Ajoutons au sujet du th. 2 que dans les conditions (18) et (20) on a
\[
\mathcal{D}(s^* \geq x) \leq 32 e^{-x/4M} \quad \text{pour} \quad x \geq B/M
\]
et dans les conditions (18 bis) et (20 bis) on a
\[
\mathcal{D}(s^* \geq x) \leq 32 e^{-x/40H} \quad \text{pour} \quad x \geq B/4H.
\]
L'inégalité (24) s'obtient de (23), en y posant \( a = 1/M \). L'inégalité (24 bis) se démontre d'une façon analogue. En même temps, si l'on remplace dans (24) et (24 bis) \( s^* \) par \( s \), le facteur 32 dans les membres droits peuvent être remplacés par 2 (cf. Kolmogoroff, loc. cit.).

La seconde partie du th. 2 permet aussi de généraliser légèrement le théorème sur le logarithme itéré, en l'étendant aux variables pas nécessairement bornées. Soit \( X_1, X_2, \ldots \) une suite infinie de variables aléatoires indépendantes, à valeurs moyennes nulles. Admettons que les \( X_i \) satisfont aux inégalités (18bis), où \( H' = H_{(v)} (v=1, 2, \ldots) \). Posons \( \bar{H}_n = \max (H_1, H_2, \ldots, H_n) \) et supposons que
\[
\bar{H}_n = o \left( \frac{B_n}{\log \log B_n} \right) \quad (B_n \to \infty).
\]
Dans ces hypothèses, la probabilité de l'inégalité (3) est égale à 1.

La démonstration de ce théorème ne diffère pas essentiellement de celle de M. Kolmogoroff. Notons seulement que la démonstration de la première partie de (3), à savoir de l'inégalité \( \lim \ldots \leq 1 \), est une conséquence facile de (21 bis).

Il résulte du th. 1 que le symbole \( o \) ne peut pas être remplacé par \( O \) dans la condition (25).

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**Algebraic Characterizations of Special Boolean Rings**

**By**

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In a paper entitled *The Theory of Representations for Boolean Algebras* \(^3\), we have introduced and discussed a certain classification of the ideals in a Boolean ring (or generalized Boolean algebra).

Here we propose to carry out a detailed study of that classification, with the particular purpose of discovering what types of Boolean ring can be characterized by properties of the ideal-structure. In order to make our examination complete, we have to consider many details of a somewhat tedious and uninteresting nature. For the convenience of the reader who prefers to pass over such details, we adopt a synthetic, rather than analytic, form of presentation; and formulate our results in a series of theorems and tables which, we hope, can be easily and rapidly surveyed.

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\(^1\) Parts of this paper (in particular §§ 1, 5, 12 and most of §§ 6, 7, 8, 10, 11) were written in 1933-4 and communicated to the Polish Mathematical Society at a meeting in Warsaw on September 12, 1935. Other parts (in particular §§ 2, 3, 4, 9 and certain aspects of §§ 6, 7, 8, 10, 11) were obtained in 1936-7 while the writer was a Fellow of the John Simon Guggenheim Memorial Foundation in residence at the Institute for Advanced Study (Princeton) as a temporary member.

\(^2\) M. H. Stone, Trans. Amer. Math. Soc. 40 (1936), pp. 37-111. A knowledge of this paper is assumed here, and references to it made by such citations as "R Th. 24", "R Def. 8", and so on.
Our central problem and the contributions made to its solution in these pages are of interest under two different aspects. In the first place, it is known \(^1\) that the classification of Boolean rings is equivalent to the classification of the totally-disconnected locally-bicompact Hausdorff spaces. Accordingly, the present investigation may be regarded as a test of the power and effectiveness of a purely algebraic attack upon a problem of topology. The fact that this attack proves to be a relatively feeble one is hardly surprising but is perhaps worthy of detailed consideration. In the second place, it is known that the structural problems of the symbolic (Aristotelian) logic of propositions are mathematically equivalent to the structural problems of the theory of Boolean rings. Speaking more precisely, we may say that the theory of deductive systems developed in recent years by Tarski \(^4\) is mathematically identical with the theory of ideals in Boolean rings (with unit). A brief digression at this point will permit us to establish this identity. The elements \(a, b, c, \ldots\) of a Boolean ring \(A\) (even one without unit) may be regarded as propositions, \(a + b\) and \(ab\) may be interpreted as the propositions "if \(a\) and only if \(b\)" and "\(a\) or \(b\)" respectively, and the equation \(a = 0\) may be interpreted as the assertion \(-a\) or \("a\) is true" \(^5\). The propositions "\(a\) and \(b\)" and "\(a\) implies \(b\)" may then be introduced by the respective definitions \(a \land b = a + b + ab = a \land \neg b\), \(a \rightarrow b = b + ab\). A subclass \(a\) of \(A\) is then called a deductive system if it has the three following properties: (1) if \(a = 0\), then \(a\); (2) if \(a\) and \(b\), then \(a \lor b, a \land b); (3) if \(a\) and \(b\), then \(a\lor b). \) Obviously, a deductive system \(a\) is non-void, by \(1\), and contains \(a \lor b\) together with \(a\) and \(b\), by \(2\). Since \(a \rightarrow b = ab + a(ab) = 0\), we see further than \(a \rightarrow ab = ab\) by \(1\) and hence that \(a\) implies \(ab\) for arbitrary \(b\) in accordance with \(3\). Thus every deductive system \(a\) is an ideal, by virtue of R Th.16. Conversely, we can show that every ideal \(a\) is a deductive system. Properties \(1\) and \(2\) are obvious from R Th.16. To establish property \(3\) we first note that \(a\) implies \(ab\) for arbitrary \(b\); and we then observe that \(a\) and \(b + ab = a \land b\) imply \(b = (b + ab) \land ab = (a \land b) \land ab\). In view of this identity between ideals and deductive systems, the present investigation bears directly on the classification of deductive systems. The fact that the purely algebraic attack proves to be relatively ineffective means, in this connection, that the profounder aspects of the theory of deductive systems must be studied by the general methods of topology. The interesting case for the theory of deductive systems is that where the Boolean ring \(A\) of propositions is countable. According to \(A\), Ch. I, the problem of classifying such Boolean rings and their ideals, considered as subrings, is the problem of classifying the closed subsets of the Cantor discontinuum and their (relatively) open subsets — or, equivalently, the problem of classifying all zero-dimensional compact metric spaces and their open subsets. In view of the special significance of the case where \(A\) is countable, we shall show (in §§ 9, 10) how our general results appear under it.

As we have already indicated, we find only a few special types of Boolean rings which can be characterized in terms of the ideal-classification of \(R\). By way of recompense, we find that most of these types can be characterized in many different, equivalent ways. Our special types fall into two main groups. On the one hand we have a series of distinct types which can all be obtained from infinite totally additive Boolean rings by appropriate combinations of the following operations: selection of a non-normal invariant subring or ideal, addition of a unit, and direct summation. These types are analysed in § 2. On the other hand, we have two types (not distinct from those in the first group) which have a fairly general structure. They can be obtained from Boolean rings with unit by the following operations: selection of a special type of prime ideal, and direct summation. These types are discussed in § 3. The only countable Boolean rings, other than the finite ones, occurring under these various types belong to three of the most restricted types in the first group.
The general plan of the paper is as follows: in § 1 we present in tabular form the essential data concerning the behavior of ideals in various specified classes under the algebraic operations and relativization; in §§ 2, 3 we discuss the special types of Boolean ring described in the preceding paragraph; in § 4 we apply results of our paper A to obtain topological constructions and interpretations related to §§ 2, 3; in §§ 5-9 we obtain ideal-structural characterizations and properties of various special Boolean rings, chiefly those described in §§ 2, 3; in §§ 10, 11 we tabulate (and complete) the earlier work according to the different possible types of ideal-structure under the fundamental classification of $R$. Finally, we show that the tables of § 1 give "best possible" results except for special types of Boolean ring occurring among those obtained in the earlier sections. The notations of the paper will be taken directly from $R$; but we shall allow ourselves on occasion to replace the term "Boolean ring" by the term "ring", since no other type of ring is considered here.

§ 1. Algebraic Operations on Ideals. From the ideals $a$ and $b$ we can form ideals $a \lor b$, $ab$, $a'$; and, if $a$ is contained in $b$, we can perform the operation — relativisation — of presenting $a$ as an ideal in the ring $b$. We shall devote the present section to a study of the behavior of ideals in the various fundamental classes under these algebraic operations. A knowledge of the results is essential in subsequent proofs. In § 12 we shall show that these results are "the best possible".

We begin by a consideration of possible inferences about the classification of $a$ and $b$ from the assumption that $a$ is contained in $b$. The only generally valid assertion we can make is the following

**Theorem 1.1.** If the ideals $a$ and $b$ in a ring $A$ have the property that $a \subseteq b$, then

1. $a \in R$ and $b \in P$ imply $a \in P$;
2. $a \in P$ and $b \in S$ imply $b \in P^* \cdot A$.

We establish (1) as follows: if $a$ is simple and $b$ principal, then $a = ab$ is principal by $R$ Th. 26. We then obtain (2) as follows: if $a$ is semiprincipal but not principal, then $a'$ is principal by $R$ Th. 32 (4b); if $b$ is simple, then $b'$ is simple by $R$ Th. 30; hence the relation $b' \subseteq a'$ of $R$ Th. 20 (1) combined with (1) above shows that $b'$ is principal; thus $b$ is semiprincipal by $R$ Th. 32; and the fact that $a$ is not principal shows by (1) that $b$ cannot be principal.

We pass now successively to various similar inferences.

**Theorem 1.2.** The three accompanying tables exhibit the dependence of the classification of the ideals $a', a \lor b$, and $ab$, respectively and in that order, upon the classifications of $a$ and $b$: the class of $a$ is shown at the left, that of $b$ at the top (in the second and third tables), and that of the resultant ideal in the appropriate row and column.

With the exception of the result that $a \in R$ and $b \in S$ imply $a \lor b \in R$, the three tables can be filled in by reference to $R$ Def. 8 and $R$ Ths. 23, 24, 26, 27, 30, 31, and 32. We prove the one result still necessary, in the following manner: if $a$ is an arbitrary element in $(a \lor b)'$, then $a(a) = a(a)(b \lor b') = a(a) b \lor a(b) b' \lor (a \lor b)' b' = b \lor (a \lor b)' \lor b = b \lor (a \lor b)' = b \lor (a' \lor b') = b \lor a' \lor b' = b \lor b' = b$, by virtue of the relations $a' = a$, $b' = b$, and we conclude that $(a \lor b)' \subseteq (a \lor b)'$, $a \lor b = (a \lor b)'$, $a \lor b \in R$, as we wished.

**Theorem 1.3.** The three accompanying tables exhibit the dependence of the classification of a relative to $b$ when $a \subseteq b$, of $ab$ relative to $b$, and of a relative to $a \lor b$, respectively and in that order, upon the classification of the ideals $a$ and $b$ in the ring $A$: the class of $a$ is shown at the left, the class of $b$ at the top, and the relative class of $a$ in $b$, $ab$ in $b$, or of $a$ in $a \lor b$, respectively, in the appropriate row and column.
If a and c are arbitrary ideals, then R Th. 22 shows that the ideal ac in c has orthocomplement a'c relative to c; and that the ideal a'c in a similarly has orthocomplement a'c relative to c. Thus 2c implies a=a'c, ac=a'c, so that ac is normal relative to c. Similarly 2c implies a=a'c, ac=a'c=ac=a, so that ac is simple relative to c. Accordingly, all entries 2c, 2c, 2c in the three tables are obviously justified; and, furthermore, we see that the remaining entries (actually 2c and 2c) cannot be "true" than 2c. Now in order that ac be principal relative to c it is necessary and sufficient that ac be principal in A. In case ac we take c=a, ac=ac=a.

With the aid of Th. 1.1 (1) we then see that all entries in the first table are justified. For the second table, we take c=a and use the table for ac in Th. 1.1, obtaining justification for all the entries 2c. Similarly for the third table, we take c=a, ac=a(a+b)=a and conclude that a is principal relative to a+b whenever it is principal in A.

We still have to justify the various entries 2c in the three tables. If a is semiprincipal and c is simple, then at least one of the ideals a and a' is principal by R Def. 8 and R Th. 32; and hence the corresponding one of the ideals ac and a'c must be principal by virtue of Th. 1.2. Thus we conclude that in c either ac or its (relative) orthocomplement a'c is principal. Since ac is simple relative to c by our preceding results it must be semiprincipal relative to c. If we take ac=c, ac=a, we obtain justification for the entries 2c in the first table. If we take c=b, ac=ac, we similarly obtain justification for the entries 2c in the second table. If we take c=a+b, ac=a, and note that 2c, 2c imply a+b=b by Th. 1.2, we obtain justification for all entries 2c in the third table, with the exception of that in the third row. In the exceptional case we know that a is simple relative to a+b from our preceding results; since the orthocomplement of a relative to a+b is a(a+b)=a+b and since 2c, 2c imply a+b=2c by virtue of Th. 1.2, we conclude that a is semiprincipal relative to a+b. This completes our discussion.

**Theorem 1.4.** The accompanying table exhibits the dependence of the classification of the ideal a in a ring A upon the classification of a relative to an ideal b containing it, and the classification of b in A: the class of a relative to b is shown at the left, the class of b in A is shown at the top, and the class of a in A is shown in the appropriate row and column.

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>a*b</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

If a is principal relative to b, then it is principal also in A, as we have observed previously. If b is principal in A, it is a ring with unit; hence if a is simple relative to b, it is principal relative to b by virtue of Th. 1.1 (1) and therefore principal in A. Thus all the entries 2c in the table are justified. The orthocomplement of a relative to b is a'b, of a'b relative to b is a''b, in accordance with R Th. 22. Thus if a is normal relative to b we have a=a''b; by Th. 1.2 the ideal a'' is normal in A; and hence, if b is normal in A, the product a''b=a is normal in A by virtue of Th. 1.2. Similarly, if a is simple relative to b and b is simple in A, we have a''b=b, b''=b, a''b''=a''b''=a''b''=b''=a''b''=e, so that a is simple in A. Thus the entries 2c and 2c are justified. Finally, if a is semiprincipal but not principal relative to b, there exists an element a in b such that a=a'(a)b, where a'(a)b is the orthocomplement relative to b of the principal ideal a(a)c; if b is semiprincipal but not principal in A, then there exists an element b in A such that b=a'(b); and hence we find that a=a'(a)b=a(a'b)=a'(a)b, so that a is semiprincipal but not principal in A by virtue of R Th. 32 or Th. 1.2. Taken with the results already established, this justifies the entries 2c.

We now consider the possible inferences about the classification of a relative to an ideal b containing it from a knowledge of the class of b in A; and also the possible inferences about the classification of b in A from a knowledge of the classification of a relative to b. The only generally valid assertion we can make is a repetition of a previous result:

**Theorem 1.5.** If a and b are ideals in a ring A such that a|b and if b is principal in A, then a is simple relative to b if and only if it is principal relative to b (and hence also in A).

We still have to study the behavior of prime ideals in a similar way. The remainder of the present section will be devoted to the necessary investigations. We first have:

**Theorem 1.6.** If a is an arbitrary ideal and p a prime ideal in a ring A, then the classification of the ideal ap is determined as follows:

1. if ap=a or if p is normal, then a and ap have the same classification in A.
2. if ap=a and p is not normal, then ap is not normal.
According to R Ths. 38, 39, 41, we may make the following preliminary remarks: one and only one of the relations $ap=a$ and $ap=\alpha$ is valid; if $ap=a$ and $p$ is normal, then $p$ is semiprincipal, $p'$ is principal, and $p\vee p'=\alpha$; if $ap=\alpha$ and $p$ is not normal, then $a'p$ and $p'=\alpha$. The case where $ap=a$ is trivial. We turn therefore to the case where $ap=\alpha$. First, let $p$ be normal. Then Th. 1.2 shows that $ap$, where $p$ is semiprincipal, belongs to the same class (or a preceding class) as does $a$ in the sequence $\mathfrak{B}, \mathfrak{B}^*, \mathfrak{E}, \mathfrak{R}, \mathfrak{Y}$. On the other hand, if we write $a=a(p\vee p')=ap\vee p'$ where $p'$ is principal in accordance with the preceding remarks, we see by Th. 1.2 again that $a$ belongs to the same class (or a preceding class) as does $ap$ in the sequence $\mathfrak{B}, \mathfrak{B}^*, \mathfrak{E}, \mathfrak{R}, \mathfrak{Y}$. Hence $a$ and $ap$ have the same classification in $A$. Finally we treat the case where $ap=\alpha$ and $p$ is not normal. Since $ap\subset a$, its orthocomplement relative to $a$ is given by $(ap)'a$ and also by $p'a=\alpha$. Hence we have $(ap)'a=a, a\subset (ap)'$. Since $ap$ is contained in a but distinct from $a$, we must therefore have $ap=\alpha(a)'$. Thus $ap$ is not normal in this case.

Theorem 1.8. If $a$ is an arbitrary ideal, $p$ a prime ideal in a ring $A$, then the classification of the ideal $ap$ relative to $a$ is determined as follows:

(1) if $ap=a$, then $ap$ is semiprincipal relative to $a$;
(2) if $ap=\alpha$, then $ap$ is a prime ideal in $A$ which is normal relative to $a$ if and only if $p$ is normal in $A$.

If $ap=a$, then $ap$ coincides with the orthocomplement $a'=ea=\alpha$ of the principal ideal $a=a(0)$ relative to $a$; and $ap$ is thus semiprincipal relative to $a$. If $ap=a$, we can show as follows that $ap$ is prime in $a$: if $a$ and $b$ are elements of $a$ such that $a\neq ap$ and hence $ap$ is not prime; now $a\alpha$ and $a\beta$ would imply $a\alpha$ and similarly $a\beta$ and $a\beta$ would imply $a\beta$; and we therefore conclude that $a\alpha, a\beta, b\alpha$ imply $a\alpha$ or $b\beta$. If $ap$ is normal in $A$, then its orthocomplement relative to $a$ is $ap'=\alpha$ in accordance with R Th. 38; and we conclude that $p'=\alpha, p\not\in R$. On the other hand, if $p$ is normal in $A$, we have $p'=\alpha, ap'=p'=\alpha$; and, by R Th. 38 again, we conclude that $ap$ is normal relative to $a$.

Theorem 1.8. If $a$ is an arbitrary ideal and $p$ a prime ideal in a ring $A$, then the classification of $ap$ relative to $p$ is connected with the classification of $a$ in $A$ in the following manner:

(1) if $p$ is normal in $A$, then the class of $ap$ relative to $p$ is the same as the class of $p$ in $A$;
(2) if $p$ is not normal in $A$ and $ap=\alpha$, then the class of $ap$ relative to $p$ is the same as the class of $\alpha$ in $A$, except for the special situations described as follows:

(i) $ap$ is never principal relative to $p$;
(ii) if $a$ is principal in $A$ and $A$ has no unit, then $ap$ is simple but not semiprincipal relative to $p$;
(iii) if $a$ is simple in $A$ and $A$ has a unit, then $ap$ is principal in $A$ and $ap$ is semiprincipal but not principal relative to $p$.

(3) if $p$ is not normal in $A$ and $ap=\alpha$, then the class of $\alpha$ in $A$ determines the class of $ap=\alpha$ relative to $p$ in accordance with Th. 1.3, first table, second column; if $ap$ is principal relative to $p$, then $a=ap$ is principal in $A$; but, even in the case where $ap$ is semiprincipal relative to $p$, the ideal $a=ap$ may be non-normal in $A$ (as in the special instance $\alpha=\gamma$).

To prove (1), we first note that, by R Th. 38, the normal prime ideal $p$ is semiprincipal. If we know the class of $a$ in $A$, we can therefore apply Th. 1.3, second table, second column, and find the class of $ap$ relative to $p$. As a result we see that, if $a$ is normal (simple, semiprincipal, principal) in $A$, then the following statement is valid for $ap$ in $p$. On the other hand, Th. 1.6 (1) shows that the class of $ap$ in $A$ is the same as the class of $a$ in $A$. Hence the second table of the column in Th. 1.4 may be regarded as yielding the class of $a$ in $A$ when the class of $ap$ relative to $p$ is known. As a result we see that, if $ap$ is normal (simple, semiprincipal, principal) in $p$, then the corresponding statement is valid for $a$ in $A$. Combining these results, we find that $ap$ is normal, simple, semiprincipal, or principal in $p$ if and only if the corresponding statement is valid for $a$ in $A$.

We now consider (2). If $a$ is normal in $A$, then $ap$ is normal relative to $p$ by Th. 1.3; and also, if $a$ is simple in $A$, then $ap$ is simple relative to $p$. On the other hand, if $ap$ is normal relative to $p$, the relation $ap=a^p$ taken together with the relation $a\vee p=e$
established in R Th. 39 yields \( a = a \lor a = a' \lor a' \lor p = a'(a \lor p) = a' \), so that \( a \) is normal in \( A \). Likewise, if \( ap \) is simple relative to \( p \), we have \( a' \lor a' \lor (a \lor a') \lor p = ap' \lor a' \lor p = p \); since \( p \) is prime, we conclude that \( a' \lor a = a', aeG \). Thus we have shown that \( ap \) is normal (simple) relative to \( p \) if and only if \( a \) is normal (simple) in \( A \). If \( ap \) were principal relative to \( p \), it would be principal, and hence normal in \( A \); but Th. 1.6 (2) shows that \( ap \) is not normal in \( A \) under the present conditions. Thus if \( ap \) is semiprincipal relative to \( p \), it is necessarily non-principal relative to \( p \); and its orthocomplement \( a' \lor p \) relative to \( p \) is therefore principal both in \( p \) and in \( A \). Since \( a' \lor p \) by R Th. 41 and since \( a \) is simple in \( A \) by preceding results, the ideals \( a', a \) are respectively principal and semiprincipal in \( A \). Moreover, \( a \) is principal in \( A \) if and only if \( A \) has a unit, as we see by reference to R Th. 25. On the other hand, if \( a \) is semiprincipal in \( A \), then \( ap \) is simple relative to \( p \) and has the orthocomplement \( a' \lor p = a' \lor p \) in \( p \). Since \( ap \) is not principal relative to \( p \), it is semiprincipal relative to \( p \) if and only if \( a' \lor p = a' \lor p \) is principal both in \( p \) and in \( A \). Thus, if \( A \) has no unit and \( a \) is not principal, then \( ap \) is semiprincipal relative to \( p \); and if \( A \) has no unit and \( a \) is principal, then \( ap \) is simple but not semiprincipal relative to \( p \). Likewise, if \( A \) has a unit and \( a \) is simple, then \( a \) and \( a' \) are both principal and \( ap \) is semiprincipal relative to \( p \). This completes the discussion of (2).

