it has property S.*

Proof. By hypothesis if ε is a positive number there exists a positive number δ_ε such that if P is a point of M and K is a circle with center at P and radius equal to δ_ε then every point within K lies together with P in a connected subset of M of diameter less than ε. Let M_K denote the set of all points [X] such that X and P lie together in a connected subset of M of diameter less than ε. The set M_K is connected and contains every point of M whose distance from P is less than δ_ε. It is easy to see that there exists a finite set G of circles $K_1, K_2, K_3, \ldots K_n$ of radius δ_ε and each having as its center some point of M such that every point of M is within some circle of the set G. It follows that every

[1]) This theorem may also be established by an easily made modification of a proof given by Sierpiński for the case when the connected sets in question are a sumed to be closed.

point of M belongs to one of the connected point-sets M_{K_1}, M_{K_2}, $M_{K_3}, \dots M_{K_n}$. But these connected point-sets are all of diameter less than ε. Theorem 3 is therefore established.

It is not true however that every bounded point-set which has property S is also uniformly connected im kleinen. Consider, for example, the set of points $K-P$ where K is a circle and P is a point belonging to K.

It follows from the above results that *Condition S, as applied to bounded point-sets is stronger than the condition of connectedness im kleinen but weaker than that of uniform connectedness im kleinen.*

In a recent paper [1]) I have shown that in order that a simply [2]) connected bounded domain should be the interior of a simple closed curve it is necessary and sufficient that it should be uniformly connected im kleinen. It is not however true that every simply connected bounded domain which has property S is the interior of a simple closed curve. Consider for example the point-set $(K+A) \dot{-} OA$ where K is the interior of a circle with center at O, A is a point on this circle and OA is the straight line interval whose endpoints are O and A. This point-set has property S but it is not uniformly connected im kleinen.

Theorem 4. *In order that the simply connected bounded domain R should have a continuous curve as its boundary it is necessary and sufficient that R should have property S.*

Proof. This condition is sufficient. For suppose on the contrary that there exists a bounded simply connected domain R which has property S but whose boundary M is not connected im kleinen. It follows by the argument beginning with the second line from the bottom of page 365 and ending in the twenty third line from the top of page 366 of a paper mentioned above [3]) that there exist a point P belonging to M, two circles K and $\overline{K}$ with center at P, such that $\overline{K}$ lies within K, and an infinite sequence of mutually

[1]) A characterization of Jordan regions by properties having no reference to their boundaries: *Proceedings of the National Academy of Sciences*, vol. 4 (1918), pp. 364—370.

[2]) A point-set R is said to be a domain if no point of R is a limit point of $S-R$, where S is the set of all points in the plane. A bounded domain R in the plane is said to be simply connected if R contains the whole of every bounded point-set whose boundary is a subset of R.

[3]) R. L. Moore, loc. cit.

exclusive closed connected point-sets $\overline{t}_1, \overline{t}_2, \overline{t}_3, \ldots$ belonging to M such that if m is any positive integer then (1) every point of $\overline{t}_m$ is either on K or on $\overline{K}$ or between K and $\overline{K}$, (2) $\overline{t}_m$ contains at least one point on K and at least one point on $\overline{K}$. Let r and $\overline{r}$ denote the radii of K and $\overline{K}$ respectively and let K^* denote a circle with center at P and a radius equal to $\frac{r+\overline{r}}{2}$. For each positive integer m the point-set t_m contains at least one point A_m on K^*. There exists an infinite sequence $B_1, B_2, B_3, \ldots$ of distinct points of the set $A_1, A_2, A_3, \ldots$, lying in the order $B_1 B_2 B_3 \ldots B_n B_{n+1} \ldots$ on the curve K^*. For each n let k_n denote that one of the point-sets $\overline{t}_1, \overline{t}_2, \overline{t}_3, \ldots$ which contains B_n. For each even integer n let K_n denote a circle with center at B_n and radius less than $\frac{r-\overline{r}}{4}$ which neither contains nor encloses any point of k_{n-1} or of k_{n+1}. Since B_n is a point of the boundary of R there exists within K_n a point C_n belonging to R. Let $\varepsilon = \frac{r-\overline{r}}{4}$. Then R is not the sum of a finite collection of connected point-sets all of diameter less than ε. For consider the points $C_2, C_4, C_6, \ldots$ If m and n are even integers such that $m < n$ then every connected point-set of diameter less than ε which contains both C_m and C_n must clearly contain a point either of k_{n-1} or of k_{n+1} and therefore of M. Thus there exist no connected subset of R of diameter less than ε which contains two distinct points of the infinite set $C_2, C_4, C_6, \ldots$. It follows that R does not have property S. Thus the supposition that the condition of Theorem 4 is not sufficient has led to a contradiction.