The statement (3) is obvious.

§ 2. Totally Additive, Totally Multiplicative, and Related Boolean Rings. In this section we shall study those Boolean rings in which it is possible to form unrestricted (logical) sums or products; and shall discuss certain special types of Boolean ring which can be constructed from them by simple algebraic operations. Each of our fundamental types is thus characterized by a certain constructive representation. We shall prove further that each such representation is unique except for isomorphisms and certain internal modifications. We introduce all our fundamental definitions at once.

First we define unrestricted sums and products as follows:

**Definition 2.1.** A non-void subclass \( a \) of a Boolean ring \( A \) is said to have the sum \( b \) if \( b \) is an element of \( A \) with the properties:

1. \( b \leq a \) for every \( a \) in \( a \);
2. if \( c \leq a \) for every \( a \) in \( a \), then \( c \leq b \).

**Definition 2.2.** A non-void subclass \( a \) of a Boolean ring \( A \) is said to have the product \( b \) if \( b \) is an element of \( A \) with the properties:

1. \( b \leq a \) for every \( a \) in \( a \);
2. if \( c \leq a \) for every \( a \) in \( a \), then \( c \leq b \).

It is immediately evident that the sum and product of \( a \) are unique whenever they exist. It is also evident that, in case \( a \) is a finite class consisting of elements \( a_1, ..., a_n \), its sum and product are the elements \( a_1 \lor ... \lor a_n \) and \( a_1 \cdot ... \cdot a_n \), respectively.

In terms of the definitions just given for sums and products, we next introduce:

**Definition 2.3.** A Boolean ring in which every non-void subclass has a sum is said to be totally additive.

**Definition 2.4.** A Boolean ring in which every non-void subclass has a product is said to be totally multiplicative.

The types of Boolean ring to be considered in the present section are now indicated in the definitions which follow.

**Definition 2.5.** An infinite, totally additive Boolean ring is said to be of type \( (a) \).

**Definition 2.6.** A Boolean ring which is isomorphic to a non-normal ideal \( a \) of a Boolean ring of type \( (a) \) is said to be of type \( (b_1) \) if \( a \) is prime, of type \( (b_2) \) if \( a \) has an atomic basis \( * \), of type \( (b_3) \) if \( a \) is neither prime nor has an atomic basis.

**Definition 2.7.** A Boolean ring which is obtained by adjunction of a unit \( \gamma \) to one of type \( (b_k) \) is said to be of type \( (b_{k+1}) \), \( k = 1, 2, 3 \).

**Definition 2.8.** A Boolean ring which is the direct sum \( * \) of two Boolean rings of respective types \( (\ast) \) and \( (\ast \ast) \) is said to be of type \( (\ast, \ast \ast) \).

We turn now to a study of totally additive and totally multiplicative rings. We begin with conditions for the existence of sum or product.
Theorem 2.1. If $a$ is an arbitrary non-void subclass of a ring $A$, the subclasses $b_1 = a'$ and $b_2 = \bigcap_{a \in a} \mathcal{P}(a)$ are normal ideals in $A$. In order that a have a sum, it is necessary and sufficient that $b_1$ be principal, the sum of a being the generating element of $b_1$ and, in order that a have a product, it is necessary and sufficient that $b_2$ be principal, the product of a being the generating element of $b_2$.

From R Ths. 19 and 27, we see that $b_1$ and $b_2$ are normal ideals. In order that $a \triangleright a$ for every $a$ in $a$, it is obviously necessary and sufficient that $a(c) \triangleright a$. Moreover, the relations $a(c) \triangleright a$ and $a(c) \triangleright b_1$ are equivalent: for $a(c) \triangleright a$ implies $a(c) = a'(c) \triangleright a'$ and $a(c) \triangleright b_1$ implies $a(c) \triangleright b_1 = a'(c) \triangleright a'$, by R Th. 20. Now if $a$ has a sum $b_1$, we must have $a(b_1) \triangleright b_1$ in accordance with the results just obtained. We can also prove the relation $a(b_1) \triangleright b_1$ as follows: if $c \triangleright b_1$, we have $a(c) \triangleright b_1$, $a(c) \triangleright b_1 = b_1$, $a(b_1) \triangleright a(c) \triangleright b_1$ since $a(b_1) \triangleright a(c)$ is a principal ideal, we must have $a(b_1) \triangleright a(c) \triangleright b_1$ by the definition of the sum $b_1$; hence we have $a(b_1) \triangleright a(c) = a$; now by virtue of the fact that $c$ may be chosen arbitrarily in $b_1$, we conclude that $a(b_1) \triangleright b_1 = a$; and it follows finally that $a(b_1) \triangleright b_1 = b_1$. Combining these results, we find that $b_1 = a(b_1)$ as we wished to prove. On the other hand, the relation $b_1 = a(b_1)$ shows that $a(b_1) \triangleright a$ and also that $a(c) \triangleright a$ implies $a(c) \triangleright b_1 = a(b_1)$ or, equivalently $a \triangleright b_1$. Hence $b_1$ is the sum of $a$ in accordance with Def. 2.1. In order that $a \triangleright a$ for every $a$ in $a$, it is evidently necessary and sufficient that $a(c) \triangleright \bigcap_{a \in a} \mathcal{P}(a) = b_2$. Thus, if $a$ has a product $b_2$, the relation $a(b_2) \triangleright b_2$ is valid; and, if $c$ is any element in $b_2$, the relations $a(c) \triangleright b_2$, $a(c) \triangleright a(b_2)$ are valid, and $b_2 \triangleright a(b_2)$. Hence we find that $b_2 = a(b_2)$, as we wished to prove. On the other hand, if $b_2 = a(b_2)$, the relation $a(b_2) \triangleright b_2$ is trivial; and $a(c) \triangleright b_2$ implies $a(c) \triangleright a(b_2)$ or, equivalently, $a \triangleright b_2$. Hence $b_2$ is the product of a in accordance with Def. 2.2.

It is now easy to characterize totally additive and totally multiplicative rings. We have

Theorem 2.2. The following properties of a Boolean ring $A$ are equivalent:

1. $A$ is totally additive;
2. Every normal ideal in $A$ is principal;
3. $A$ has a unit and is totally multiplicative.

In particular, every finite Boolean ring is totally additive.

It is obvious from Th. 2.1 that (2) implies (1). On the other hand, if we assume (1) and take $a$ as an arbitrary normal ideal in $A$, Th. 2.1 shows that $a = a'$ is principal; hence (1) implies (2). Since $A$ is a normal ideal relative to itself, (2) implies that $A$ has a unit; and, moreover, (2) also implies in accordance with Th. 2.1 that $A$ is totally multiplicative. Thus (2) implies (3). It is also easy to show that (3) implies (2). If $A$ has a unit, every normal ideal $a$ is the product of the principal ideals containing it, by virtue of R Th. 27. Thus if (3) holds, Th. 2.1 can be applied with the result that $a$ is principal, as we wished to show. The equivalence of (1), (2) and (3) is thereby fully established. A finite Boolean ring is obviously totally additive, since it contains only finite subclasses.

Theorem 2.3. The following properties of a Boolean ring $A$ are equivalent:

1. $A$ is totally multiplicative;
2. Every normal ideal in $A$ is simple;
3. Every principal ideal in $A$ is totally additive.

In particular, every Boolean ring with an atomic basis is totally multiplicative.

First let us prove that (1) implies (3). Let $a(a)$ be an arbitrary principal ideal in $A$. Then it is evident that, considered as a ring, $a(a)$ is totally multiplicative. Since $a(a)$ has $a$ as its unit, Th. 2.2 shows that $a(a)$ is totally additive. Next we show that (3) implies (2). If $a$ is a normal ideal in $A$ and $a(a)$ is an arbitrary principal ideal in $A$, then $a(a)$ is a normal ideal relative to $a(a)$ in accordance with Th. 1.3. Thus (3) implies by Th. 2.2 that $a(a)$ is principal in $a(a)$ and hence also in $A$. By R Th. 26, we find that $a$ is a simple ideal. Finally, we show that (2) implies (1). Let $a$ be an arbitrary non-void subclass of $A$, let $a_0$ be a selected element of $a$, and let $b_2 = \bigcap_{a \in a} \mathcal{P}(a)$ be the normal ideal considered in Th. 2.1. By hypothesis, $b_2$ must be simple. Moreover, since $b_2 = b_2(a_0)$, we see by R Th. 26 that $b_2$ must even be principal. Hence $A$ is totally multiplicative in accordance with Th. 2.1. The equivalence of (1), (2), and (3) is thus established. If a Boolean ring has an atomic basis, then every principal ideal is obviously finite. Hence (3) above is satisfied by virtue of Th. 2.2; and it follows that the ring is totally multiplicative.
We next prove two fundamental imbedding theorems.

**Theorem 2.4.** Let \( A \) be a Boolean ring; \( \mathcal{P} \) the class of all its principal ideals, considered as a Boolean ring in accordance with \( \text{R. Th. 31} \); and \( R \) the class of all its normal ideals, considered as a Boolean ring in accordance with \( \text{R. Th. 29} \).

Then \( R \) is a totally additive Boolean ring; and the correspondence \( a \mapsto a(a) \) carries \( A \) isomorphically into the subring \( \mathcal{P} \) of \( R \) in such a way that the sum (product) of a subclass of \( A \) is carried, when it exists, into the sum (product) in \( R \) of the corresponding subclass of \( \mathcal{P} \).

In particular, \( A \) is totally additive if and only if it is isomorphic to \( R \).

In \( \text{R. Ths. 29 and 31} \), we have already shown that the indicated correspondence carries \( A \) isomorphically into \( \mathcal{P} \), that \( \mathcal{P} \) is a subring of \( R \), and that \( R \) has the property that its normal ideals are all principal. By reference to \( \text{Th. 2.2} \), we now see that \( R \) is totally additive. If \( A \) were isomorphic to \( R \), it would obviously be totally additive; on the other hand, if \( A \) were totally additive, then the relation \( A = \mathcal{P} \) would hold by \( \text{Th. 2.2} \) and the isomorphism between \( A \) and \( \mathcal{P} \) would reduce to one between \( A \) and \( R \). Now let \( a \) be an arbitrary non-void subclass of \( A \), and let \( \mathcal{P} \) be its correspondent in \( \mathcal{P} \) under the isomorphism \( A \rightarrow \mathcal{P} \). Then the normal ideals \( b_1 \) and \( b_2 \) associated with \( a \) in the manner described in \( \text{Th. 2.1} \) are the sum and product, respectively, of the class \( \mathcal{P} \) in \( R \). First let us consider \( b_1 \). A member \( a(a) \) of \( A \) obviously satisfies the relation \( b_1 \equiv a(a) \) since \( b_1 \equiv a(a) \); and, if \( a \) is a normal ideal with the property that \( c \equiv a(a) \) for every \( a(a) \) in \( A \), we have \( c \equiv c \equiv a(a) \equiv b_1 \).Thus \( b_1 \) is identified as the sum of \( \mathcal{P} \) in \( R \). The discussion of \( b_2 \) is similar. It is obvious that \( b_2 \equiv a(a) \) for every \( a(a) \) in \( \mathcal{P} \); and if \( a \) is any ideal, whether normal or not, the relation \( c \equiv a(a) \) holding for every \( a(a) \) in \( \mathcal{P} \) implies \( c \equiv \bigcap_{a \in A} a(a) = b_2 \). Thus \( b_2 \) is the product of \( \mathcal{P} \) in \( R \). Th. 2.1 now shows that, if \( a \) has sum \( b_1 \), then \( b_1 \equiv a(b_1) \) is the correspondent of \( b_1 \) under the isomorphism \( A \rightarrow \mathcal{P} \); and that, if \( a \) has product \( b_2 \), then \( b_2 \equiv a(b_2) \) is the correspondent of \( b_2 \) under this isomorphism.

The theorem just established is significant in two senses. In the present context, it is important because it provides us with a construction for totally additive rings and shows that all possible totally additive rings can be obtained by the construction described: one has merely to start with an arbitrary ring and pass to the ring of its normal ideals. It has an additional interest in that it shows that an arbitrary Boolean ring can be imbedded in a totally additive ring with preservation of all sums and products, even the infinite ones, which are already present \(^4\). From \( \text{R. Th. 29} \) we can now read off the algebraic behavior of sums and products without further difficulty.

**Theorem 2.5.** Let a Boolean ring \( A \) be contained as an ideal \( a \) in a totally additive Boolean ring \( B \). Then the ideal \( a^* \) in \( B \) is a Boolean ring \( A^a \) containing \( A \) as the ideal \( a \); \( A^a \) is totally additive; \( A \) is totally multiplicative; and in \( A^a \) the ideal \( a \) has the property that \( a^* = a \). Next, let a Boolean ring \( A \) be contained as an ideal \( a \) with \( a = 0 \) in a totally additive Boolean ring \( B \); and let \( A_0 \) be a subring of \( B \) containing \( A \). Then \( A_0 \) contains \( A \) as the ideal \( a \); \( A_0 \) is totally multiplicative; and in \( A_0 \) the ideal \( a \) has the property that \( a^* = a \). Finally, let \( A \) be a totally multiplicative Boolean ring contained as an ideal with \( a = 0 \) in a Boolean ring \( A_0 \) and let \( \mathcal{P} \) and \( R \) be the Boolean rings of the principal ideals and of the normal ideals, respectively, in \( A_0 \). Then the correspondence \( a \mapsto a(a) \) carries \( A \) isomorphically into \( \mathcal{P} \) and \( A_0 \) isomorphically into a subring \( \mathcal{P}_0 \) of \( R \); \( \mathcal{P} \) is an ideal in \( R \) with the property that \( \mathcal{P} \equiv 0 \); and \( \mathcal{P}_0 \) contains \( \mathcal{P} \). In particular, if \( A = A_0 \), this correspondence embeds \( A \) as the ideal \( \mathcal{P} \) in \( R \) with \( \mathcal{P} \equiv 0 \). In order that \( A \) be totally multiplicative, it is necessary and sufficient that \( \mathcal{P} \equiv 0 \) be an ideal in \( R \). In order that \( A \) be totally additive, it is necessary and sufficient that \( \mathcal{P} \equiv 0 \) and \( A \) is then isomorphic to \( R \).

Consequently, if \( A_1 \) and \( A_2 \) are totally multiplicative Boolean rings contained as ideals \( a_1 \) and \( a_2 \) with \( a_1 = 0 \) and \( a_2 = 0 \), in totally additive Boolean rings \( B_1 \) and \( B_2 \) respectively, then any isomorphism \( A_1 \rightarrow A_2 \) can be extended to an isomorphism \( B_1 \rightarrow B_2 \). Taken together the preceding results characterize the totally multiplicative Boolean rings as the ideals \( a^* \), with \( a^* = a \), in totally additive Boolean rings; and show further that, except for isomorphisms, a totally multiplicative Boolean ring has essentially only one representation as such an ideal.

If \( A \) is contained as the ideal \( a \) in the totally additive Boolean ring \( B \), we show as follows that \( A \) is totally multiplicative: if \( b \) is any ideal in \( A \), it is an ideal in \( a \) and hence also in \( B \); if \( b \) is

\(^4\) This result is given by Mac Neile, The Theory of Partially Ordered Sets, Harvard doctorial dissertation (1935); a summary is given in Proceedings of the National Academy, U. S. A., vol. 22 (1936), pp. 45-50. From letters, I understand that Tarski has obtained this result independently.
normal relative to $A$, then $b=5\alpha_0$ since $b''$ is normal in $R$; it must be principal in accordance with Th. 2.2; by Th. 1.3 it must therefore be simple relative to $A$; and hence $A$ is totally multiplicative in accordance with Th. 2.3 (2). Considered as a ring, the ideal $a''$ is totally multiplicative by the result just established. As a normal ideal in $B$, it is principal and therefore has its generating element as unit. By Th. 2.2, it is a totally additive ring. Obviously $A$ is contained as an ideal in $a''$; and its orthocomplement relative to $a''$ is $a'^{**}=o$. Hence the first part of the theorem is established.

The second part of the theorem offers no difficulty. If $A$ is an ideal $a$, with $c'_0=0$, in a totally additive Boolean ring $B$, then $A$ is totally multiplicative, as we have already seen. If now $A_0$ is a subring $b_0$ of $B$ such that $a_0=a_0$, the class $a=a_0$ is an ideal in $a_0$; and the relation $a_0=a$ permits us to calculate the orthocomplement of a relative to $a_0$ as $a'_0=0$, as we see by direct use of $R$ Def. 7.

The second part of the theorem remains to be discussed. Under the assumptions made, we proceed as follows. By Th. 1.3, the ideal $a(a)$ is normal relative to $a$. The correspondence $a\rightarrow a(a)$ therefore carries $A_0$ into a subclass $B_0$ of $B_1$; and, since $a\rightarrow a$ implies $a(c)c'=c$ or $a(c)=a(a)$, it carries $A_0$, in particular, into $B$. Thus $B_0$ contains $B$. The relations $a(a)c'=a(c)c'=a(a)c'$ show that the correspondence from $A_0$ to $B_0$ is a homomorphism in accordance with Th. 42. It follows that $A_0$ is a subring of $B$. In order to show that the homomorphism $A_0\rightarrow B_0$ is actually an isomorphism, we have only to observe that $a\rightarrow a(a)=a$ implies $a(a)c'=a$ and hence $a=0$. We must now verify the assertion that the orthocomplement of $B$ in $R$ is the class $c$ consisting of the zero element alone. If $b$ is in $B$, it is a normal ideal relative to $a$ and satisfies the relation $b(a)=a$ for every $a$ in $a$. Thus, taking $a$ as an arbitrary element $b$ of $B$, we find that $a(b)=b(a)=a$, $b=0$ and hence conclude that $b=0$, as we wished to show. Up to this point we have not used the hypothesis that $A$ is totally multiplicative. Now, in order to prove that $B$ is an ideal in $R$, we bring it into play. When $A$ is totally multiplicative, Th. 2.3 shows that $R$ coincides with the Boolean ring $\mathbb{S}$ of all simple ideals in $A$. From Th. 30, we know that $B$ is an ideal in $R=\mathbb{S}$. Since all our hypo-

\[1/\] It must be observed that the sum in $R$ is the normalized sum of $R$ Def. 9; but, when, as here, the sum of two ideals is normal, it is equal to their normalized sum.

theses are fulfilled by taking $A$ as a totally multiplicative ring and putting $A_0=A$, we can then represent $A$ isomorphically as the ideal $B$, with $B'=\mathbb{S}$, in the ring $R=\mathbb{S}$; and Th. 2.4 shows that $R$ is totally multiplicative. We thus obtain the imbedding theorem described above. Now, in general, if $B_0$ is an ideal in $R$, it is totally multiplicative, by results established above; and its isomorph $A_0$ is also totally multiplicative. On the other hand, if $A_0$ is totally multiplicative, we can show that $B_0$ is an ideal in $R$. Since $B_0$ is a subring, we have to prove that, if $a(a)$ is an arbitrary element of $B_0$ and $b$ an arbitrary element of $R$, then $a(a)b$ is an element of $B_0$ — that is, is representable in the form $a(b)a$ where $b\in A_0$. Since $b$ is a normal ideal in $a$, it is an ideal in $A_0$ with $a\rightarrow b$. Thus we may regard $a(a)b=a(a)b'$ as an ideal in the principal ideal $a(a)$ in $A_0$. By combining our hypothesis with Th. 2.3 (3), we see that the class $a(a)b$ has a sum $b$ in $a(a)$. According to Th. 2.1 the principal ideal $a(b)$, considered in $a(a)$, is given by $a(b)=a(a)b'$, since $a(a)b'$ is the second orthocomplement of $a(b)$ relative to $a(a)$ in accordance with Th. 22. Since $b$ is normal relative to $a$, we have $b=\alpha$. Thus we find that $a(b)=a(a)b'=a(a)b=a(a)b$, as we wished to prove. Using the result just established, it is easy to determine under what circumstances $A_0$ is totally additive. By Th. 2.2 (3), we see that $A_0$ is totally additive if and only if $B_0$ has a unit and is an ideal in $R$. Obviously $B_0$ has these properties if and only if it is a principal ideal in $R$. If $B_0$ is a principal ideal in $R$, then the relation $B_0\supseteq B$ implies $B_0=B\supseteq B\supseteq R$ and hence $B_0=R$; and, on the other hand, if $B_0=R$, then $B_0$ is obviously a principal ideal in $R$, with $c$ as its generating element. Thus $A_0$ is totally additive if and only if $B_0=R$. When the latter relation holds, $A_0$ and $R$ are evidently isomorphic. Now let us consider the case of two totally multiplicative Boolean rings $A_1$ and $A_2$ contained as ideals $a_1$ and $a_2$ with $a_1=0$ and $a_2=0$, in totally additive rings $B_1$ and $B_2$ respectively. We denote by $B(A_1), B(A_2), B(A_1), R(A_1)$ the associated rings of ideals. By the preceding results there exist isomorphisms $B_1\rightarrow R(A_1)$ and $B_2\rightarrow R(A_2)$ carrying $A_1$ into $R(A_1)$ and $A_2$ into $R(A_2)$ respectively. Now any isomorphism $A_1\rightarrow A_2$ obviously establishes an isomorphism $B(A_1)\rightarrow B(A_2)$ and an extension of it to an isomorphism $R(A_1)\rightarrow R(A_2)$. Thus if we combine the isomorphisms $B_1\rightarrow R(A_1), B_2\rightarrow R(A_2)$ and $R(A_1)\rightarrow R(A_2)$, we obtain an isomorphism $B_1\rightarrow B_2$ which carries $A_1$ into $A_2$ in the same way as the postulated isomorphism $A_1\rightarrow A_2$. 
The characterization of all totally multiplicative Boolean rings follows at once from the preceding results, as stated in the theorem. So likewise does the essential uniqueness of the representation in terms of ideals.

The results up to this point are sufficient to settle the logical status of the types \((a), (\beta_1), (\beta_2), (\beta_3)\); and are also essential to an analysis of the composite types formed from them. Before proceeding to this further analysis, it is convenient to state the information now available about the simple types. We have:

**Theorem 2.6.** The four types \((a), (\beta_1), (\beta_2), (\beta_3)\) of Definitions 2.5 and 2.6 are distinct and exhaust the infinite totally multiplicative Boolean rings. A Boolean ring \(A\) belongs to type \((a)\) if and only if it is an infinite totally multiplicative ring with unit. It belongs to one of the three types \((\beta_1), (\beta_2), (\beta_3)\) if and only if it is a totally multiplicative ring without unit. It belongs to the type \((\beta_1)\) if and only if it satisfies one of the following two equivalent criteria:

1. \(\mathfrak{P}\) is a prime ideal in \(\mathcal{R}\);
2. \(\mathfrak{P} + \mathfrak{P}^* = \mathcal{S} = \mathcal{R}\).

It belongs to the type \((\beta_2)\) if and only if it has an infinite atomic basis.

If a ring \(A\) belongs to any one of the four types \((a), (\beta_1), (\beta_2), (\beta_3)\), then Th. 2.2 (3) and Th. 2.5 show that \(A\) is totally multiplicative. If it belongs to type \((a)\) it has a unit, by Th. 2.2 (3), and is infinite. If it belongs to any one of the types \((\beta_1), (\beta_2), (\beta_3)\), it is isomorphic to a certain non-normal ideal \(a\). Since \(a\) is not normal, it is not principal and therefore has no unit. Consequently \(A\) has no unit and, by R Th. 1, must be infinite. We thus see that the type \((a)\) is distinct from the aggregate of the three types \((\beta_1), (\beta_2), (\beta_3)\); and that in studying these types we may confine our attention to infinite totally multiplicative rings.

If \(A\) is such a ring, the condition that it be infinite being automatically satisfied if it has no unit, we shall show that it belongs to just one of the four types. First, if \(A\) has a unit, then Th. 2.2 shows at once that it is of type \((a)\). With this characteristic of type \((a)\) is completed; and it is evident that the only rings of types \((\beta_1), (\beta_2), (\beta_3)\) are totally multiplicative rings without unit. If \(A\) is such a ring, we proceed to imbed it as an ideal \(a\) with \(a^* = a\) in a totally additive ring \(B\) in accordance with Th. 2.5. We can do so in essentially only one way; and the most convenient way is to identify \(A\) with the ideal \(\mathfrak{P}\) in \(\mathcal{R}\). Since \(A\) and its isomorph \(\mathfrak{P}\) are infinite, \(\mathcal{R}\) is also infinite and hence of type \((a)\). Since \(A\) has no unit, \(\mathfrak{P}\) is not principal in \(\mathcal{R}\) and by Th. 2.2 (2), cannot be normal in \(\mathcal{R}\). The preliminary conditions of Def. 2.6 are thus satisfied. Now it is evident that the given ring \(A\) is of type \((\beta_3)\) if and only if \(\mathfrak{P}\) is a prime ideal in \(\mathcal{R}\); moreover, if we recall our earlier result that a ring of type \((\beta_1)\) is necessarily a totally multiplicative ring without unit, we see that the necessity of criterion (1) for type \((\beta_3)\) is now established. On the other hand the sufficiency of this criterion is so far established only for totally multiplicative rings without unit. A brief digression will enable us to remove this difficulty. If \(A\) is any ring with \(\mathfrak{P}\) a prime ideal in \(\mathcal{R}\), the known relation \(\mathfrak{P} = \mathcal{S}\) shows that \(\mathfrak{P}\) is not normal and, in particular, not principal. Thus \(A\) has no unit and \(\mathfrak{P} = \mathfrak{P}^*\). By R Th. 32 we know that \(\mathfrak{P}^*\) is a subring of \(\mathcal{R}\). Since it contains the prime ideal \(\mathfrak{P}\) but does not coincide with it, we must have \(\mathfrak{P}^* = \mathcal{R}\). In consequence \(\mathcal{S} = \mathcal{R}\), and \(A\) must be totally multiplicative by virtue of Th. 2.3 (2). It now follows that \(A\) is of type \((\beta_3)\). In this proof, we have found that criterion (1) implies the relations \(\mathfrak{P} = \mathfrak{P}^* = \mathcal{S} = \mathcal{R}\) of criterion (2); but, conversely, these relations show by R Th. 32 that \(\mathfrak{P}\) is a prime ideal in \(\mathfrak{P}^*\) and hence also in \(\mathcal{R}\). Having justified our two criteria for type \((\beta_3)\), we resume our discussion of the case of a totally multiplicative ring \(A\) without unit. Our next step is to show that if \(A\) has an atomic basis, it is not of type \((\beta_3)\). Turning our attention to the ideal \(\mathfrak{P}\) in \(\mathcal{R}\), we see that \(\mathfrak{P}\) has an atomic basis \(\mathfrak{B}\). Obviously the ideal generated by \(\mathfrak{B}\) in \(\mathcal{R}\) is \(\mathfrak{P}\). By R Th. 20, we see that \(\mathfrak{B} = \mathfrak{P}\) and hence that \(\mathcal{R} \subseteq \mathcal{B} = \mathcal{S} = \mathcal{R}\). By R Def. 5 and 7, the latter relation means that \(\mathcal{R}\) is a complete atomic system in \(\mathcal{R}\). Since every normal ideal in \(\mathcal{R}\) is principal, R Th. 62 shows that \(\mathcal{R}\) is the isomorphism to the Boolean ring of all subclasses of a fixed (here necessarily infinite) class \(E\), the correspondents of \(\mathfrak{B}\) being the system of all one-element subclasses of \(E\). If we now choose \(b\) and \(c\) as elements of \(N\) corresponding to two disjoint infinite subclasses of \(E\), we see that \(b = c\) but that neither \(b\) nor \(c\) is in \(\mathfrak{B}\). Hence \(\mathfrak{P}\) is not a prime ideal in \(\mathcal{R}\), and \(A\) is not of type \((\beta_3)\). It follows that the types \((\beta_1)\) and \((\beta_3)\) are distinct. It follows also that in a ring of type \((\beta_3)\) the relation \(\mathfrak{P}^* = \mathcal{S}\) must hold; for the relation \(\mathfrak{P}^* = \mathcal{S} = \mathcal{R}\) would imply
that \( B \) is prime in \( R = B^* \). To determine whether \( A \) is of type \((\beta_3)\) or not, we therefore have only to ascertain whether \( A \) has an atomic basis or not. Obviously a totally multiplicative ring without unit, being infinite, cannot have a finite atomic basis. On the other hand, any ring with an atomic basis is totally multiplicative by Th. 2.3; and, if the atomic basis is infinite, the ring obviously cannot have a unit. Hence any ring with infinite atomic basis is of type \((\beta_3)\). The characterization of rings of type \((\beta_3)\) is thereby completed. Finally we observe that the remaining type \((\beta_2)\) was so defined as to take in all those totally multiplicative rings without unit which are not of types \((\beta_1)\) or \((\beta_2)\). The proof of the theorem is thus brought to a close.

The study of the various composite types depends not only upon the preceding results but also upon some further information, which we shall present next. We first give a few elementary properties of direct sums.

**Theorem 2.7.** If a Boolean ring \( A \) is represented as a direct sum \( A_1 \sqcup A_2 \), then:

1. \( A_1 \) and \( A_2 \) are simple ideals in \( A \);
2. \( A \) is totally multiplicative if and only if \( A_1 \) and \( A_2 \) are totally multiplicative;
3. \( A \) has a unit if and only if \( A_1 \) and \( A_2 \) both have units;
4. \( A \) is totally additive if and only if \( A_1 \) and \( A_2 \) are totally additive.

If a Boolean ring \( A \) is represented as a direct sum \( A^* \sqcup A_2 \) where \( A^* \) is obtained from a ring \( A \) without unit by the adjunction of a unit, then:

1. \( A_1, A_2, A^* \), and \( A_1 \sqcup A_2 \) are ideals in \( A \); the ideals \( A^* \) and \( A_2 \) are simple; the ideals \( A_1 \) and \( A_1 \sqcup A_2 \) are non-normal; and the ideal \( A_1 \sqcup A_2 \) is prime;
2. \( A \) is totally multiplicative if and only if \( A_1 \) is of type \((\beta_2)\) and \( A_2 \) is totally multiplicative;
3. \( A \) has a unit if and only if \( A_2 \) has a unit;
4. \( A \) is totally additive if and only if \( A_1 \) is of type \((\beta_2)\) and \( A_2 \) is totally additive.

In particular, the ring \( A^* \) obtained by adjunction of a unit to a ring \( A \) without unit is totally additive if and only if \( A \) is of type \((\beta_2)\). By R. Th. 51, the summands \( A_1 \) and \( A_2 \) are simple ideals \( a_1 \) and \( a_2 \) respectively in the direct sum \( A = A_1 \sqcup A_2 \). If \( a \) is an arbitrary ideal in \( A \), then \( a = a_1 \sqcup a_2 \). If \( a \) is normal in \( A \), then \( a_1 \) and \( a_2 \) are also normal in \( A \) by Th. 1.2. Conversely, if \( a_1 \) and \( a_2 \) are normal in \( A \), then so is \( a \). To prove this, we begin by calculating \((a_1)^{\prime\prime}\) and \((a_2)^{\prime\prime}\). We write \((a_1)^{\prime\prime} = (a_1)\cap (a_1)\cup (a_2)\cap (a_2)\). Here we may regard the first term on the right as the second orthocomplement of the ideal \( a_1 = (a_1)\cap (a_1)\cup (a_2)\cap (a_2)\) relative to \( a_1 \) in accordance with R. Th. 22 and hence find that \((a_1)^{\prime\prime} = a_2 \cdot a_1 ; \) and similarly we may regard the remaining term as the second orthocomplement of the ideal \( a_2 = (a_2)\cap (a_1)\cup (a_2)\cap (a_2)\) relative to \( a_2 \) and hence find that \((a_2)^{\prime\prime} = a_2 \cdot a_1 \). Thus we have \((a_1)^{\prime\prime} = a_1 \cdot a_2 ; \) and, in the same way, \((a_2)^{\prime\prime} = a_1 \cdot a_2 \). Now, if \( a_1 \) and \( a_2 \) are normal, we have \( a = a_1 \sqcup a_2 = (a_1)^{\prime\prime} \cup (a_2)^{\prime\prime} = a_1 \cdot a_2 \). So that \( a \) is also normal. Next Ths. 1.2 and 1.3 show that \( a_1 \) is normal in \( A \) if and only if it is normal relative to the simple ideal \( a_1 \) containing it; and likewise that \( a_2 \) is normal in \( A \) if and only if it is normal relative to \( a_2 \). Thus, a is normal if and only if \( a_1 \) and \( a_2 \) are normal relative to \( a_1 \) and \( a_2 \) respectively.

It is now easy to discuss the conditions under which \( A \) is totally multiplicative, using the test of Th. 2.3 (2). If \( a \) is any ideal contained in \( a_1 \) and normal relative to \( a_1 \), then the preceding results show that \( a = a_1 \sqcup a_2 \) is normal in \( A \). Hence, if \( A \) is totally multiplicative, \( a \) is simple in \( A \) and, according to Th. 1.2, is also simple relative to \( a_1 \). Thus, \( a_1 \) and \( a_2 \) are, likewise, totally multiplicative rings. On the other hand, if \( a \) is any normal ideal in \( A \), \( a_1 \) and \( a_2 \) are normal relative to \( a_1 \) and \( a_2 \) respectively. Hence, if \( a \) and \( a_2 \) are totally multiplicative, \( a_1 \) and \( a_2 \) are simple relative to \( a_1 \) and \( a_2 \) respectively; and, according to Th. 1.3, are also simple in \( A \). It follows that \( a = a_1 \sqcup a_2 \) is simple in \( A \). Thus \( A \) is totally multiplicative.

We have thereby proved that \( A \) is totally multiplicative if and only if \( A_1 \) and \( A_2 \) both are. If \( A \) has a unit, both simple ideals \( a_1 \) and \( a_2 \) are principal so that \( A_1 \) and \( A_2 \) both have units; and, if \( A_1 \) and \( A_2 \) both have units, then so does their direct sum \( A \). An easy application of Th. 1.2 (2) to the preceding results now shows that \( A \) is totally additive if and only if \( A_1 \) and \( A_2 \) both are.

In the direct sum \( A^* \sqcup A_2 \), the summands \( A^* \) and \( A_2 \) are simple ideals as before. Moreover, by R. Th. 37, \( A_1 \) is a non-normal prime ideal in \( A^* \). Hence we see that \( A_1 \) is a non-normal ideal in \( A \); and also that \( A_1 \sqcup A_2 \) is a non-normal ideal in \( A \), by virtue of the results established in the preceding paragraph.
To show that \( A_1 \lor A_2 \) is prime, we proceed as follows. Let \( a \) and \( b \) be elements of \( A \) with \( ab \in A_1 \lor A_2 \). We can then write \( a = a_1 + b_1 \), \( b = a_2 + b_2 \) where \( a_1, a_2 \) are in \( A_1 \) and \( b_1, b_2 \) are in \( A_2 \). It is then clear that \( ab = a_1a_2 + b_1b_2 \) where \( a_1a_2 \) is in \( A_1 \) and \( b_1b_2 \) is in \( A_2 \). By virtue of the fact that \( A_1 \) is prime in \( A_1^+ \), at least one of the elements \( a_1, a_2 \) is in \( A_1 \); and then the corresponding element \( a \) or \( b \) must belong to \( A_1 \lor A_2 \). Accordingly \( A_1 \lor A_2 \) is a prime ideal in \( A \). If we apply the results of the preceding paragraph, we see that \( A_1^+ \lor A_2^+ \) has a unit if and only if \( A_1 \) does; and, since \( A_1^+ \) being a ring with unit, is totally multiplicative if and only if it is totally additive, that \( A \) is totally multiplicative (totally additive) if and only if \( A_1 \) is totally multiplicative (totally additive) and \( A_1^+ \) totally additive.

To complete our discussion we must show that a ring \( A^* \) obtained from a ring \( A \) without unit by the adjunction of a unit is totally additive (or, equivalently, totally multiplicative) if and only if \( A \) is of type \( (\beta_1) \). Since \( A \) is a non-normal prime ideal in \( A^* \), the assumption that \( A^* \) is totally additive immediately identifies \( A \) as of type \( (\beta_1) \) in accordance with Definition 2.6 and Theorem 2.6. On the other hand, if \( A \) is of type \( (\beta_1) \), it can be imbedded as a non-normal prime ideal in a totally additive ring \( B \). It is evident that \( B \) coincides with its subring generated by the prime ideal \( A \) and the unit of \( B \). Hence Th. 1 shows that \( A^* \) is isomorphic to \( B \) and thus totally additive.

In further applications of direct sum representations we find the following definition useful:

**Definition 2.9.** If a Boolean ring \( A \) is represented in two ways as a direct sum:

\[
A = A_1 \lor A_2, \quad A = A_3 \lor A_4
\]

where

\[
A_1 = B_1 \lor B_2, \quad A_3 = B_1 \lor B_4, \\
A_2 = B_3 \lor B_4, \quad A_4 = B_3 \lor B_2
\]

then each representation is said to be obtained from the other by interchange of the direct summands \( B_2 \) and \( B_4 \).

The connection between the results of Th. 2.5 and representations by direct sums is now easily discussed. We have:

**Theorem 2.8.** The following assertions concerning a Boolean ring \( A \) are equivalent:

1. \( A \) contains an ideal \( a \) with \( a^2 = 0 \) which, considered as a ring, is totally multiplicative and which together with an element \( a_0 \) not in a generates \( A_0 \);
2. there exists a totally additive Boolean ring \( B \) containing a subring \( B_0 \) which is isomorphic to \( A_0 \) and which is generated by an ideal \( b \), with \( b^2 = 0 \), and some element \( b_0 \) not in \( b \);
3. there exist totally multiplicative Boolean rings \( A_1 \) and \( A_2 \), where \( A \) has no unit, such that \( A_0 \) is isomorphic to a direct sum \( A_1^+ \lor A_2^+ \) of the kind discussed in Th. 2.7.

In (1) the ideal \( a \) is not uniquely determined in general but is necessarily prime, so that \( a_0 \) is free to vary outside \( a \); and in (2) the ring \( B \) is necessarily isomorphic to \( \Psi(A_0) \) in such a way that the correspondents of \( B_0 \) is \( \Psi(A_0) \), but the correspondent of \( b \) in \( \Psi(A_0) \) is not uniquely determined in general. The representation (3) is likewise not uniquely determined. It is possible, however, to pass reversibly from a representation of any kind to one of any other by the following processes: if a representation (1) is given, we put \( B = \Psi(A_0) \), \( B_0 = \Psi(A_0) \), \( b = \Psi(a) \) and \( b_0 = \Psi(a_0) \) to obtain a representation (2); and, \( A_1 = a(a_0) \), \( A_2 = a^2(a_0) \) to obtain a representation (3); if a representation (2) is given, we take \( a \) and \( a_0 \) as the respective correspondents of \( b \) and \( b_0 \) under the isomorphism \( A_0 \rightarrow B_0 \) to obtain a representation (1); and put \( A_1 = a(b_0)b, A_2 = a^2(b_0)b \) to obtain a representation (3); and, if a representation (3) is given, we put \( a = A_1 \lor A_2, a_0 = a_0^+ \), where \( e^+ \) is the unit of \( A_1^+ \), to obtain a representation (1); and \( B = \Psi(A_0) \lor \Psi(A_2) \), \( b = \Psi(A_1) \lor \Psi(A_2) \), \( b_0 = e_1 \) where \( e_1 \) is the unit of \( \Psi(A_2) \), to obtain a representation (2). In order that a Boolean ring representable in these equivalent forms be totally multiplicative, the following conditions are separately necessary and sufficient:

1. \( a(a_0)a \) is a ring of type \( (\beta_1) \);
2. \( a(b_0)b \) is a ring of type \( (\beta_1) \);
3. \( e^+ \) is a ring of type \( (\beta_1) \).

First, let \( A_0 \) have a representation (1). Then the subring of \( A_0 \) generated by \( a \) and \( a_0 \) consists of all elements \( a + a_0 \), where \( a \) and \( a_0 \) are in \( A_0 \). Consequently \( A_0/a \) is a two-element ring, and \( A \) is prime in \( A \). The relation \( a^2 = 0 \) shows that \( a \) is not normal.
By Th. 2.5 the correspondence \( a \rightarrow a(a) a \) carries \( A_0 \) isomorphically into a subring \( \mathfrak{R}_0 \) of \( \mathfrak{R}(a) \), a isomorphically into the ideal \( \mathfrak{P}(a) \) in \( \mathfrak{R}(a) \). On the other hand the correspondence \( e \rightarrow a(a) \) defines the isomorphism \( A_0 \rightarrow \mathfrak{P}(A_0) \). Combining these correspondences we obtain an isomorphism \( \mathfrak{P}(A_0) \leftarrow \mathfrak{R}_0 \) which leaves \( \mathfrak{P}(a) \) as a common part of \( \mathfrak{P}(A_0) \) and \( \mathfrak{R}_0 \), invariant. We shall show now that this isomorphism can be extended to the rings \( \mathfrak{R}(A_0) \), \( \mathfrak{R}(a) \).

From Th. 1.3, we know that the correspondence \( b \rightarrow b(a) \) carries a normal ideal \( a \) in \( A_0 \) into a normal ideal relative to \( a \). Moreover, if \( b \) is a normal ideal relative to \( a \), the relation \( b = b'' a = \mathfrak{P}(a) \) is valid and the normal ideal \( b'' \) in \( A_0 \) is carried by the above correspondence into the prescribed ideal \( b \) in \( a \). If we apply Th. 22 to the ideal \( (b_1 \vee b_2) a = (b_1 a \vee b_2 a) a \) in \( a \) we find that \( (b_1 \vee b_2) a'' = (b_1 a \vee b_2 a) a'' \). Hence the relations \( b_1 \rightarrow b_2 a, \ b_2 \rightarrow b_3 a \) imply

\[
\begin{align*}
\quad b_1 b_2 & \rightarrow b_3 a, \quad (b_1 \vee b_2) a'' = (b_3 a \vee b_4 a) a''.
\end{align*}
\]

In words the second relation becomes: the normalized sum is carried by the above correspondence into the normalized sum, relative to \( a \), of the corresponding elements of the original summands. This correspondence therefore defines a homomorphism \( \mathfrak{R}(A_0) \rightarrow \mathfrak{R}(a) \). If \( b \rightarrow b = a \), then \( bC = a \). Hence the indicated homomorphism is an isomorphism, as we wished to prove. We now see that \( \mathfrak{P}(a) \) is an ideal in \( \mathfrak{R}(A_0) \) as well as in \( \mathfrak{P}(A_0) \). Hence we obtain a representation (2) for \( A_0 \) by putting \( B = \mathfrak{P}(A_0), \ B_0 = \mathfrak{P}(A_0), \ b = a(a) e \).

Assuming still that \( A_0 \) has a representation (1), we note that \( a \) is a prime, must contain \( a(a) a \) in accordance with R Th. 41. Thus we may represent \( A_0 \) as the direct sum \( a(a) a \vee a(a) \) and as the direct sum \( a(a) a \vee a(a) a \) in \( A_0 \). Since \( a \) is totally multiplicative, both \( a(a) a \) and \( a(a) a \) are totally multiplicative by Th. 2.7. Since \( a \) is prime and non-normal in \( A_0 \), Th. 1.7 shows that \( a(a) a \) is prime and non-normal relative to \( a(a) a \). Hence \( a(a) a \) is generated by \( a(a) a \) and \( a(a) a \) — that is, arises from the ring \( a(a) a \) without unit by the addition of the element \( a(a) a \) as unit. Thus if we put \( A_1 = a(a) a, \ A_2 = a(a) a \), we have \( \{A_1 \} = a(a) a \); and find a representation (3) for \( A_0 \).