The condition of Theorem 4 is also necessary. For suppose on the contrary that there exists a simply connected bounded domain R which does not have property S but has for its boundary a point-set M which is a continuous curve. By hypothesis there exists a positive number ε such that R cannot be expressed as the sum of a finite number of connected subsets each of diameter less than ε. There exists a square K that encloses R. Let k be the length of a side of K. Subdivide K into 4^n equal squares $K_1, K_2, K_3, \ldots K_{4^n}$ where n is so large that the length of a diagonal of K_1 is less than $\frac{\varepsilon}{3}$. For each positive integer $m (1 \leq m \leq n)$ let T_m denote the square which is concentric with K_m and whose sides are respectively parallel to, but three times as long as, those of K_m. For each point P that belongs to R and lies on or within K_m let R_{mP} denote the greatest connected subset of R which contains P and lies wholly within T_m. Since R does not have property S and each set R_{mP} is of diameter less than ε there must exist at least one value $\overline{m}$ of m and infinitely many distinct points $P_1, P_2, P_3, \dots P_n \dots$ belonging to R and lying on or within $K_{\overline{m}}$ such that no two of the connected sets $R_{\overline{m}P_1}, R_{\overline{m}P_2}, R_{\overline{m}P_3}, \dots$ have a point in common. Since R is a connected domain every $R_{\overline{m}P_i}$ has at least one limit point on the square $T_{\overline{m}}$. Let $H_{\overline{m}}$ denote a square concentric with the squares $K_{\overline{m}}$ and $T_{\overline{m}}$ and such that $a = \dfrac{b+c}{2}$ where a, b and c are lengths of sides of $H_{\overline{m}}, K_{\overline{m}}$ and $T_{\overline{m}}$ respectively. For each i the connected point set $R_{\overline{m}P_i}$ contains a point without $H_{\overline{m}}$ and a point P_i within $H_{\overline{m}}$. Hence it must contain a point X_i on $H_{\overline{m}}$. The infinite sequence of distinct points $X_1, X_2, X_3, \dots$ contains an infinite subsequence of distinct points $Y_1, Y_2, Y_3, \dots$ lying in the order $Y_1 Y_2 Y_3 \dots$ on the square $H_{\overline{m}}$. For each n let R_n denote that one of the sets $R_{\overline{m}P_1}, R_{\overline{m}P_2}, \dots$ to which Y_n belongs. For each n the boundary of R_n contains a closed connected subset $\overline{R}_n$ such that (1) $\overline{R}_n$ contains at least one point on $K_{\overline{m}}$ and at least one point on $H_{\overline{m}}$, (2) every point of $\overline{R}_n$ lies either on $K_{\overline{m}}$ or on $H_{\overline{m}}$ or between them. It can easily be seen that no two of the sets $\overline{R}_2, \overline{R}_4, \overline{R}_6, \dots \overline{R}_{2n}, \overline{R}_{2(n+1)}, \dots$ have a point in common and that indeed if Z and W are points belonging to different sets of this infinite collection then Z and W do not lie together in any connected subset of M which is contained in the point-set I composed of $K_{\overline{m}}, H_{\overline{m}}$ and all points that lie between them. Let $\varepsilon = \dfrac{b-c}{4}$. No matter how small positive number δ may be there exist two even integers m and n such that the distance from Y_m to Y_n is less than δ. It follows that there exists no connected subset of M of diameter less than ε that contains both Y_m and Y_n. Therefore M is not connected im kleinen. Thus the supposition that the condition of Theorem 4 is not necessary has led to a contradiction.