Next we suppose that \( A_0 \) has a representation (2). According to Th. 2.5, the representation of \( A_0 \) in terms of \( A_1 \) and \( A_2 \) is a representation (1); and there is an isomorphism \( B \rightarrow \mathfrak{R}(b) \) carrying \( b \) into \( \mathfrak{P}(B) \) and \( B \) into \( \mathfrak{R}(b) \), where \( \mathfrak{P}(b) \) is an ideal not only in \( \mathfrak{R}(b) \) but also in \( \mathfrak{R}(B) \). The isomorphism \( A_0 \rightarrow A_0 \) shows that the correspondents \( a(a) a \) and \( a(a) a \) respectively provide a representation (1) for \( A_0 \). This isomorphism induces an isomorphism \( \mathfrak{R}(A_0) \rightarrow \mathfrak{R}(A_0) \) carrying \( \mathfrak{R}(A_0) \) into \( \mathfrak{R}(A_0) \) and thus leads to an isomorphism \( B \rightarrow \mathfrak{R}(A_0) \) carrying \( B \) into \( \mathfrak{P}(A_0) \) and \( b \) into the ideal \( \mathfrak{P}(a) \). We therefore see that the given representation (2) is isomorphic to the one constructed from \( A_0 \) in the manner described in the preceding paragraph. Since \( B \) is not a ideal in \( B \), except in special cases, we have a representation (2) for \( A_0 \) by Th. 15 (5), but cannot replace the inclusion by equality without further argument. Since \( a(a) a \) and \( a(a) a \) contains \( a(a) b \) and \( a(a) b \) and \( b \), it is evident that it contains, and hence coincides with, \( B \).

Accordingly we see that \( B \) is represented as the direct sum \( a(a) b \) of \( B \) and \( B \) where \( a(a) B \) is defined as its unit. The ideal \( b \) in \( B \) is represented at the same time as the direct sum \( a(a) b \) of \( b \). Since \( b \) and \( b \) provide a representation (1) for \( B \), the results of the preceding paragraph show that, putting \( A_1 = a(a) B, \ A_2 = a(a) b \), we obtain \( A_1 \rightarrow a(a) B_0 \) and \( B_0 = A_1 \). Thus \( B_0 \) and its isomorphism \( A_0 \) have a representation (3).

We start now with the assumption that \( A_0 \) has a representation (3). We, without loss of generality, identify \( A_0 \) with \( A_1 \). Th. 2.7 shows immediately that on putting \( e = A_1 A \) and \( e = A_2 \), where \( e = A_1 \) is the unit of \( A_1 \), we obtain a representation (1) for \( A_0 \). If we now use this representation to reconstruct a representation (3) of \( A_0 \) as in the preceding paragraphs, it is obvious that we recover the given representation (3). By Th. 2.5 we know that \( A_0 = A_1 \) isomorphic to the subring of \( \mathfrak{P}(a) \) of \( \mathfrak{P}(A_1) \) generated by the ideal \( \mathfrak{P}(a) \) and the element \( a(a) a \). Since \( A_1 \) and \( A_2 \) are simple ideals in the direct sum \( A_1 \) of \( A_1 \) they may be regarded as elements of \( A_1 \) of \( A_1 \). Since they satisfy the relations \( a(a) a = a(a) a = A_1 \), \( a(a) a = a(a) a = A_1 \), they define a direct sum representation of \( \mathfrak{R}(A_1) \). By Th. 2.7 and our present hypothesis, we know that \( \mathfrak{R}(A_1) \) isomorphic to \( \mathfrak{R}(A_1) \) and \( \mathfrak{R}(A_1) \) isomorphic to \( \mathfrak{R}(A_1) \). Now the direct sum representation of \( \mathfrak{P}(A_1) \) is easily calculated on the basis of Th. 51 and Th. 1.2, 1.3, and 1.4. We find that each element of \( \mathfrak{P}(A_1) \) is represented as the sum of components which are unrestricted elements of \( \mathfrak{P}(A_1) \) and \( \mathfrak{P}(A_1) \) respectively. In particular, an element of \( \mathfrak{P}(A_1) \) is represented...
as the sum of components which are unrestricted elements of $\mathcal{P}(A_1)$ and $\mathcal{P}(A_2)$ respectively. Thus we find that $\mathcal{P}(A_1 \cup A_2) = \mathcal{P}(A_1) \cup \mathcal{P}(A_2)$. Moreover the element $a \in \mathcal{P}(A_1)$ is easily identified with the unit $e_1$ of $\mathcal{P}(A_1)$. Hence, if we put $B = \mathcal{P}(A_1) \cup \mathcal{P}(A_2)$, $b = \mathcal{P}(A_1) \cup \mathcal{P}(A_2)$, $b_0 = e_0$, we obtain a representation (2) for $A_0$. Since we obviously have $a(b_0)b = \mathcal{P}(A_1)$ and $a'(b_0)b = \mathcal{P}(A_2)$, the reconstruction of a representation (3) from the representation (2) just found yields a representation essentially the same as that assumed at the outset.

In view of Th. 2.7 we see that a ring $A_0$ represented in the form (3) is totally multiplicative if and only if $A_1$ is of type (2), the correspondence between the three different representations thus lead to the equivalent conditions stated above for $A_0$ to be totally multiplicative.

The effect of Th. 2.8 is to show that all the Boolean rings $A_0$ obtainable from totally multiplicative rings without unit by a least possible proper extension can be constructed in either of two equivalent ways: as specific subrings of infinite totally additive rings or as direct sums of the form (3). Naturally, our interest now centers on the rings $A_0$ of this form which are not totally multiplicative. We have:

**Theorem 2.9.** A non-totally-multiplicative Boolean ring $A_0$ which is representable in any of the three equivalent forms (1), (2), (3) of Th. 2.8 has in each case a representation which is unique in the following sense:

1. for a representation (1), the ideal $a$ is uniquely determined;
2. for a representation (2), the isomorphism $B_0 \rightarrow \mathcal{P}(A_0)$ carrying $B_0$ into $\mathcal{P}(A_0)$ carries $b$ into a uniquely determined ideal in $\mathcal{P}(A_0)$;
3. for two representations (3), $A_0 = A_1^* \cup A_2 = A_3^* \cup A_4$, the direct sums $A_1 \cup A_2$ and $A_3 \cup A_4$ are obtained from each other by an exchange of totally additive direct summands.

In case $A_0$ also has a unit, the various representations can be reduced to unique forms by the following normalizing conditions:

1. for a representation (1), $a_0 = e$;
2. for a representation (2), $b_0$ is the unit in $B$;
3. for a representation (3), $A_2$ is a one-element ring so that $A_0 = A_1^* \cup A_2$.

Let there be given two representations (1) for a ring $A_0$ in terms of ideals $a_1$ and $a_2$ respectively; and assume that $A_0$ is not totally multiplicative. Then by Th. 2.3 there exists a non-simple normal ideal $c$ in $A_0$. According to Th. 1.3, the ideals $a_1c, a_2c$ are normal, and hence simple, relative to the totally multiplicative rings $a_1$ and $a_2$ respectively. Thus we have the relations:

$$c \cap c' = e, \quad a_1c \cap a_2c = e, \quad a_1c \cap a_2c = e.$$

Since $a_1$ and $a_2$ are both prime we conclude that $a_1 = a_2$. With this result we have established the uniqueness-assertion of the theorem. In view of the results of Th. 2.8, the uniqueness assertion for representations of the form (2) follows immediately; and that for representations of the form (3) is proved in part, to the extent that $A_0 = A_1^* \cup A_2 = A_3^* \cup A_4$ is now seen to imply $a_1 = a_2 \cup A_2 = a_2 \cup A_4$. Since the two representations $A_1^* \cup A_2$ and $A_3^* \cup A_4$ correspond to representations (1) given by $a_1$ and $a_2$, where $a_1$ and $a_2$ are any suitable elements not in $a_0$, we have:

$$A_1 = a(a_0)a_1, \quad A_3 = a(a_0)a_1,$$

$$A_2 = a'(a_0)a_1, \quad A_4 = a'(a_0)a_1,$$

by the results of Th. 2.8. If we now introduce

$$B_1 = a(a_0)a(a_0), \quad B_2 = a'(a_0)a'(a_0),$$

$$B_3 = a(a_0)a'(a_0), \quad B_4 = a'(a_0)a(a_0),$$

we see immediately that $A_0 \cup A_2$ is obtained from $A_0 \cup A_4$ by exchange of the direct summands $B_1$ and $B_2$ in accordance with Def. 2.9. By Th. 1.2, $B_1$ and $B_2$ are principal ideals in $A_0$ and in $a_0$ hence both are totally additive rings by Th. 2.3 (3). Thus the uniqueness-assertion for representations of the form (3) is established. In view of Th. 2.7, any exchange of totally additive direct summands within $a = A_1^* \cup A_2$ is permissible.

In case $A_0$ has a unit we may obviously take $a_0 = a$ in the representation (1). It then follows that in any representation (2), the subring $B_0$ contains the unit of $B$: for the isomorphism $B_0 \rightarrow \mathcal{P}(A_0)$ carrying $B_0$ into the subring $\mathcal{P}(A_0)$ of $\mathcal{P}(A_0)$ obviously takes the unit of $B$ into the unit $a_0 \in \mathcal{P}(A_0)$ of $\mathcal{P}(A_0)$. Accordingly, we may take $b_0$ as the unit of $B$. Finally the representation (3) corresponding to the choice $a_0 = e$ is evidently given by $A_0 = A_1^* \cup A_2$ where $A_1 = a(a_0)a = a$ and $A_2 = a'(a_0)a = a(0) = e$. 
We can now give our series of fundamental types in full. We have:

**Theorem 2.10.** The infinite Boolean rings constructed from (infinite) totally additive Boolean rings by the selection of ideals, finite direct summation, and at most one unit-adjunction applied to a ring without unit fall into exactly nine distinct types — namely, the types \((\alpha), (\beta_1), (\beta_2), (\beta_3), (\beta_3^1), (\beta_3^1, \beta_2), (\beta_3, \beta_2), (\beta_3^1, \beta_2^1), \) and \((\beta_3^1, \beta_2^1).\) A ring of type \((\alpha), (\beta_1), (\beta_2), (\beta_3), (\beta_3^1), (\beta_3^1, \beta_2), (\beta_3^1, \beta_2^1), \) or \((\beta_3^1, \beta_2^1);\) a ring of type \((\beta_3^1, \beta_2)\) or \((\beta_3^1, \beta_2^1);\) has a unique representation as a member of this type; a ring of type \((\beta_3^1, \beta_2)\) or \((\beta_3^1, \beta_2^1)\) has a representation as a member of this type which is unique except for an exchange of totally additive direct summands between the two underlying components; and a ring of type \((\beta_3^1, \beta_2^1)\) has a representation as a member of this type which is unique except for an exchange of finite direct summands between the two underlying components. As a special case under type \((\beta_2),\) we note the type \((\beta_2, \beta_1);\) any ring of this type has a representation as a member of this type which is unique except for an exchange of totally additive direct summands between the underlying components. Other constructions of these types of ring are given in Th. 2.8.

The types \((\alpha), (\beta_1), (\beta_2), (\beta_3), (\beta_3^1), (\beta_3^1, \beta_2), (\beta_3^1, \beta_2^1), (\beta_3^1, \beta_2^1, \beta_2),\) and \((\beta_3^1, \beta_2^1, \beta_2^2);\) have already been discussed in Th. 2.6. Since they exhaust the infinite totally multiplicative rings, we have to show that the five remaining types are distinct and exhaust the rings treated in Th. 2.9. Th. 2.7 enables us to discuss the rings of type \((*, **)\) more exactly. They are all infinite and totally multiplicative. Such a ring is of type \((\alpha)\) if and only if it is of type \(\alpha, \alpha.\) Similarly such a ring is of type \((\beta_1)\) if and only if it is of type \(\beta_1, \alpha.\) It is easily seen that such a ring is of type \((\beta_2)\) if and only if each of its components has an atomic basis; in other words, if and only if the ring is of type \(\beta_2, \alpha.\) Consequently, the rings of types \((\beta_2, \alpha), (\beta_2, \beta_1), (\beta_2, \beta_2), (\beta_2, \beta_3), (\beta_2, \beta_3^1), (\beta_2, \beta_3^1, \beta_2), (\beta_2, \beta_3^1, \beta_2^1).\) All included under the type \((\beta_2).\) It is obvious that we could retain some of these special types for the purpose of subdividing the type \((\beta_2).\) As none of them except the type \((\beta_2, \beta_1)\) has any further interest as an individual type in this paper, we shall not do so. So far as the type \((\beta_2, \beta_1)\) is concerned, it appears as a special case under types discussed in the following section. We shall therefore prove the uniqueness-assertion concerning the representation as a member of this special type. If \(A_1 \vee A_2 = A_3 \vee A_4\) are two representations of a ring \(A\) of type \((\beta_2, \beta_1)\) we put:

\[
A_1 = B_1 \vee B_3, \quad A_3 = B_1 \vee B_4,
\]

\[
A_2 = B_2 \vee B_4, \quad A_4 = B_3 \vee B_2,
\]

where

\[
B_1 = A_1, \quad B_2 = A_4, \quad B_3 = A_2, \quad B_4 = A_3.
\]

Since \(A_3\) and \(A_4\) are simple ideals in \(A\) their intersections \(B_1, B_2, B_3, B_4\) with the simple ideals \(A_1\) and \(A_4\) are simple in \(A\) and also in whichever of the ideals \(A_1, A_2\) contains them, as we see by reference to Ths. 1.2 and 1.3. Now according to Th. 2.6, the fact that \(A_1\) and \(A_2\) are rings of type \((\beta_1)\) implies that the ideals \(B_1, B_2, B_3, B_4\) are semi-principal in whichever of the ideals \(A_1, A_2\) contains them. Since neither \(A_1\) nor \(A_2\) has a unit, only one of the two ideals contained in each of \(A_1\) and \(A_2\) can be principal. Our notation can be so adjusted, by proper assignment of the indices \(3\) and \(4\), that \(B_2\) is principal in \(A_1\) and \(B_3\) in \(A_2\). Thus \(B_2\) and \(B_4\) are totally additive rings in accordance with Th. 2.3 (3). We have therefore proved that each of the given representations arises from the other by an exchange of totally additive direct summands. Th. 2.7 shows furthermore that any such exchange is permissible, in that it leaves each component of type \((\beta_2)\). Next we consider the types \((\beta_1^1), (\beta_2^1), (\beta_3^1), (\beta_3^1, \beta_2), (\beta_3^1, \beta_2^1);\) Th. 2.7 shows that a ring of one of these types is totally multiplicative if and only if it is of type \((\beta_1^1)\) and is then of type \((\alpha)\). Thus Th. 2.9 shows that the two types \((\beta_2^1, \beta_2)\) and \((\beta_2^1, \beta_2)\) are distinct and together exhaust the rings with unit considered there.

We now consider the composite types \((\beta_2^1, \beta_2), (\beta_3^1, \beta_2), (\beta_3^1, \beta_2), (\beta_3^1, \beta_2), (\beta_3^1, \beta_2).\) Th. 2.7 shows that no ring of any of these types is totally multiplicative or has a unit. It is obvious therefore that these types exhaust the rings without unit considered under Th. 2.9. Since the representation of one of the latter rings in the form \(A_1^1 \vee A_2\) is unique except for exchanges of totally additive direct summands between the components \(A_1\) and \(A_2\) and since Th. 2.7 shows any such exchange to be permissible — we find that the type \((\beta_3^1, \beta_1)\) is included under the type \((\beta_2^1, \beta_2)\) and \((\beta_3^1, \beta_2)\) under the type \((\beta_3^1, \beta_2)\). As we discuss the case of type \((\beta_3^1, \beta_2);\) this direct summand of type \((\alpha)\) from the component \(A_3\) of type \((\beta_2)\) to the component \(A_4\) of type \((\beta_2)\), we obtain a new representation in which, according to Th. 2.7, the components \(A_1\) and \(A_2\) are now of types \((\beta_3, \alpha)\) and \((\beta_2)\) respectively. Since the type \((\beta_3, \alpha)\) is included under type \((\beta_3)\), the new representation
exhibits the original ring as a member of type \((\beta^*_1, \beta_1)\). Th. 2.7 shows that no ring of type \((\beta^*_2, \beta_1)\) is a member of either of the types \((\beta^*_2, \beta_2)\) and \((\beta^*_3, \beta_3)\): for an exchange of totally additive direct summands between the components of a direct sum replaces a component of type \((\beta_2)\) by one of type \((\beta_3)\). For a similar reason no ring of type \((\beta^*_3, \beta_1)\) is a member of either of the types \((\beta^*_2, \beta_2)\) or \((\beta^*_3, \beta_3)\). In fact, a totally additive direct summand in a ring of type \((\beta_2)\) is necessarily a principal ideal in the ring and must therefore be a finite ring. Hence the only exchange of totally additive direct summands between the underlying components of a ring of type \((\beta^*_2, \beta_1)\) is an exchange of finite direct summands; and such an exchange obviously leaves both components with finite atomic bases. We thus conclude that the three types \((\beta^*_2, \beta_1), (\beta^*_2, \beta_2), (\beta^*_3, \beta_3)\) are distinct and together exhaust the rings without unit considered in Th. 2.9. We have proved incidentally that the representation of a ring as a member of any of these three types is unique in the sense described above.

The results established in the preceding paragraphs show that the operations admitted here cannot produce any composite type other than the nine explicitly investigated: for example the composite type \(((\beta^*_1, \beta_2)^*, \beta_3), (\alpha_3, \beta_3, (\beta_1, \beta_2))\) is seen to reduce first to the type \(((\beta^*_1, \beta_3), (\alpha_3, \beta_3, (\beta_1, \beta_2)))\), then to the type \(((\beta^*_2, \beta_3), (\alpha_2, \beta_3)), \text{ or } (\beta^*_3, (\alpha_2, \beta_3))\), then to the type \(((\beta^*_2, \beta_3), (\alpha_2, \beta_3))\), and finally to the type \(((\beta^*_1, \beta_3), (\alpha_2, \beta_3))\). It is essential, of course, that not more than one unit-adjunction is allowed in the process of composition.

A further refinement of type could be introduced by the consideration of the systems of atomic elements, if any exists, in totally multiplicative rings. While we shall not consider such a refinement in the remaining sections of the paper, it seems appropriate to complete our investigations of totally multiplicative and related rings by examining the part played by atomic systems. We have:

**Theorem 2.11.** With respect to the existence of atomic elements, the totally multiplicative Boolean rings may be classified as follows:

1. rings without atomic element;
2. rings with complete atomic systems;
3. direct sums of the preceding two kinds of ring.

Any totally multiplicative ring \(A\) containing an incomplete atomic system \(s\) is uniquely representable as a direct sum \(s' \vee s''\) where \(s'\) and \(s''\) are totally multiplicative rings belonging to classes (1) and (2) respectively.

The proof consists in establishing the last statement of the theorem. By Th. 19 the classes \(s'\) and \(s''\) in \(A\) are ideals; by Ths. 26 and 27 both are normal in \(A\); and by Th. 23 (2) both are simple in \(A\). By hypothesis \(s'\neq a, s''\neq a\). It is clear that \(s'\) contains no atomic element; and it is easily shown that the orthocomplement of \(s\) relative to \(s''\vee s\) is \(s'\vee s'\) and hence that \(s\) is a complete atomic system in \(s''\). Since \(s'\) and \(s''\) are simple ideals in \(A\), we can represent \(A\) as the desired direct sum \(A=s''\vee s''\) in accordance with Th. 51 and Th. 27.

We conclude by recalling a known algebraic criterion for the existence of a complete atomic system in a totally additive Boolean ring.\(^{11}\)

**Theorem 2.12.** In a totally additive Boolean ring \(A\) the following properties are equivalent:

1. \(A\) contains a complete atomic system \(s\);
2. if \(\mathfrak{V}\) is any non-void family of two-element subclasses \(a\) of \(A\) and if \(\mathfrak{B}\) is the family of all those subclasses \(b\) of \(A\) which are contained in \(\sum a\) and have exactly one element in common with each \(a\) in \(\mathfrak{V}\), then the distributive law is valid:

\[
P \sum a = \sum_b P a.
\]

In order that a Boolean ring \(A\) be isomorphic to the Boolean ring of all subclasses of a fixed class \(E\), it is necessary and sufficient that it be totally additive and have the equivalent properties (1) and (2). In such a Boolean ring, the following general form of the distributive law is valid: if \(\mathfrak{V}\) is any non-void family of non-void subclasses \(a\) of \(A\) and if \(\mathfrak{B}\) is the family of all those subclasses \(b\) of \(A\) which are contained in \(\sum a\) and which have exactly one element in common with each \(a\) in \(\mathfrak{V}\), then

\[
P \sum a = \sum_b P a.
\]

We here use the symbols \(\sum\) and \(P\) to indicate product and sum as defined in Def. 2.1 and 2.2. First let us show that (2) implies (1). Let each class consist of an element \(a\) and the corresponding element \(a'\), and define the family \(\mathfrak{V}\) by permitting \(a\) to run over the entire ring \(A\). Of course each class occurs twice in \(\mathfrak{V}\).

\(^{11}\) A. Tarski, Fund. Math. 24 (1935), pp. 177-198 especially p. 186. The result was found in collaboration with A. Lindenbaum.
We then have \( e = \sum_{a \in A} P_a \). Hence the subclass \( s \) of \( A \) consisting of those elements \( b \) such that \( b = P_a \) for any \( a \) is certainly not void. We shall show that \( s \) is a complete atomic system. From our definition of the classes \( b \), we see immediately that, if \( c \) is an element of \( A \) and \( b \) any member of \( B \), then \( b \) contains either \( c \) or \( c' \); and that, accordingly, either \( a \in B \) or \( a \notin B \). In particular, if \( a \in A \) and \( b \in S \), then either \( cb = b \) or \( cb = 0 \). Hence \( s \) is an atomic system in \( A \) in accordance with D. Def. 3. Now it is evident that \( e = \sum_{b \in B} b \). By Th. 3.1 we have \( s'' = e \), \( s' = o \). It follows that the atomic system \( s \) is complete in accordance with D. Def. 5.

By R. Th. 62, a Boolean ring \( A \) which is totally additive and satisfies condition (1) is isomorphic to the Boolean ring of all sub-classes of a fixed class \( E \); and conversely, in such a ring the final statement of the theorem is easily proved: an element of \( E \) belongs to \( \sum_{a \in A} P_a \) if and only if it belongs to some \( a \) in each class \( a \); an element of \( E \) belongs to \( \sum_{b \in B} P_a \) if and only if it belongs to every \( a \) in some class \( b \); and thus the definition of the classes \( b \) in terms of the classes \( a \) implies the desired equality. It follows that in a totally additive Boolean ring (1) implies (2) as a special case of the equality just proved.

§ 3. Barrier Ideals and Associated Types of Boolean Ring. In the present section we shall introduce a new class of ideals — the barrier ideals — and two associated special types of Boolean ring. Our choice of terminology is dictated by topological reasons which will be developed in the following sections. The two definitions fundamental for the purposes of the present section are:

**Definition 3.1.** An ideal \( a \) in a Boolean ring \( A \) is said to be a barrier ideal if \( a = c \), \( a' = 0 \), and there exist normal ideals \( b \) and \( c \) such that \( a = b \lor c \), \( bc = 0 \). The class of all barrier ideals in \( A \) is denoted by \( B \), the class of all other ideals in \( A \) by \( C \).

**Definition 3.2.** A Boolean ring \( A \) is said to be of type (a) if it is isomorphic to a prime ideal \( p \) in a Boolean ring with unit, such that \( p \) non \( e \).
with Th. 28. It is evident that
\[ b \lor c = (b_1 \lor b_2)(b_2 \lor b_2) = a_1 a_2 = a, \quad bc = b. \]

Hence it is sufficient for us to prove that \( b \) and \( c \) are normal. We recall that the orthocomplement of any ideal product \( c_1 c_2 \) relative to \( c_2 \) may be calculated either as \( (c_1 c_2) c_1 \) or as \( c_1 c_2 \) so that \((c_1 c_2) c_1 = c_1 c_2\) in particular, \( c_1 c_2 = 0 \) implies \( c_1 = c_2 \). Applying these remarks we obtain the relations
\[
\begin{align*}
  b_1 b_2 &= (b_1 b_2) b_1 = (b_1 b_2) b_2 = (b_1 b_2) b_1, \\
  b_2 b_2 &= (b_2 b_2) b_2 = (b_2 b_2) b_2, \\
  b_1 b_2 &= (b_2 b_2) b_1 = (b_2 b_2) b_2, \\
  (b_1 b_2)(b_1 \lor b_2) &= b_1 b_2, \\
  (b_1 b_2)(b_1 \lor b_2) &= b_1 b_2, \\
  (b_1 b_2)(b_1 \lor b_2) &= b_1 b_2, \\
  (b_1 b_2)(b_1 \lor b_2) &= b_1 b_2.
\end{align*}
\]

Combining the four sets of equations on the left, we have
\[
b = (b_1 b_2 \lor b_1 b_2) \lor (b_1 b_2 \lor b_1 b_2) = (b_1 b_2) \lor (b_1 b_2) \lor (b_1 b_2) = \!(c_1 \lor a_2) = \!c_1;
\]
and combining the four sets of equations on the right we have, similarly, \( c = b \). Thus \( b = c = b' \), as we wished to prove.

If the theorem holds for \( N = 2, 3, \ldots, M \), it holds also for \( N = M + 1 \). In fact, the ideal \( a_1 \ldots a_M \) is a barrier ideal; and the relation \( a_1 \ldots a_M \lor a_{M+1} = (a_1 \lor a_{M+1}) \ldots (a_M \lor a_{M+1}) = \varepsilon \) is valid. Hence the result of the preceding paragraph shows that \( a_1 \ldots a_{M+1} \) is a barrier ideal. The theorem is therefore established for \( N = 2, 3, \ldots \).

It is also possible to establish a converse of the preceding result. We have:

**Theorem 3.** If \( a_1, \ldots, a_N \) are ideals such that \( a_n = \varepsilon \) for \( n = 1, \ldots, N \) and if \( a_1 \ldots a_N \) is a barrier ideal, then the condition

(1) for \( m = n \) there exists a simple ideal \( b_{mn} \) satisfying the relations
\[
a_m \lor b_{mn} = \varepsilon, \quad a_0 \lor b_{mn} \implies b_{mn} \lor a_n = a_n,
\]

implies that \( a_1, \ldots, a_N \) are barrier ideals.

The condition (1) is equivalent to the condition

(1') there exist simple ideals \( b_n, \quad n = 1, \ldots, N \), such that
\[
a_n \lor b_n = \varepsilon, \quad b_0 \lor \cdots \lor b_N = \varepsilon, \quad b_{mn} = 0 \quad \text{for} \quad m = n, \quad m, n = 1, \ldots, N.
\]

The conditions (1) and (1') are satisfied if \( a_1, \ldots, a_N \) are distinct prime ideals; and also if \( A \) is a ring with unity and the ideals \( a_1, \ldots, a_N \) have the property that \( a_m \lor a_n = \varepsilon \) for \( m = n, \quad m, n = 1, \ldots, N \).

First let us prove the equivalence of the conditions (1) and (1'). If (1') holds, we may put \( b_{mn} = b_m \) since \( a_m \lor b_m = \varepsilon \) and \( b_m = (a_m \lor b_0) b_m = a_m b_m \lor a_m \) for \( m = n, \quad m, n = 1, \ldots, N \); and we thus obtain (1). On the other hand, if (1) holds, we obtain (1') by induction. For \( N = 2 \) we obtain the desired result by setting \( b_1 = a_1, \quad b_2 = a_2 \); we obviously have
\[
a_1 \lor b_1 = a_1 \lor b_2 = \varepsilon, \quad a_2 \lor b_2 \lor b_2 = \varepsilon, \quad b_1 \lor b_2 = \varepsilon, \quad b_2 \lor b_2 = \varepsilon.
\]

If we have established the desired result for \( N = 2, \ldots, M \), we treat the case \( N = M + 1 \) as follows. By hypothesis there exist simple ideals \( c_1, \ldots, c_M \) such that \( a_0 \lor c_0 = \varepsilon, \quad c_1 \lor \cdots \lor c_M = \varepsilon \), \( c_n = \varepsilon \) for \( m = n, \quad m, n = 1, \ldots, M \). We introduce the ideals \( b = b_1, b_{M+1} \lor \cdots \lor b_M, b_{M+1} \lor b_M, b_{M+1} \lor b_n, n = 1, \ldots, M, \quad b_{M+1} = b' \). By Th. 1.2, the ideals \( b, b', b_n, n = 1, \ldots, M \) are all simple. It is evident that
\[
b \lor \cdots \lor b_{M+1} = (c_1 \lor \cdots \lor c_M) b \lor b' \lor b' = \varepsilon
\]
and that \( b \lor b_n = \varepsilon \) for \( m = n, \quad m, n = 1, \ldots, M \). Since \( b_n b_{M+1} \subsetneq b \) for \( n = 1, \ldots, M \), we have \( a_n \lor b_n = (a_n \lor c_n)(a_n \lor b) = a_n \lor b_n \lor b_{M+1} = \varepsilon, \quad a_n \lor b_n \lor b_{M+1} \lor b_M = \varepsilon \), \( a_n \lor b_n \lor b_{M+1} \lor b_M \lor b = \varepsilon \). Hence (1') follows from (1) in the case \( N = M + 1 \). The proof of the equivalence of (1) and (1') is now complete.

If the condition (1), or the equivalent condition (1'), is satisfied and \( a_1, \ldots, a_N \) is a barrier ideal, we can write \( a_1 \ldots a_N = b \lor c \), \( b = \varepsilon \), \( c \in R \), \( c \in R \) in accordance with Def. 3.1. Using condition (1') we put \( b = b \lor c \); \( c = c_n \) for \( n = 1, \ldots, N \). By Th. 1.2 it is clear that \( b \) and \( c \) are normal ideals. It is obvious that
\[
b_n \lor c_n = b \lor c \lor c_n = b \lor c \lor c_n = (a_n \lor b_n)(a_n \lor b) = a_n \lor b_n \lor a_n = a_n \lor b_n \lor a_n = \varepsilon, \quad a_n \lor b_n \lor a_n = \varepsilon
\]
and hence the relations \( a_n \lor b_n = (a_n \lor b_n) (a_n \lor b) = a_n \lor b_n \lor b_n = a_n \lor b_n \lor b_n = a_n \lor b_n \lor b_n = \varepsilon, \quad a_n \lor b_n \lor b_n = a_n \lor b_n \lor b_n = \varepsilon \). It is evident that \( a_n \lor b_n \lor b_n = a_n \lor b_n \lor b_n = \varepsilon \). Accordingly, \( a_n \) is a barrier ideal if \( a_n = \varepsilon, \quad n = 1, \ldots, N \).

If \( a_1, \ldots, a_N \) are distinct prime ideals, then for \( m = n \) there exists an element \( a_m a_n \) belonging to \( a_n \) but not to \( a_m \). If we set \( b_{mn} = a(a_m a_n) \), we have \( a_m \lor b_{mn} = \varepsilon, \quad a_0 \lor b_{mn} \) for \( m = n, \quad m, n = 1, \ldots, N \). Thus condition (1) is satisfied in this case.

If \( A \) has a unit \( e \) and the ideals \( a_1, \ldots, a_N \) satisfy the relations \( a_m \lor a_n = e \) for \( m = n, \quad m, n = 1, \ldots, N \), we know that for \( m = n \) there exist elements \( a_m a_n \) such that \( a_m a_n = e \), \( a_n e = a_n \), \( a_n e = a_n \). If \( a_n \lor b_{mn} \), we have \( a_m \lor a_n \lor b_{mn} \lor a_n = a(e) = e, \quad a_0 \lor b_{mn} \). Hence condition (1) is satisfied in this case also.
Ths. 3.1-3.5 give us information about the existence and construction of barrier ideals. While it will not be applied in detail, it provides a background for the further discussion.

We shall now characterize the Boolean rings of type (\(\omega\)). We have:

**Theorem 3.6.** The following properties of a Boolean ring \(A\) are equivalent:

1. \(A\) is a ring without unit in which every simple ideal is semiprincipal;
2. \(A\) is a ring of type (\(\omega\)).

A ring of type (\(\omega\)) is totally multiplicative if and only if it is of type (\(\beta_1\)).

Starting with the remark that a ring of type (\(\omega\)) cannot have a unit, we may proceed under either of the conditions (1) and (2) to adjoin a unit to \(A\) as in R Th. 1; in this way we obtain a ring \(A^*\) which is uniquely determined except for isomorphisms and which contains \(A\) as a prime ideal \(p\) in accordance with R Th. 38. If (1) holds, then R Th. 39 shows that \(p\) is not normal; and if (2) holds then by definition \(p\) is not normal. If (1) holds, then \(p\) cannot be a barrier ideal. If it were we could write \(p = b \lor b'\), \(b \in R\), in accordance with Th. 3.1. Thus \(b\) would be simple relative to \(p\) and by (1) would thus be semiprincipal relative to \(p\). Accordingly, one of the ideals \(b\) and \(b'\) would be principal in \(p\) and hence also in \(A^*\). The fact that \(b = b'' = b''\lor b'\) and the fact that \(A^*\) has a unit would thus imply that both \(b\) and \(b'\) were principal ideals. In this way we would reach the contradiction that \(b \lor b' = e\). On the other hand, if \(p\) is not a barrier ideal, then (1) holds. If \(b \lor p\) is any ideal simple relative to \(p\), then \(b\) is an ideal in \(A^*\) and

\[ b'' \lor b' \lor (b'' \lor b')p = b'' \lor b' \lor bp = b \lor bp = p \]

by virtue of R Th. 22. Since \(p\) is not a barrier ideal, Th. 3.1 shows that \(b'' \lor b' = p\). The fact that \(p\) is prime therefore implies \(b'' \lor b' = e\). Accordingly the ideals \(b''\) and \(b'\) are simple, and hence principal, in the ring \(A^*\) with unit. By R Th. 41, we have either \(b'' \lor p\) or \(b \lor p\). In the first alternative, \(b'' = b''p = b'(b \lor bp) = b \lor b'bp = b\) so that \(b\) is principal both in \(A^*\) and in \(p\). In the second alternative, \(b'' = b'bp\) is principal both in \(A^*\) and in \(p\). Hence we see that \(b\) is semiprincipal relative to \(p\). The equivalence of (1) and (2) is thus established.

By combining the preceding results with Ths. 2.3 (2) and 2.6, we obtain the final assertion of the present theorem.

A characterization of rings of type (\(\omega, \omega\)) is given by the following result:

**Theorem 3.7.** The following properties of a Boolean ring \(A\) are equivalent:

1. \(A\) is isomorphic to the product of prime ideals \(p_1\) and \(p_2\) in a ring with unit neither of which is a normal ideal or a barrier ideal;
2. \(A\) is of type (\(\omega, \omega\)).

A ring satisfying (1) or (2) has no unit. A ring of type (\(\omega, \omega\)) is totally multiplicative if and only if it is of type (\(\beta_1, \beta_2\)).

First we show that (1) implies (2). Identifying \(A\) with the product \(p_1p_2\) in the given ring \(A^*\) with unit, we select an element \(a\) of \(A^*\) belonging to \(p_1\) but not to \(p_2\) and note that by R Th. 41 the element \(a'\) belongs to \(p_1\) but not to \(p_2\). We can then represent \(A^*\) as the direct sum \(a(a)\lor a(a')\), and \(A\) as the direct sum

\[ p_1p_2a(a(a)\lor a(a')) = p_1(a(a)\lor p_2a(a'), \]

in accordance with R Th. 51 and Th. 1.3. We wish to prove that the rings \(p_1a(a)\) and \(p_2a(a')\) are of type (\(\omega\)). Evidently it is enough to treat the first. Here we have to show that \(p_1a(a)\) is an ideal which is prime in \(a(a)\) but is neither a normal ideal nor a barrier ideal relative to \(a(a)\). Th. 1.7 shows that \(p_1a(a)\) is prime and non-normal relative to \(a(a)\). If \(p_1a(a)\) were a barrier ideal in \(a(a)\), we could write \(p_1 = b \lor c\), \(bc = a\), where \(b\) and \(c\) are normal relative to \(a(a)\). We shall then have \(p_1 = p_1(a(a)\lor a(a')) = p_1(a(a)\lor a(a') = b \lor c\lor a(a'))\) where \(b, c\) and \(c\lor a(a')\) are normal absolute to \(A^*\), by virtue of Ths. 1.4 and 1.1. Since \(b(c \lor a(a')) = a\), the ideal \(p_1\) would be a barrier ideal in \(A^*\), contrary to hypothesis. The proof that (1) implies (2) is therefore complete.

On the other hand (2) implies (1). If \(A = A_1 \lor A_2\) where \(A_1\) and \(A_2\) are of type (\(\omega\)), we adjoin units to \(A_1\) and \(A_2\) obtaining rings \(A_1\) and \(A_2\). We then introduce the ring with unit \(A^* = A_1 \lor A_2\) and the subrings \(p_1 = A_1 \lor A_2\), \(p_2 = A_1 \lor A_2\). By Th. 2.7 both \(p_1\) and \(p_2\) are non-normal prime ideals in \(A^*\). Their product is obviously the given ring \(A\). We thus have to prove that neither \(p_1\) nor \(p_2\) is a barrier ideal in \(A^*\).
It is enough to treat $p_1$. If $p_1$ were a barrier ideal, we could write $p_1 = b \lor c$, $b \in A$, $c \in A$. Considering $A'$ as a principal ideal of $A'$, the ideals $b(a)$ and $c(a)$ would be normal relative to $a(a)$ by Th. 1.3. Moreover they would satisfy the relations $p_1 a(a) = b(a) \lor c(a)$, $b(a): c(a) = 0$. Since $p_1 a(a)$ coincides with $A'_1$, it is a non-normal prime ideal in $A'$. We thus conclude that $p_1 a(a)$ would be a barrier ideal relative to $a(a)$, against our assumption concerning $A_1$ as an ideal in $A'$, hence $p_1$ cannot be a barrier ideal in $A'$. With this we complete our proof that (2) implies (1).

It is evident from Ths. 2.7 and 3.6 that no ring of type $(\omega, \omega)$ has a unit; and, from Ths. 2.7 and 3.6, it is clear that a ring of type $(\omega, \omega)$ is totally multiplicative if and only if it is of type $(\beta_1, \beta_1)$. According to Th. 2.10 the rings of type $(\beta_1, \beta_1)$ appear as a special case under type $(\beta_2)$.

The types $(\omega)$ and $(\omega, \omega)$ are by no means so sharply limited as the various types discussed in § 2, as we see by virtue of the following result:

**Theorem 3.8.** If $A_1$ is a Boolean ring of type $(\omega)$ or $(\omega, \omega)$, and if $A_2$ is an arbitrary Boolean ring with unit, then the ring $A = A_1 \lor A_2$ is of the same type as $A_1$.

First, let $A_1$ be of type $(\omega)$. By Th. 2.7, the ring $A = A_1 \lor A_2$ is a non-normal prime ideal in the direct sum $A_1 \lor A_2$. If it were a barrier ideal, then the arguments used in the second paragraph of the proof of Th. 3.7 could be applied, since $A_2$ has a unit, to show that $A_1$ is a barrier ideal in $A_1'$; but by hypothesis $A_1$ is of type $(\omega)$ and therefore cannot be a barrier ideal in $A_1'$. Hence $A$ is of type $(\omega)$. If $A_1$ is of type $(\omega, \omega)$, then $A = A_1 \lor A_2$ where $A_1$ and $A_2$ are of type $(\omega)$. If we write $A = A_1 \lor A_2 = A_1 \lor (A_2 \lor A_2)$ and apply the result just proved, we find that $A$ is also of type $(\omega, \omega)$.

Later, in § 8, we give a further characterization of rings of type $(\omega, \omega)$. For the present we content ourselves with the following remark:

**Theorem 3.9.** The types $(\omega)$ and $(\omega, \omega)$ are distinct.

By definition a ring $A$ of type $(\omega, \omega)$ is the direct sum $A_1 \lor A_2$ of rings of type $(\omega)$. Since neither $A_1$ nor $A_2$ has a unit, each is a non-essential principal simple ideal in $A$ by virtue of Th. 2.7. By Th. 3.6 the ring $A$ cannot be of type $(\omega)$.

### § 4. Topological Aspects of the Two Preceding Sections.

In the present section, we shall add some topological comments upon the general concepts introduced in §§ 2, 3. We shall presuppose an acquaintance with our paper A establishing the fundamental connections between Boolean rings and general topology.

We first indicate the topological origin of the totally additive rings.

**Theorem 4.1.** If $A_\infty$ is the complete basic ring of a $T_1$-space $\mathcal{R}$ and $a_{\infty}$ is the ideal of nowhere dense sets in $A_\infty$, then the quotientring $A(\mathcal{R}) = A_\infty/a_{\infty}$ is totally additive. Conversely, if $A$ is a totally additive ring, then there exists at least one $T_1$-space $\mathcal{R}$ such that $A(\mathcal{R})$ is isomorphic to $A$.

The complete basic ring $A_\infty$ is the ring generated by $a_{\infty}$ and the open sets in $\mathcal{R}$. It obviously contains $a_{\infty}$ as an ideal. If $a$ is any non-void subclass of $A_\infty$, we define a corresponding open set $a_\infty$ as $\sum a^{-'}$. Since $a = a^{-'} (\mod a_{\infty})$, the relation $a a^{-'} = a^{-'}$ implies $a_\infty = a (\mod a_{\infty})$ for every $a$ in $a$. If $b$ is an element of $A_\infty$ such that $b a = a (\mod a_{\infty})$ for every $a$ in $a$, we have $b a = b a_\infty = 0 (\mod a_{\infty})$; hence $b = 0$. If $b$ and $a$ are open sets the relation $b = a^{-'} = 0 (\mod a_{\infty})$ implies $b = 0$. Hence $b = a_{\infty}$; and, since $b^- = b (\mod a_{\infty})$, we have $b = 0 (\mod a_{\infty})$ or, equivalently, $b = a_{\infty} = a_\infty = 0 (\mod a_{\infty})$. Now if $a(\mathcal{R})$ is a non-void subclass of $A(\mathcal{R})$, its antecedent $a$ in $A_\infty$ under the homomorphism $A_\infty \rightarrow A(\mathcal{R})$, defines an element $a_\infty$ in the manner just described. Let $a_\infty(\mathcal{R})$ be the image of $a_\infty$ in $A(\mathcal{R})$. We shall show that $a_\infty(\mathcal{R})$ is the sum of $a(\mathcal{R})$ in the sense of Def. 3.4. If $a(\mathcal{R}) \in A(\mathcal{R})$, its antecedent $a_\infty$ in $A_\infty$ satisfies the relation $a_\infty a = a (\mod a_{\infty})$, and we see therefore that $a_\infty a(\mathcal{R}) = a(\mathcal{R})$ or, equivalently, $a_\infty(\mathcal{R}) = a(\mathcal{R})$ for every $a(\mathcal{R})$ in $a(\mathcal{R})$. If $b(\mathcal{R})$ is an element of $A(\mathcal{R})$ such that $b(\mathcal{R}) > a(\mathcal{R})$ or, equivalently, $b(\mathcal{R}) a(\mathcal{R}) = a(\mathcal{R})$ for every $a(\mathcal{R})$ in $a(\mathcal{R})$, then its antecedent $b$ in $A_\infty$ satisfies the relation $b a = a (\mod a_{\infty})$ for every $a$. By the preceding results $b a = a_{\infty} (\mod a_{\infty})$. Hence we find that $b(\mathcal{R}) a(\mathcal{R}) = a(\mathcal{R})$ or, equivalently, $b(\mathcal{R}) a(\mathcal{R}) = a(\mathcal{R})$. Thus $a_\infty(\mathcal{R})$ is the sum of $a(\mathcal{R})$; and $A(\mathcal{R})$ is a totally additive ring.

13) For a detailed discussion of the complete basic ring, see A, Ch. II, Ths. 24 and 25.
We now let $A$ be an arbitrary totally additive Boolean ring, and take $R$ as its representative bicomplete Boolean space. Then $A$ is isomorphic to the Boolean ring of all closed-and-open sets in $R$; and the ideals in $A$ are represented by the open sets in $R$. The normal ideals in $A$ are precisely those represented by regular open sets in $R$. Now Th. 2.2 (2) shows that all the normal ideals in $A$ are principal. Hence the space $R$ has the property that its regular open sets are precisely the closed-and-open sets. Thus we see that, in $A_R$, the regular open sets constitute a subring isomorphic to $A$. On the other hand, we know that each residual class (mod $e_A$) in $A_R$ contains exactly one regular open set. It follows immediately that $A(R) = A_R/e_A$ is isomorphic to the ring of regular open sets in $R$ and hence to the given ring $A$.

Th. 4.1 provides a method of construction capable of yielding all totally additive Boolean rings. It also raises the interesting problem of determining all the different $T_v$-spaces such that the corresponding rings $A(R)$ are isomorphic to a given totally additive ring $A$. Our paper 4 provides methods which should suffice for a deep investigation, if not actually for a complete solution, of this problem.

In connection with his theory of continuous geometries, v. Neumann has had occasion to point out that the ring $A(R)$ is totally additive, although he was not in a position to prove that all totally additive rings could be obtained as rings $A(R)$. He has observed also that a similar construction — namely, reduction of the ring of all measurable sets on the unit interval modulo the ideal of null sets — yields a totally additive Boolean ring. When this construction is generalized in the obvious way, it provides a means of obtaining a variety of totally additive Boolean rings. That it cannot suffice for the treatment of the entire category of totally additive rings was also pointed out by v. Neumann. In fact, it is easily seen that when $R$ is the unit interval the ring $A(R)$ cannot support the kind of numerical measure which would have to be defined in it if it were obtainable by the construction under consideration. Thus an interesting problem arises in this connection: to determine which totally additive Boolean rings can be constructed in terms of the general theory of measure as indicated above.

We turn now to a topological study of the concept of barrier ideals introduced in § 3. Our results throw new light on the nature of totally additive Boolean rings. We begin with some general theorems of topology which are parallel to those given in the first part of § 3 and do, in fact, contain the latter as special cases.

**Definition 4.1.** A set $\mathcal{F}$ in a $T_v$-space $R$ is said to be a barrier if it is the non-void common boundary of two disjoint open sets $\mathcal{G}_1$ and $\mathcal{G}_2$.

**Theorem 4.2.** If $\mathcal{F}$ is a barrier in a $T_v$-space $R$ and $\mathcal{G}_1$, $\mathcal{G}_2$, are associated open sets, then $\mathcal{F}$ is closed, $\mathcal{G}_1$ and $\mathcal{G}_2$ are non-void regular open sets, and $\mathcal{G}_1 \cup \mathcal{G}_2 \supseteq \mathcal{F}$. Conversely, if $\mathcal{G}_1$ and $\mathcal{G}_2$ are disjoint non-void regular open sets such that $(\mathcal{G}_1 \cup \mathcal{G}_2)^- = R$; they have a common boundary $\mathcal{F}$ which is a barrier in $R$ whenever it is non-void.

If $\mathcal{F}$ is a barrier and $\mathcal{G}_1$, $\mathcal{G}_2$ are associated open sets we have $\mathcal{F} = \mathcal{G}_1 \cap \mathcal{G}_2 = \mathcal{G}_1 \cap \mathcal{G}_2 = 0$, and hence $\mathcal{G}_1 = 0$, $\mathcal{G}_2 = 0$. Since $\mathcal{G}_2$ and $\mathcal{G}_1$ are closed, $\mathcal{F}$ is also closed. The relation $\mathcal{G}_1 \cap \mathcal{G}_2$ implies $\mathcal{G}_1 \cap \mathcal{G}_2 = 0$. Hence to prove the equality $\mathcal{G}_1 = \mathcal{G}_2$, which identifies $\mathcal{G}_1$ as a regular open set, it is sufficient to prove the inclusion relation $\mathcal{G}_1 \supseteq \mathcal{G}_2$. Since $\mathcal{G}_2 \mathcal{G}_3 = \mathcal{G}_2 \mathcal{F} = 0$, we have

\[
\begin{align*}
\mathcal{G}_1 &= \mathcal{G}_1 \mathcal{G}_1 \mathcal{G}_1 = \mathcal{G}_1 \mathcal{G}_2 = (\mathcal{G}_1 \cup \mathcal{G}_2) \mathcal{G}_2, \\
\mathcal{G}_2 &= \mathcal{G}_2 \mathcal{G}_2 \mathcal{G}_2 = \mathcal{G}_2 \mathcal{G}_2 = (\mathcal{G}_1 \cup \mathcal{G}_2) \mathcal{G}_2, \\
\mathcal{G}_1' &= (\mathcal{G}_1 \mathcal{G}_2)^- \supseteq \mathcal{G}_2, \\
\mathcal{G}_2' &= (\mathcal{G}_1 \mathcal{G}_2)^- \supseteq \mathcal{G}_2, \\
\mathcal{G}_1' &\subseteq (\mathcal{G}_1 \mathcal{G}_2)^- \mathcal{G}_2 = \mathcal{G}_2 \mathcal{G}_2 \mathcal{G}_2 \subseteq \mathcal{G}_2.
\end{align*}
\]

Hence $\mathcal{G}_1$ is a regular open set. The same argument applies to show that $\mathcal{G}_2$ is a regular open set. If it is evident that $\mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G} = (\mathcal{G}_1 \cup \mathcal{G}_2) \cup (\mathcal{G}_2 \cup \mathcal{G})$, then $\mathcal{G}_1 \cup (\mathcal{G}_2 \cup \mathcal{G}) = (\mathcal{G}_1 \cup \mathcal{G_2})^-$ is a closed set.

If $\mathcal{G}$ is the open set $(\mathcal{G}_1 \cup \mathcal{G}_2)^-$, we shall show that the open sets $\mathcal{G}_1$ and $\mathcal{G}_2 = \mathcal{G}_2 \cup \mathcal{G}$ are an associated pair for $\mathcal{G}$. It is evident that $\mathcal{G}_1 \mathcal{G}_2 = \mathcal{G}_1 \mathcal{G}_2 \mathcal{G}_2 = \mathcal{G}_2 \mathcal{G}_2 \mathcal{G}_2 = 0$. Now

\[
\begin{align*}
\mathcal{G}^- &= (\mathcal{G}_1 \cup \mathcal{G}_2)^- \mathcal{G}_2 = \mathcal{G}_2 \mathcal{G}_2 \mathcal{G}_2 = \mathcal{G}_2 \mathcal{G}_2 = \mathcal{G}_2.
\end{align*}
\]
Hence
\[ G_1 = G_1^c \cup G_2 = \overline{\mathcal{F}} \cup G_1 \cup G_2 = \overline{\mathcal{F}} \cup \mathcal{G}_1, \]
\[ G_2 = G_2^c = \overline{\mathcal{F}} \cup G_2^c = \overline{\mathcal{F}}^c = (G_1 \cup G_2^c)^c = \mathcal{F}. \]
Thus the open sets \( G_1, G_2 \) are disjoint and have \( \mathcal{F} \) as their common boundary. Moreover
\[(G_1 \cup G_2)^c = \overline{G_1} \cup \overline{G_2} = G_1 \cup G_2 \cup \mathcal{F} = G_1 \cup G_2 \cup \mathcal{F} \cup \mathcal{F} = \mathcal{R}.\]
We have thereby proved that \( \mathcal{F} \) is a barrier the associated open sets may be chosen as stated in the theorem.

We now prove the converse part of the theorem. If we start with open sets \( G_1 \) and \( G_2 \) having the indicated properties, we first prove that \( \overline{G_1} \supseteq G_1, \overline{G_2} \supseteq G_2 \). By symmetry, it is enough to establish the first relation. Since \( \overline{G_1} \cup \overline{G_2} = (G_1 \cup \overline{G_2})^c = \mathcal{R} \), we have \( \overline{G_1} \supseteq G_2^c \) and hence \( \overline{G_1} \supseteq G_2^c \). Since \( G_2 \) is a regular open set we have \( G_2^c = \overline{G_2} = G_2 \). Hence the relation \( \overline{G_1} \supseteq G_2 \) is valid. Since \( G_1 \subset \mathcal{G}_1 \), we have \( G_1 \cup \mathcal{G}_1 = (G_1 \cup G_1^c) = G_1 \), and similarly \( G_2 \cup G_2^c = G_2 \). Hence we see that the set \( \overline{G_1} \cup \overline{G_2} = \overline{G_1} \) is closed and contains \( \mathcal{F} \). Thus we have \( \overline{G_1} \subset \mathcal{G}_1 \cup \mathcal{G}_2 \). On the other hand, we also have \( \overline{G_1} \cup \overline{G_2} \subset \overline{G_1}, \overline{G_2} = \overline{G_1} \) by the earlier results. We therefore find that \( \overline{G_1} = \overline{G_1} \cup \mathcal{G}_1 \); and, in similar fashion, that \( \overline{G_2} = \overline{G_2} \cup \mathcal{G}_2 \). Hence we conclude that \( G_1 \cup \mathcal{G}_1 = G_2 \cup \mathcal{G}_2 \). Thus the closed set \( \overline{\mathcal{F}} = \overline{\mathcal{G}_1} \) is the common boundary of \( G_1 \) and \( \mathcal{F} \). Obviously \( \mathcal{F} \) is 0 if and only if \( G_1 \cup \mathcal{G}_1 = \mathcal{R} \); that is, if and only if \( G_1 \) and \( G_2 \) are closed as well as open. When \( \mathcal{F} \) is not void, it is a barrier by Def. 4.1.

**Theorem 4.4.** If \( \mathcal{F} \) is a barrier in the \( T_0 \)-space \( \mathcal{R} \) and \( \mathcal{G} \) is a closed-and-open set such that \( \mathcal{F} \mathcal{G} \neq \emptyset \), then \( \mathcal{F} \mathcal{G} \) is a barrier in \( \mathcal{R} \).

Let \( G_1 \) and \( G_2 \) be open sets associated with \( \mathcal{F} \) and introduce the open sets \( G_3 = G_1 \cup G_2 \). Since \( (G_1 G_2)^c \subset G_1 \cup G_2 \) and \( (G_1 G_2^c)^c \subset G_2 \), we have
\[(G_1 G_2)^c = 0, \quad (G_1 G_2^c)^c = 0.\]
\[G_1 = (G_1 G_2 G_2^c)^c = (G_1 G_2)^c \cup (G_2 G_2^c)^c = (G_1 G_2^c)^c.\]
Hence \( \overline{G_1} = (G_1 G_2^c)^c = G_1 \cup G_2 \cup \mathcal{G}_1 \cup \mathcal{G}_2 = \mathcal{R}. \)

In the same way \( G_2 \) is \( \mathcal{R} \). Thus, as the common boundary of \( G_1 \) and \( G_2 \), the set \( \mathcal{F} \mathcal{G} \) is a barrier when it is non-void.

**Theorem 4.5.** If \( \mathcal{F}_1, \ldots, \mathcal{F}_N \) are non-void sets in a \( T_0 \)-space \( \mathcal{R} \) and if \( \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_N \) is a barrier, then the condition
\[(1) \exists \text{a closed-and-open set } \mathcal{D}_{nn}, \text{such that } \mathcal{F}_m \subset \mathcal{D}_{nn}, \mathcal{D}_{nn} \mathcal{F}_m = 0 \quad \text{for } m = n, m, n = 1, \ldots, N \]
implies that \( \mathcal{F}_1, \ldots, \mathcal{F}_N \) are barriers.

The set \( \mathcal{D}_m = \mathcal{D}_{n_1} \mathcal{D}_{n_1,m} \cdots \mathcal{D}_{n_m,m}, m = 1, \ldots, N \) is a closed-and-open set such that \( \mathcal{F}_m \subset \mathcal{D}_m, \mathcal{F}_m \mathcal{D}_m = 0 \) for \( m = n, m, n = 1, \ldots, N \). Hence we have \( \mathcal{F}_m = \mathcal{F}_m \mathcal{F}_m \mathcal{F}_m = 0 \) for \( m = 1, \ldots, N \). Th. 4.3 now shows that, if \( \mathcal{F} = 0 \) and if \( \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_N \) is a barrier, then \( \mathcal{F}_m \) is also a barrier, \( m = 1, \ldots, N \).
We now indicate the connection between barriers and barrier ideals. We have:

**Theorem 4.6.** Let a be an ideal in a Boolean ring \( A \); and let \( \mathcal{B}(A) \) be a representative Boolean space for \( A \), \( \mathcal{G}(a) \) the representative open set for \( a \) in \( \mathcal{B}(A) \). Then \( a \) is a barrier ideal if and only if \( \mathcal{G}(a) \) is a barrier.

We base the proof on results of our paper \( A \) and Th. 4.2 above. If \( a \) is a barrier ideal, we have \( a = e, \ a = b \lor c, \ b \land c = a, b \in \mathfrak{R}, \ c \in \mathfrak{R} \). Thus the corresponding representative open sets \( \mathcal{G}(a), \mathcal{G}(b), \mathcal{G}(c) \) in \( \mathcal{B}(A) \) have the properties

\[
\mathcal{G}(a) = 0, \quad \mathcal{G}^{-1}(a) = 0, \quad \mathcal{G}(a) = \mathcal{G}(b) \cup \mathcal{G}(c), \quad \mathcal{G}(b) \cap \mathcal{G}(c) = 0;
\]

and \( \mathcal{G}(b) \) and \( \mathcal{G}(c) \) are regular open sets. From Th. 4.2 we see at once that \( \mathcal{G}(a) = \mathcal{G}(b) \cap \mathcal{G}(c) \) is the common boundary of \( \mathcal{G}(b) \) and \( \mathcal{G}(c) \) and is a barrier in \( \mathcal{B}(A) \). On the other hand if \( \mathcal{G}(a) \) is a barrier in \( \mathcal{B}(A) \) there exist associated open sets \( \mathcal{G}_{1}, \mathcal{G}_{2} \), which we may suppose, by virtue of Th. 4.2, to satisfy the relation \( \mathcal{G}_{1} \cup \mathcal{G}_{2} \cap \mathcal{G}(a) = \mathcal{B}(A) \). Hence there exist ideals \( b \) and \( c \) in \( A \) such that \( \mathcal{G}_{1} = \mathcal{G}(b) \land \mathcal{G}_{2} = \mathcal{G}(c) \). The relations \( \mathcal{G}_{1} \cup \mathcal{G}_{2} = \mathcal{G}(a) \) and \( \mathcal{G}_{1} \cap \mathcal{G}_{2} = 0 \) imply that \( a = b \lor c \), \( b \land c = a \). Since \( \mathcal{G}_{1} \) and \( \mathcal{G}_{2} \) are regular open sets by Th. 4.2, the ideals \( b \) and \( c \) are normal. Since \( \mathcal{G}(a) = 0 \), we have \( a = e \); and since

\[
\mathcal{G}^{-1}(a) = (\mathcal{G}_{1} \cup \mathcal{G}_{2})^{-1} = (\mathcal{G}_{1}^{-1} \cup \mathcal{G}_{2}^{-1}) = 0
\]
in accordance with Th. 4.2, we see also that \( a' = a \). Hence \( a \) is a barrier ideal. We may remark that if \( \mathcal{G} \) is any barrier in \( \mathcal{B}(A) \), then the open set \( \mathcal{G} \) is the representative of a barrier ideal in \( A \).

As a consequence of Ths. 3.2 and 4.6 we immediately have the following result, the proof of which is obvious:

**Theorem 4.7.** The following properties of a Boolean space \( \mathcal{B} \) are equivalent:

1. There exists no barrier in \( \mathcal{B} \);
2. \( \mathcal{B} \) is the representative space for a totally multiplicative Boolean ring.

It is plain that the presence of a barrier in a \( T_{0} \)-space \( \mathfrak{R} \) indicates some sort of connectedness for \( \mathfrak{R} \). Accordingly, Th. 4.7 proves that the Boolean spaces, except for those which are the representatives of totally multiplicative Boolean rings, still show traces of connectedness in spite of the fact that they are totally disconnected in the usual sense (given any two distinct points of the space, there exist disjoint open sets, one about each point, of which the entire space is the union). In consequence a certain interest attaches to the problem of determining the \( T_{0} \)-spaces which exhibit the extreme of disconnectedness implied by the absence of barriers. It is easily seen that an \( H \)-space without barriers is totally disconnected in the usual sense; and the methods of our paper \( A \) appear sufficient to characterize the semiregular \( H \)-spaces without barrier, topologically, as the dense subsets of bicomplete Boolean spaces without barrier.

The preceding results throw a certain amount of light on the facts developed in § 3. Thus Ths. 3.3, 3.4, and 3.5 follow directly from the corresponding Ths. 4.3, 4.4, and 4.5 with the help of Th. 4.6. Furthermore the behavior of prime ideals can be deduced from Th. 4.6, if we recall that an open set in \( \mathcal{B}(A) \) represents a prime ideal in \( A \) if and only if its complement consists of a single point. If the removal of this point divides \( \mathcal{B}(A) \) into two disjoint nonvoid sets open in \( \mathcal{B}(A) \) which have it as their common point of accumulation, then the corresponding prime ideal is a barrier ideal. If the removal of this point does not so divide \( \mathcal{B}(A) \), then the corresponding prime ideal is not a barrier ideal. Thus we see that the rings of type \( (\omega, \omega) \) have representative Boolean spaces which are obtained by the removal of single points of the latter kind from bicomplete Boolean spaces; and are characterized by this property. We see also that the rings of type \( (\omega, \omega) \) arise similarly by removal of two such points.

§ 5. Algebraic Characterizations of Finite Rings. In this section we shall give some equivalent algebraic characterizations of the finite Boolean rings.

We first prove a theorem about infinite rings.

**Theorem 5.1.** The following assertions concerning a Boolean ring \( A \) are equivalent:

1. \( A \) is infinite — that is, \( A \) contains a sequence \( \{a_{n}\} \) such that \( a_{m} = a_{n} \) for arbitrarily great \( m \) and \( n \);
2. \( A \) contains a sequence \( \{b_{n}\} \) such that \( b_{m} < b_{n+1}, b_{n} + b_{n+1} \) for \( n = 1, 2, 3, \ldots \);
3. \( A \) contains a sequence \( \{c_{n}\} \) such that \( c_{m}c_{n} = 0 \) for \( m \neq n, c_{n} = 0, m, n = 1, 2, 3, \ldots \).
It is evident that (1) follows from (2) or (3). It is easily seen that (2) follows from (3); we have only to define \( b_0 = c_1 \lor \ldots \lor c_n \lor \ldots \lor c_n \) to obtain the desired sequence. Now suppose that (1) implies (2) for the case of a ring with unit. We then deduce (3), and hence (2) also, for the general case in the following manner. Starting with an arbitrary infinite ring \( A \) we adjoin a unit, obtaining an infinite ring \( A^* \) which contains \( A \) as a prime ideal. This process can be carried out whether \( A \) happens to have a unit or not. Then, under our present hypothesis, \( A^* \) contains a sequence \( \{b_n\} \) with the properties given in (2). If we define \( c_m = b_{m+1} + b_m \) we have \( c_n = 0 \) and

\[
c_m = (b_{m+1} + b_m)(b_m + b_n) = b_{m+1} + b_n + b_m b_{m+1} + b_m b_n = 0
\]

for \( m+1 \leq n \) by virtue of the relations \( b_m < b_{m+1} < b_n < b_{n+1} \). Consequently the sequence \( \{c_n\} \) has the properties desired under (3); but its elements are elements of \( A^* \) rather than of \( A \). We can show however that at most one member of the sequence \( \{c_n\} \) fails to belong to \( A \). If the element \( c_m \) is not in \( A \), the relation \( c_m = 0 \) for \( m = n \) and the fact that \( A \) is a prime ideal together show that \( c_m \notin A \) for \( n = m \). Hence, by rejecting at most one member of the sequence \( \{c_n\} \), we obtain a sequence in \( A \) which has all the properties desired under (3). Thus the truth of the theorem depends upon the deduction of (2) from (1) under the assumption that \( A \) has a unit. We proceed now to this remaining step, constructing the desired sequence \( \{b_n\} \) from the given sequence \( \{a_n\} \).

The given sequence has the property (P) of containing infinitely many unequal elements. In our construction we shall make repeated use of the following principle: if \( c \) is any element of a ring with unit and \( \{d_n\} \) is a sequence with the property (P), then at least one of the two sequences \( \{cd_n\} \), \( \{c'd_n\} \) has the property (P). The proof of this principle is obvious: if \( cd_n = cd_k \) and \( c'd_n = c'd_k \) for \( n \geq N \), then \( d_n = c'd_n \lor c'd_n = cd_n \lor c'd_n = d_n \) for \( n \geq N \), contrary to hypothesis. Thus, starting with the sequence \( \{a_n\} \) we see that at least one of the sequences \( \{a_n d_n\} \), \( \{a_n a_0\} \) has the property (P). If both have the property (P), we define \( b_n = a_1 \); if one fails to have the property (P), we define \( b_n \) as equal to the associated element \( a_n \) or \( a_0 \). Suppose now that \( b_1, \ldots, b_k \) have been determined so that:

1. \( b_1, \ldots, b_k \) are unequal;
2. \( b_i < b_i < \ldots < b_k < b_k \);
3. the sequence \( \{b_n a_n\} \) has the property (P).

By \((\gamma)\) there exists a least integer \( l \) such that \( b_0, b_0, a = b_0 \). At least one of the sequences \( \{a_n b_n a_n\} \), \( \{a'_n a_n\} \) has the property (P). If both \( d_0 \), define \( c_0 = a_1 b_0 \); and if one fails to have the property (P), we define \( c_0 \) as equal to the associated element \( a_1 b_0 \) or \( a'_1 b_0 \). We then define \( b_{k+1} = b_k \lor c_n \). Since \( b_k a_1 b_0 = b_1 a_1 b_0 = 0 \), we have \( b_k c_n = 0 \). Moreover \( c_0 b_0 = 0 \) for \( a_0 b_0 = 0 \). By our determination of \( b \) and \( a_0 b_0 = 0 \) would imply \( a_0 b_0 + a_0 b_0 = 0 \), contrary to our determination of \( b \). It is obvious then that the properties \((a)\) and \((\beta)\) above can be extended to the elements \( b_1, \ldots, b_{k+1} \); they are unequal and satisfy the relations \( b_i < b_i < \ldots < b_k < b_{k+1} \). The property \((\gamma)\) can also be extended to \( b_{k+1} \); the sequence \( \{b_n a_n\} \) has the property \((P)\). If \( c_0 = a_1 b_0 \), we have \( b_{k+1} a_0 = b_k a_n a_0 \lor b_0 a_n a_0 = a_1 b_0 a_n \); and, if \( c_0 = a_1 b_0 \), we have similarly \( b_{k+1} a_0 = b_k a_n a_0 \). By virtue of our choice of \( c_0 \), we see that the sequence \( \{b_n a_n\} \) has the property \((P)\). The principle of mathematical induction therefore establishes the existence of the \( \{b_n\} \) desired under (2); and our proof is brought to a close.

With the aid of this theorem it is easy to characterize the finite Boolean rings.

**Theorem 5.2.** A Boolean ring \( A \) is finite if and only if it has one of the following four equivalent properties:

1. \( A \) is a ring with unit in which every ideal is normal;
2. every ideal in \( A \) is principal;
3. \( A \) has a finite atomic basis or consists of the element 0 alone;
4. \( A \) is isomorphic to the Boolean ring of all subclasses of a fixed finite class.

The equivalence of (1) and (2) follows at once from \( R \) Ths. 23, 24, 25. The equivalence of (3), (4), and the property of finiteness follows from \( R \) Ths. 11, 12, 13. If \( A \) is finite every ideal in \( A \) is a finite Boolean ring and hence has a unit by \( R \) Th. 1; and hence every ideal in \( A \) is principal. On the other hand (2) implies that \( A \) is finite. If \( A \) were not finite, there would exist a sequence \( \{b_n\} \) in \( A \) of the kind described in Th. 5.1 (2). The ideal generated by the class consisting of all elements of \( \{b_n\} \) would then be a principal ideal \( a(a) \), by hypothesis. Obviously, we would then have \( b_n \lor a \) for \( n > 1, 2, 3, \ldots \); and, on the other hand, \( R \) Th. 17 shows that for some integer \( k \) we must have \( a < b \) for \( n > k \). We thus reach the contradiction that \( b_n < a \lor b_k \) for \( n > k \). With this the proof of the theorem is complete.
We shall consider briefly the part played by the divisor chain condition ("Tellerkettensatz") and by the maximal and minimal conditions on ideals. The great emphasis which has been laid upon such conditions in the general theory of rings justifies an explicit formulation of the facts in the special case before us.

**Theorem 5.3.** In order that a Boolean ring $A$ be finite, each of the following four assertions is both necessary and sufficient:

1. $A$ satisfies the ascending chain condition;
2. $A$ satisfies the maximal condition;
3. $A$ satisfies the descending chain condition;
4. $A$ satisfies the minimal condition $^{12)}$.

It is obvious that a finite Boolean ring has properties (1), (2), (3), (4) since its ideals are finite in number. We can show at once that (1) and (3) separately imply the finiteness of $A$. If $A$ were infinite, then Th. 5.1 (2) would provide us with a sequence $\{b_n\}$ such that $b_n < b_{n+1}, b_n \neq b_{n+1}$. The corresponding sequences $\{a(b_n)\}$ and $\{a'(b_n)\}$ would then have the properties:

$$
a(b_n) \cap a(b_{n+1}), \quad a(b_n) = a(b_{n+1}), \quad n = 1, 2, \ldots
$$

$$
a'(b_n) \cap a'(b_{n+1}), \quad a'(b_n) = a'(b_{n+1}), \quad n = 1, 2, \ldots
$$

These properties contradict (1) and (3) respectively. Since (2) implies (1) and (4) implies (3), in obvious ways, our proof is complete.

§ 6. Algebraic Characterizations of Boolean Rings of Types $(\beta_1), (\beta_2), (\beta_3), (\beta_4)$. In the present section we shall give characterizations of the indicated types of ring, including some already obtained in earlier sections and some new ones.

We shall begin with the rings of type $(\beta_4)$, finding

**Theorem 6.1.** A Boolean ring $A$ is of type $(\beta_4)$ if and only if it has one of the following four equivalent properties:

1. $A$ is a ring without unit in which every ideal is normal;
2. $A$ is a ring without unit in which every ideal is simple;
3. $A$ has an infinite atomic basis;
4. $A$ is isomorphic to the Boolean ring of all finite subclasses of a fixed infinite class.

$^{12)}$ For definitions of the terms used, see B. L. van der Waerden, *Moderne Algebra II* (Berlin 1931), pp. 23–30, 151–152.

In a Boolean ring of type $(\beta_3)$ there exist simple ideals which are not semiprincipal.

The equivalence of (1) and (2) follows at once from Ths. 23 and 34. The equivalence of (3) and (4) is proved in Th. 11. A ring of type $(\beta_4)$ has property (3) by Th. 26. By the same theorem, a ring with property (3) is of type $(\beta_4)$. Th. 26 enables us to deduce (2) from (3). For, if $a$ is any ideal in $A$ and $a(a)$ any principal ideal, we see that $a(a)$ is a finite ring generated by the atomic elements $b_i$, finite in number, such that $ba + \emptyset$; and hence that the ideal $a(a)$, being a finite ring, has a unit and is principal. On the other hand, we can deduce (3) from (2). If $a(a)$ is any principal ideal in $A$ and if any ideal in $a(a)$, then $a$, considered as an ideal in $A$, must be simple by (2). Th. 1.1 shows that $a$ is principal in $A$ and hence in $a(a)$ also. Thus Th. 6.1 shows that $a(a)$ has a finite atomic basis. It follows that every element $a$ in $A$ is a finite sum of atomic elements in $A$. Hence $A$ has an atomic basis. If the basis were finite, $A$ would be finite by Th. 6.1 and would have a unit contrary to (2). Hence $A$ has an infinite atomic basis, as we wished to prove. In the demonstration of Th. 2.6 we have already shown that a ring of type $(\beta_4)$ contains a non-semiprincipal simple ideal.

**Theorem 6.2.** A Boolean ring $A$ is of type $(\alpha)$ if and only if $\mathcal{P} = \mathcal{P}^* = \mathcal{S} = \mathcal{R} = \mathcal{J}$.

By Th. 2.2 (2), the totally additive Boolean rings are characterized by the equation $\mathcal{P} = \mathcal{R}$. By Th. 5.2, the finite rings among these are characterized by the further equation $\mathcal{R} = \mathcal{J}$. Hence the rings of type $(\alpha)$ are characterized by the relations $\mathcal{P} = \mathcal{P}^* = \mathcal{S} = \mathcal{R}$. The relations $\mathcal{P} = \mathcal{P}^* = \mathcal{S} = \mathcal{R}$ then follow in accordance with Th. 23.

**Theorem 6.3.** A Boolean ring $A$ is of type $(\beta_1)$ if and only if $\mathcal{P} + \mathcal{P}^* = \mathcal{S} = \mathcal{R} = \mathcal{J}$.

In Th. 2.6 we proved that $A$ is of type $(\beta_1)$ if and only if $\mathcal{P} + \mathcal{P}^* = \mathcal{S} = \mathcal{R}$. We must show that these relations imply $\mathcal{R} = \mathcal{J}$. By reference to Ths. 5.2 and 6.1 we see that the relation $\mathcal{R} = \mathcal{J}$ would imply either $\mathcal{P} = \mathcal{J}$ or $\mathcal{P}^* = \mathcal{S}$; and it therefore follows that $\mathcal{R} = \mathcal{J}$ in the present case.

**Theorem 6.4.** A Boolean ring $A$ is of type $(\beta_2)$ if and only if $\mathcal{P} + \mathcal{P}^* + \mathcal{S} = \mathcal{R} = \mathcal{J}$. 
By Th. 2.3 (2) the totally multiplicative rings are characterized by the equation \( S = \mathbb{Z} \). The totally multiplicative rings are all distributed among the finite rings, and the rings of distinct types \((a), (b), (c), (d), (e)\). Since among these, the finite rings and the rings of type \((b)\) are characterized by the equation \( R = \mathbb{Z} \) in accordance with Th. 5.2 and 6.1, the rings belonging to one of the three types \((a), (b), (c)\) are characterized by the relations \( S = R + \mathbb{Z} \). According to Th. 5.2 and 6.3 the relations \( \mathbb{P}^* = S = R + \mathbb{Z} \) characterize the rings belonging to one of the types \((a), (b)\). Hence the rings of type \((b)\) are characterized by the relations \( \mathbb{P}^* = S = R + \mathbb{Z} \); and these relations imply that \( \mathbb{P} = \mathbb{R}^* \) by Th. 25. With this the proof of the present theorem is complete.

§ 7. Algebraic Characterizations of Boolean Rings of Types \((b^2), (b^3), (b^2, b), (b^2, b), (b^2, b)\). In this section we continue the studies of §§ 5, 6, 7, obtaining new characterizations of some of the indicated types, either individually or in groups.

We begin with a discussion of types \((b^2)\) and \((b^2, b)\).

**Theorem 7.1.** The Boolean rings of types \((b^2)\) and \((b^2, b)\) are collectively characterized as those rings \( A \) with \( \mathbb{Z} = \mathbb{S} \) possessing one of the following four equivalent properties:

1. There exists a prime ideal \( p \) in \( A \) such that \( p \cap a \lor a' \) whenever the ideal \( a \);
2. There exists a prime ideal \( q \) in \( A \) such that every ideal contained in \( q \) is simple relative to \( q \);
3. There exists a non-void class \( s \) of atomic elements in \( A \) such that the ideal \( a(s) \) is prime;
4. The ring \( A \) is isomorphic to the Boolean ring generated by all the finite subclasses of a fixed infinite class \( E \) and a single infinite subclass \( I \) of \( E \).

In such a ring, the relations \( p = q = a(s) \) hold, \( s \) is a complete atomic system, and the ideal \( p \) is not normal; and the ring \( A \) belongs to type \((b^2)\) or \((b^2, b)\) according as \( A \) has or has not a unit. In the representation given by (4), the ring \( A \) has a unit if and only if the class \( E + I \) is finite; and in that case is characterized by the cardinal number of \( E \). When \( A \) has no unit, it is characterized by the cardinal numbers of the two classes \( E + I \) and \( I \), both infinite. In a ring of type \((b^2)\), the relations \( \mathbb{P} = \mathbb{P}^* = S = R + \mathbb{Z} \) are valid; and in a ring of type \((b^2, b)\), the relations \( \mathbb{P} = \mathbb{P}^* = S = R + \mathbb{Z} \).

We may remark that in a ring with \( \mathbb{Z} = \mathbb{S} \), the properties (1) and (2) hold for arbitrary prime ideals \( p \) and \( q \), as we see by reference to Ths. 5.2 and 6.1. Similarly, in a finite ring or a ring of type \((b)\), we can satisfy condition (3) by defining \( s \) through the suppression of exactly one element of the atomic basis. Thus the condition \( \mathbb{S} = \mathbb{Z} \) is essential in connection with (1), (2), and (3), if we wish to eliminate types already discussed. Condition (4), on the other hand, implies \( \mathbb{Z} = \mathbb{S} \) as we shall see; hence the condition \( \mathbb{Z} = \mathbb{S} \) is superfluous as far as (4) is concerned. By Th. 2.8, we know that the rings of type \((b^2)\) or type \((b^2, b)\) are not totally multiplicative. Thus we have for them the relations \( R = \mathbb{Z} \) and hence \( \mathbb{Z} = \mathbb{S} = R + \mathbb{Z} \). Since a ring of type \((b^2)\) has a unit, the relations \( \mathbb{P} = \mathbb{P}^* = S = R + \mathbb{Z} \) must hold for it. In a ring of type \((b^2, b)\), we have \( \mathbb{P} = \mathbb{P}^* \) and \( \mathbb{Z} = \mathbb{S} = R + \mathbb{Z} \); and we shall presently that \( \mathbb{P} = \mathbb{P}^* \) also.

We first show that (1), (2), and (3) are equivalent and that the condition \( \mathbb{Z} = \mathbb{S} \) implies the uniqueness and equality of the ideals \( p, q, a(s) \). To prove that (1) implies (2), we consider an arbitrary ideal \( a \) in the ring \( p \). Since \( a \) is then an ideal in \( A \) and the relation \( a \lor a' \cap p \) is valid by (1), we have \( a \lor a' \cap p = (a' \lor a') \cap p \). Thus \( a \) is simple relative to \( p \), its orthocomplement relative to \( p \) being the ideal \( a' \). Accordingly, (1) implies (2) with \( q = p \). Next (2) implies (3) with \( a(s) = q \). For Th. 6.1 shows that \( q \) has an atomic basis \( s \); and the ideal \( a(s) \) generated by \( s \), considered as an atomic system in \( A \), obviously coincides with \( q \). Now (3) implies (1) with \( p = a(s) \). The ideal generated by a non-void class of atomic elements \( s \) is easily seen to consist of all finite sums of elements of \( s \), in accordance with R Def. 3 and R Th. 17. We know from Ths. 5.2 and 6.1 that every ideal in \( a(s) \) is simple relative to \( a(s) \). Hence, if \( a \) is any ideal in \( A \), we have \( a \lor a' \cap a(s) = a(s) \) or, equivalently, \( a \lor a' \cap a(s) \). Consequently (1) holds for \( q = a(s) \). Our discussion shows that, if any one of the ideals \( p, q, a(s) \) is not uniquely determined, then the others are not. Now the condition \( \mathbb{Z} = \mathbb{S} \) implies that the ideal \( p \) of (1), and hence also the ideals \( q \) and \( a(s) \), is uniquely determined. We note first that the ideal \( p \) must be non-normal under the present circumstances: for there exists an ideal \( a \) with \( a \lor a' \cap p \). The relation \( a \lor a' \cap p \) therefore implies \( p = a' \lor a' \cap a \). According to R Th. 58, the prime ideal \( p \) is not normal. Thus if \( p_1 \) and \( p_2 \) are two ideals with the properties required in (1), we have \( p_1 \lor p_2 \lor p \), \( p_1 \lor p \), and \( p_2 \lor p \). We conclude therefore that \( \mathbb{Z} = \mathbb{S} \) implies the uniqueness and also the equality of \( p, q, a(s) \).
In particular, we have shown incidentally that \( s \) is an atomic basis for \( a(s) \) and that the ideals \( p=q=a(s) \) are non-normal when \( \mathfrak{Z} \neq \mathfrak{S} \).

We prove next that a ring is of type \( (\beta^2) \) or \( (\beta^2, \alpha) \) if and only if it is a ring with \( \mathfrak{Z} = \mathfrak{S} \) satisfying the equivalent conditions \( (1), (2), \) and \( (3) \). By Ths. 2.8–2.10 a ring of type \( (\beta^2) \) or \( (\beta^2, \alpha) \) is representable as a ring \( A \) containing a prime ideal \( a' \) with \( a' = a \) which is of type \( (\beta^2) \). According to Th. 6.1, the ideal \( a \) is then a prime ideal with the properties demanded of \( q \) in \( (2) \). We have already observed that a ring of type \( (\beta^2) \) or \( (\beta^2, \alpha) \) satisfies the condition \( \mathfrak{Z} = \mathfrak{S} \). On the other hand, if a ring \( A \) with \( \mathfrak{Z} = \mathfrak{S} \) contains an ideal \( q \) with the properties demanded in \( (2) \), the fact that \( q \) is not normal shows that \( q \) has no unit and is of type \( (\beta^2) \) in accordance with Th. 6.1. Thus \( A \) must be either a totally multiplicative ring or a ring of one of the types \( (\beta^2), (\beta^2, \alpha) \), by virtue of Ths. 2.8–2.10. If \( A \) were totally multiplicative, then we could obtain a contradiction as follows. Let \( a' \) be an element of \( A \) not in \( q \). Then the principal ideal \( a(a') \) would be totally additive by Th. 2.3; and \( a(a')q \) would be a non-normal prime ideal relative to \( a(a') \). Thus \( a(a')q \) would be a ring of type \( (\beta^2) \). On the other hand \( q \) has an atomic basis by the preceding results. It follows that \( a(a')q \) has an atomic basis likewise, as an ideal in \( q \). Thus \( a(a')q \) is of type \((\beta^2)\); it is a ring without unit, being a non-normal prime ideal in \( a(a')q \), and has an atomic basis. Since the types \( (\beta^2) \) and \( (\beta^2) \) are distinct, we have the desired contradiction. We see therefore that \( A \) is of type \( (\beta^2) \) or of type \( (\beta^2, \alpha) \).

The characteristic representation described in \( (4) \) is now easily established. First, we shall prove that, when \( \mathfrak{Z} = \mathfrak{S} \), \( (3) \) implies \( (4) \). From what has been proved already, we know that \( a(s) \) is a non-normal prime ideal in \( A \) with \( s \) as an atomic basis. In particular we have \( a'(s)=a \). Since \( a(s) \) is a totally multiplicative ring of type \( (\beta^2) \), Ths. 2.5 and 6.1 show that \( a(s) \) is isomorphic to the ring of all finite subclases of a fixed infinite class \( E \); and that \( A \) is isomorphic to the subring of the totally additive ring of all subclases of \( E \) generated by a fixed subclase \( e \) and the finite subclases. Conversely, we show that \( (4) \) implies \( (3) \). A ring \( A \) of the kind described in \( (4) \) is obtained by the process analyzed in Th. 2.5: in the totally additive ring of all subclases of \( E \), the one-element subclases constitute a complete atomic system \( s \); the ideal \( a(s) \) generated by \( s \) is the system of all finite subclases of \( E \) and obviously has the property that \( a'(s)=a \); and \( A \) arises as the subring generated by \( e \) and \( a(s) \). Hence we see that \( a(s) \) is a non-normal prime ideal in \( A \) and that it is generated by the atomic system \( s \), regarded now as a subclase of \( A \). In view of the fact that \( a(s) \) is non-normal, we have \( \mathfrak{Z} = \mathfrak{S} \), \( \mathfrak{Z} = \mathfrak{S} \). Thus \( A \) is a ring with \( \mathfrak{Z} = \mathfrak{S} \) satisfying \( (3) \). Incidentally, the equivalence just shown for \( (3) \) and \( (4) \) implies that, when \( \mathfrak{Z} = \mathfrak{S} \), the atomic system \( s \) in \( A \) is complete and serves as an atomic basis for \( a(s) \).

It is obvious that a ring \( A \) represented in the form \( (4) \) has a unit if and only if the class \( E \) is a member of it; but \( E \) is a member if and only if \( E = \Gamma \) (mod \( a(s) \)) or, equivalently, if and only if \( E \Gamma \) is finite. Since such a ring is generated by \( E \) and its finite subclases, it is completely characterized by the cardinal number of \( E \).

The theory of cardinal numbers shows that \( E \) and \( A \) have the same infinite cardinal number. In order that \( A \) should have no unit, it is necessary and sufficient that \( E \Gamma \) be infinite. In this case, the theory of cardinal numbers shows that \( A \) is characterized by the infinite cardinal numbers of \( E \) and \( E \Gamma \); and also that \( A \) and \( E \) have the same cardinal number, equal to the greater of the cardinal numbers of \( E \) and \( E \Gamma \).

It is trivial that, under the foregoing conditions, a ring \( A \) belongs to type \( (\beta^2) \) or to type \( (\beta^2, \alpha) \) according as it has a unit or not.

We now wish to show that in a ring \( A \) of type \( (\beta^2, \alpha) \), the relation \( \mathfrak{S} = \mathfrak{S} \) is valid. Since such a ring is representable in the form \( A \Gamma \Delta \) where \( A_1 \) and \( A_2 \) are of type \( (\beta^2) \), and since \( A_2 \) can be represented in the form \( A_3 \Delta \) where \( A_3 \) and \( A_4 \) are without unit, by virtue of the fact that \( A_4 \) contains simple ideals which are not semiprincipal; — we see that \( A \) is represented as a direct sum of the rings \( A_3 \Delta \) and \( A_4 \), neither of which has a unit. By Th. 51, the two summands in this representation are non-semiprincipal simple ideals in \( A \).

If we make use of the existence and divisibility properties of prime ideals, we can add to the list of equivalent properties set forth in the preceding theorem. We have:

**Theorem 7.2.** In a Boolean ring \( A \) with \( \mathfrak{Z} = \mathfrak{S} \), the following properties are equivalent:

1. there exists a prime ideal \( p \) in \( A \), such that \( p \mathfrak{S} a \mathfrak{S} a' \), whatever the ideal \( a \);
2. there exists a prime ideal \( q \) in \( A \) such that \( q \mathfrak{S} a \mathfrak{S} a' \), whatever the ideal \( a \), and \( q \mathfrak{S} a \mathfrak{S} a' \), whatever the non-simple ideal \( a \);
3. there exists exactly one non-normal prime ideal \( r \) in \( A \);
4. the sum of all non-simple ideals in \( A \) is a prime ideal \( s \).

In such a ring \( A \), the relations \( p=q=r=s \) are valid.
It is evident that (1) implies (2) with \( q = p \) for \( a'' \lor a' \supset p \), whatever the ideal \( a \); and the relations \( a \lor a' = e, a'' \lor a' \supset p \), holding for any non-simple ideal \( a \), imply \( p = a \lor a' \) and hence \( p \lor a \supset a' \). It is also easily seen that (2) implies (3) with \( r = q \). If \( r \) is any non-normal prime ideal, we have \( r' = a, r = r' \lor r' \supset a, e, \) and hence \( r = q \).

Secondly, \( q \) is not normal: if it were, every prime ideal in \( A \) would be normal; hence every ideal, being the product of its prime ideal divisors (unless it is the normal ideal \( e \)), would be normal in \( A \), by virtue of Ths. 29 and 66; and hence every ideal in \( A \) would be simple in accordance with R. Th. 24. We assumed, however, that \( \exists \neq 0 \). Next we show that (3) implies (4) with \( s = r \). Since the ideal \( r \) is not normal and hence not simple, we see that \( s \), the sum of all non-simple ideals in \( A \), contains \( r \). If \( a \) is any non-simple ideal, then the ideal \( a \lor a' \) is not normal since \((a \lor a')'' = (a'' a') = a'' + r \lor a' \supset a' \); and if \( t \) is any prime ideal divisor of \( a \lor a' \), the relation \( a \lor a' \supset t \lor (a \lor a') = a, t = a, \) so that \( t \) is non-normal and must coincide with \( r \). By R. Th. 66, we conclude that \( a \lor a' \supset r \).

Thus we find that \( s \lor r \), and hence that \( s = r \). Finally, we show that (4) implies (1) with \( p = s \). If \( a \) is any simple ideal we have \( a'' a' = c \lor s \). On the other hand, if \( a \) is any non-simple ideal, we know that \( a \lor a' \supset a' \supset a'' \). It follows that \( c = a \lor a' \supset a'' \), and hence that \( t = s \). R. Th. 66 now shows that \( a \lor a'' = a' \lor s \). We conclude that \( a \lor a' \supset a'' \), whatever the ideal \( a \) in \( A \).

Since the ideals \( r \) and \( s \) of (3) and (4) are uniquely determined, our argument shows that \( p, q, r, s \) are unique and equal.

We proceed now with the discussion of the remaining types.

**Theorem 7.3.** The Boolean rings of types \((\beta_1^2, \beta_2, \beta_1)\) and \((\beta_2, \beta_2)\) are characterized as those rings \( A \) with \( R + \mathbb{S} \), possessing one of the following equivalent properties:

1. \((1)\) there exists a prime ideal \( p \) in \( A \) such that \( p \lor a'' = a' = a \lor a' \); while for some ideal \( a \) the relation \( p \lor a \supset a' \) is false;

2. \((2)\) there exists a prime ideal \( q \) in \( A \) such that every ideal in \( q \) which is normal relative to \( q \) is simple relative to \( q \) while some ideal in \( q \) is not normal relative to \( q \).

The ideals \( p \) and \( q \) of (1) and (2) respectively are unique and equal. Among the rings of the kind described by these various equivalent properties, those of type \((\beta_1^2, \beta_2, \beta_1)\) are characterized by the presence of a unit, and those of type \((\beta_2, \beta_2, \beta_2)\) by membership in the type \((\alpha)\).

We begin by showing that (1) implies (2) with \( q = p \). If \( a \) is any ideal in \( p \) normal relative to \( p \), then

\[a = a' \lor p \text{ and } a'' \lor a' = a' \lor a'' = p.

Hence \( a \) is simple relative to \( p \). On the other hand, let \( a \) be an ideal in \( A \) such that \( a \lor a' \) does not contain \( p \). Then the ideal \( qa \), considered as an ideal in \( p \) has the properties \( a \lor a' \supset p, a' \lor a'' \supset p \).

Hence \( a \lor a' \supset p \) and \( a \) is not normal relative to \( p \). Likewise, (2) implies (1) with \( p = q \). If \( a \) is an arbitrary ideal in \( A \), then \( a \lor a' \) is an ideal in \( q \) with orthocomplement \( a' \lor a'' \) in \( q \). Since \( a' \lor a'' \) is normal relative to \( q \), it is simple relative to \( q \) so that \( a'' \lor a' \lor a'' = q \lor a' \lor a'' \). On the other hand, if \( a \) is a non-normal ideal relative to \( q \), it is not simple relative to \( q \) and \( a' \lor a'' \lor a' \lor a'' \). Thus \( a \) is an ideal in \( A \) with the property that \( a \lor a' \) does not contain \( q \). Our argument shows that if either \( p \) or \( q \) is uniquely determined then the other is also, and \( p \) and \( q \) are equal. If \( R \lor \Sigma \), we can show that \( p \) is uniquely determined. In fact, let \( a \) be any non-simple normal ideal in \( A \). Then \( a'' = a' \lor a' \lor a'' = a' \lor a'' \lor a' \lor a'' \), so that \( a \lor a' \lor a'' \). Our assertion is thus established. Incidentally, we see that the relation \( p'' = a' \lor a'' \lor a'' \lor a'' = p \) implies that \( p \), and hence \( q \) also, is not normal.

If \( A \) is a ring of any of the types \((\beta_1^2, \beta_2, \beta_1), (\beta_2, \beta_2, \beta_2), \) or \((\beta_2, \beta_2)\), then \( A \) is representable in terms of a prime ideal \( a \) with \( a' \lor a \), where \( a \) is totally multiplicative, and an element \( a \) in \( A \), with \( R \lor \Sigma \) in accordance with Ths. 2.8-2.10; and this representation is essentially unique. Moreover \( A \) is totally multiplicative, so that \( R \lor \Sigma \) in accordance with Th. 2.3; and \( a \) is of type \((\beta_2)\) of or one of the special types \((\beta_2, \beta_2), (\beta_2, \beta_2)\) included under \((\beta_2)\) and hence is not of type \((\beta_2)\). Th. 6.1 now shows that \( a \) contains at least one non-normal ideal; and Th. 2.3 that every normal ideal in \( A \) is simple relative to \( a \). Thus we find that \( A \) is a ring with \( R \lor \Sigma \) possessing property (2) with \( q = a \). On the other hand, let \( A \) be a ring with \( R \lor \Sigma \) possessing property (2). Then the ideal \( q \) has the property that \( q = a \); as we noted above; and \( a \) is a totally multiplicative ring in accordance with Th. 2.3. Since \( q \) is prime in \( A \) and the relation \( R \lor \Sigma \) implies that \( A \) is not totally multiplicative, we see that \( A \) must be one of the types \((\beta_1^2, \beta_2, \beta_2), (\beta_2, \beta_2, \beta_2), (\beta_2, \beta_2)\). Since the ideal is uniquely determined in \( A \), by virtue of Th. 2.9, we see that Th. 7.1 (2) excludes the possibility of membership in either of the types \((\beta_1^2, \beta_2, \beta_2), (\beta_2, \beta_2, \beta_2)\). Hence we see that \( A \) is one of the three types \((\beta_1^2, \beta_2, \beta_2), (\beta_2, \beta_2), (\beta_2, \beta_2)\).
It is evident that among the rings belonging to these three types those of type \((\beta^2)\) are characterized by the presence of a unit. Ths. 3.6 and 3.8 show that every ring of type \((\beta^2, \beta, \delta)\), being the direct sum of a ring with unit and a ring of type \((\beta)\), is of type \((\omega)\). On the other hand, we can show that a ring \(A\) of type \((\beta^1)\) or of type \((\beta^1, \beta)\) is not of type \((\omega)\). In the case in the first of these types, the presence of a unit shows at once that \(A\) is not of type \((\omega)\). In the other case, we express \(A\) as the direct sum \(A_1 \oplus A_2\) of a ring with unit and a ring \(A_2\) of type \((\beta)\). Th. 2.6 shows that \(A_2\) contains non-semiprincipal simple ideals. Hence, by virtue of R Th. 51, we can express \(A_2\) as the direct sum \(A_1 \oplus A_3\) of rings without unit. Writing \(A\) as the direct sum \((A_1 \oplus A_2) \oplus A_3\), by an exchange of direct summands, we see that neither of the new summands has a unit. Using R Th. 51, we see further that both the new summands are non-semiprincipal simple ideals in \(A\). The relation \(\mathbb{S} = \mathbb{P}^*\) now shows by Th. 3.6 that \(A\) is not of type \((\omega)\). With this the proof of the theorem is complete.

Again employing the existence and divisibility properties of prime ideals, we can add to the list of equivalent properties given in Th. 7.3. We have:

**Theorem 7.4.** In a Boolean ring \(A\) with \(\mathbb{R} = \mathbb{S}\), the following properties are equivalent:

1. There exists a prime ideal \(p\) in \(A\) such that \(p \cap a'' \cap A'\), whatever the ideal \(a''\), while for some ideal \(a\) the relation \(p \cap A\) is false;
2. The sum of all non-normal ideals is the ideal \(e\), the sum of all normal non-simple ideals is a prime ideal \(e\).

The ideals \(p\) and \(r\) in such a ring are unique and equal.

We first prove that (1) implies (2) with \(r = p\). By hypothesis there exist non-simple normal ideals in \(A\). If \(a\) is such an ideal, we have \(a'' \cap A = a' \cap a' = e\) and hence \(p = a' \cap a'\). Since all such ideals are contained in \(p\), by the last relation, their sum also is contained in \(p\). On the other hand, if \(a\) is such an ideal, \(a'\) is normal and the relation \(a' \cap a' = p + e\) shows that \(a'\) is not simple. Since \(a' \cap a'\) is then contained in the sum of all non-simple normal ideals, we conclude that this sum coincides with \(p\). Since \(3 = \mathbb{S}\) follows from \(\mathbb{R} = \mathbb{S}\), the assumption that there exist no non-normal prime ideal other than \(p\) would imply that \(p \cap a'\) for every ideal \(a\), contrary to (1); for the relation \(p = a' \cap a'\) obtained above leads to the equation \(p' = a' \cap a' = e\) and hence implies that \(p\) is non-normal.

Th. 7.2 would then establish the desired contradiction. Hence we see that there exists a non-prime ideal \(a\) distinct from \(p\). It is now clear that the sum of all non-normal ideals in \(A\) contains \(p' \cap a = e\) and thus coincides with \(e\).

On the other hand (2) implies (1) with \(p = r\). Let \(a\) be any element of \(r\) and let \(a(a)\) be the principal ideal generated in \(A\) and in \(r\) by \(a\). If \(a\) is any ideal in \(a(a)\) normal relative to \(a(a)\), then \(a\) is normal in \(A\). The relation \(a \cap a(a) = a' \cap a(a)\). By Th. 41 shows that \(a'\) is not contained in the prime ideal \(r\). Hence \(a'\) is not contained in \(r\) and, being normal, must be simple by virtue of the definition of \(r\). Hence \(a'' = a'\) is also simple; but, being contained in the principal ideal \(a(a)\), the ideal \(a\) must even be principal by virtue of Th. 1.1. Since every normal ideal in \(a(a)\) is principal, \(a(a)\) is a totally additive ring by Th. 2.2 (2); and thus \(r\) is a totally multiplicative ring by Th. 2.3 (3). Consequently every ideal normal relative to \(r\) is simple relative to \(r\) by Th. 2.3 (2). If now \(a\) is an arbitrary ideal in \(A\), then \(a''\) is normal in \(A\) and \(a'\) is normal relative to \(r\) in accordance with Th. 1.3. Hence we see that \(a'\) is simple relative to \(r\) and conclude that \(a' \cap a(a) = a' \cap a(a) = r\). Now our assumption that the sum of all non-normal ideals in \(A\) coincides with \(e\) shows that there exists at least one non-normal ideal \(a\) not contained in \(r\). Since \(a' \cap a' = e\), by virtue of the fact that \(a\) is not simple, we see that the relation \(a' \cap a' = r\) is false: for otherwise we would have \(a \cap a = r\), against our choice of \(a\). Hence (2) implies (1) with \(p = r\).

Finally, it is evident that the ideals \(p, r\) of (1) and (2) respectively are uniquely determined and are equal to one another.

### § 8. Algebraic Characterizations of Boolean Rings of Types \((\omega)\), \((\omega, \omega)\).

In this section we proceed to characterize rings of the indicated types.

**Theorem 8.1.** A Boolean ring \(A\) is of type \((\omega)\) if and only if \(\mathbb{P} = \mathbb{P}^* = \mathbb{S}\).

We now prove:

**Theorem 8.2.** A Boolean ring \(A\) is of type \((\omega, \omega)\) if and only if it has the properties:

1. \(A\) contains a non-semiprincipal simple ideal \(a\);
2. If \(a\) is any non-semiprincipal simple ideal in \(A\), then every ideal \(b\) contained in \(a\) and simple relative to \(a\) is semiprincipal relative to \(a\).
According to R Th. 51, the relation $\Psi^*+\subseteq$ is equivalent to the existence of a representation of $A$ as a direct sum of two rings without unit, the summands being simple, but not semiprincipal, ideals in $A$. Hence we shall consider $A$ as such a direct sum, $c\vee c'$, in the remainder of the discussion. If $A$ is of type $(\omega,\omega)$, we may choose $c$ and $c'$ as rings of type $(\omega)$. Let $a$ be a non-semiprincipal simple ideal in $A$; and let $b$ be an ideal in $a$, simple relative to $a$. By Th. 1.3, $b$ is simple in $A$; and $a\supset b$ implies $b'\subseteq a'$. If $b$ were principal in $a$, it would be semiprincipal relative to $a$ without further discussion. Hence we may suppose that $b$ is not principal in $a$ or in $A$. Now at least two of the ideals $abc=bc$, $ab'c'=a'c$ must be principal. For example, if $abc=bc$ is not principal, it is not principal in $c$. Being a product of simple ideals, it is simple both in $A$ and in $c$ by Ths. 1.2 and 1.3. Since $c$ is of type $(\omega)$, the orthocomplement of $abc=bc$ relative to $c$ must be principal in $c$ and hence in $A$. Thus $b'c$ is a principal ideal. Consequently the ideals $ab'c'$ and $a'b'c'$, being the products of the simple ideals $a$ and $a'$ with the principal ideal $b'c$, are principal by Th. 1.2. Similarly, if $ab'c$ is not principal, $a'c'bc=(ab')'c'$ is principal, and $a'c$ and $bc$ are principal by Th. 1.1; and if $a'c$ is not principal, then $ac=a'c$ is principal and so also are $abc$ and $ab'c$. By similar arguments, we see that at least two of the ideals $abc=bc'$, $ab'c'$, $a'b'c'=a'c$ are principal. On the other hand, the ideals $bc$ and $bc'$ are not both principal since we have assumed that their sum $b=bc\vee bc'$ is non-principal. For the same reason, the ideals $ab'c$ and $ab'c'$ are not both principal. We conclude therefore that the ideals $ab'c$ and $ab'c'$ are both principal. Hence the orthocomplement of $b$ relative to $a$ is found to be the principal ideal $ab'=ba'c\setminus ab'c'$. Thus we have shown that $b$ is semiprincipal relative to $a$. This completes our proof of the assertion that a ring of type $(\omega,\omega)$ has properties (1) and (2). On the other hand, a ring with these properties can be represented as a direct sum $c\vee c'$, as we noted above. Since (2) shows that every ideal $b$ contained in $c$ or in $c'$ which is simple relative thereto is semiprincipal relative thereto, we see by reference to Th. 8.1 that $c$ and $c'$ are rings of type $(\omega)$. Hence the given ring is of type $(\omega,\omega)$.

We have already given some investigations into the relation of the types $(\omega)$ and $(\omega,\omega)$ to the other types considered. It is convenient to complete and summarize our results in the following terms:

**Theorem 8.3.** Of the nine types $(\omega)$, $(\beta_1)$, $(\beta_2)$, $(\beta_3)$, $(\beta_4)$, $(\beta_5)$, $(\beta_6)$, $(\beta_7)$, $(\beta_8)$ only $(\beta_1)$ and $(\beta_5)$ are included under type $(\omega)$; and only the special types $(\beta_1)$ and $(\beta_5)$, included under $(\beta_1)$ and $(\beta_5)$ respectively, are included under type $(\omega,\omega)$.

The rings of types $(\omega)$, $(\beta_5)$, and $(\beta_5)$ all have units and hence belong to neither of the types $(\omega)$ and $(\omega,\omega)$. We have already seen, in Ths. 3.6 and 7.3, that the rings of types $(\beta_1)$ and $(\beta_5)$ are of type $(\omega)$ while those of types $(\beta_2)$, $(\beta_3)$, and $(\beta_5)$ are not. Comparison of Th. 8.1 with the result of Th. 7.1 that a ring of type $(\beta_7,\beta_8)$ has the property $\Psi^*+\subseteq$ shows that such a ring is not of type $(\omega)$. Since the types $(\omega)$ and $(\omega,\omega)$ are distinct, by Th. 3.9, the rings of types $(\beta_1)$ and $(\beta_5)$ are not of type $(\omega,\omega)$. Furthermore, Th. 3.7 shows that the rings of types $(\beta_1)$ are not of type $(\omega,\omega)$, and that among the rings of type $(\beta_1)$ only those of the special type $(\beta_1,\beta_1)$ are of type $(\omega,\omega)$. Consequently we see that among the nine types only $(\beta_1)$ and $(\beta_5)$ are included under type $(\omega)$; and that only $(\beta_1,\beta_1)$, $(\beta_5,\beta_5)$, and $(\beta_5,\beta_5)$ could possibly be included under type $(\omega,\omega)$. Since the type $(\beta_7,\beta_8)$ is included under the type $(\omega,\omega)$, we have only two types left to consider.

Now let a Boolean ring $A$ be represented in the form $A_1\vee A_2$ where $A_2$ has no unit. According to Th. 3.8, $A$ is of type $(\omega,\omega)$ if $A_2$ is of that type. Conversely, we shall show that, if $A$ is of type $(\omega,\omega)$, then so is $A_2$. First, $A_2$ cannot be of type $(\omega)$: for, if it were, Th. 3.8 would show that $A$ is of type $(\omega)$, distinct from the type $(\omega,\omega)$. Thus Th. 8.1 shows that $A_2$ contains simple ideals which are not semiprincipal. If $a$ is such an ideal in $A_2$, the fact that $A_2$ is a simple ideal in the direct sum $A_1\vee A_2$ shows that $a$ is simple in $A$ by virtue of Th. 1.4. Hence, if $b$ is any ideal contained in $a$ and simple relative to $a$, then $b$ must be semiprincipal relative to $a$ by an application of Th. 8.2 together with the assumption that $A$ is of type $(\omega,\omega)$. We see therefore that $A_2$ has the properties (1) and (2) of Th. 8.2 and must consequently be of type $(\omega,\omega)$. Now a ring of either of the types $(\beta_7,\beta_8)$, $(\beta_7,\beta_8)$ has an essentially unique representation in the form $A_1\vee A_2$ where $A_1$ and $A_2$ are totally multiplicative rings without unit. In particular $A_2$ is of type $(\beta_7)$ or $(\beta_8)$ according as $A$ is of type $(\beta_7,\beta_8)$ or $(\beta_7,\beta_8)$. If $A$ is of type $(\omega,\omega)$, then so is $A_2$, by the preceding results. Hence Th. 3.7 shows that $A$ cannot be of type $(\omega,\omega)$ except in the case where $A_1$ is of type $(\beta_7,\beta_1)$, included under type $(\beta_7)$. Hence no ring of type $(\beta_7,\beta_1)$ is
of type \((\omega, \omega)\); and the only rings of type \((\beta_1^*, \beta_2^*)\) which can possibly be included under type \((\alpha, \omega)\) are those of the special type \((\beta_1^*, \beta_1, \beta_1)\). Of course, Th. 3.7 and 3.8 show that the rings of types \((\beta_1, \beta_2^*)\) and \((\beta_1^*, \beta_1, \beta_1)\) are of type \((\alpha, \omega)\).

§ 9. Countable Boolean Rings. We shall now consider which of the various types of Boolean ring introduced in §§ 2, 3 and discussed in §§ 4–8 include countable rings. We shall find that only the types \((\beta_1, \beta_2^*)\) and \((\beta_1^*, \beta_2^*)\) have this property.

Our fundamental result is the following

Theorem 9.1. An infinite totally additive Boolean ring \(A\) has cardinal number not less than \(2^{\aleph_0}\), the cardinal number of the continuum.

Let \(A\) be a ring of the kind described. Th. 5.1 then shows that \(A\) contains a sequence \(\langle a_n \rangle\) such that \(a_nM=0\), \(a_mM=0\) for \(m\neq n\). If \(s\) is any non-void subring of the sequence \(a(s)\) is the generating element of the principal ideal \(a(s)\) in accordance with Th. 2.1. Now \(sM=0\) implies \(a(s)a_m=0\); and \(sM=0\) implies \(a(s)=a(s)\) and hence \(a(s)a_m=0\). It follows that the class of all sums \(a(s)\) is in bimorphic correspondence with the non-void subrings formed from the sequence \(\langle a_n \rangle\). Since such subrings constitute a collection with the cardinal number \(2^{\aleph_0}\), we conclude that the cardinal number of \(A\) is not less than \(2^{\aleph_0}\).

As an immediate consequence of this theorem, we have:

Theorem 9.2. A countable Boolean ring \(A\) is totally additive if and only if it is finite; a countable Boolean ring \(A\) is totally multiplicative if and only if it has a countable atomic basis.

By Th. 9.1, a countable Boolean ring \(A\) cannot be infinite if it is totally additive. Hence the result stated here. If a Boolean ring \(A\) is countable and totally multiplicative, then every principal ideal of \(A\) is obviously countable and in addition is totally additive by virtue of Th. 2.3 (3). Hence every principal ideal of \(A\) is finite. It follows that \(A\) has a countable atomic basis, by virtue of Th. 6.1.

On the other hand a ring \(A\) with countable atomic basis is finite if the basis is finite, and is isomorphic to the ring of all finite subclasses of a fixed countably infinite class \(E\) in accordance with Th. 6.1 if the basis is countably infinite. Hence \(A\) is countable and totally multiplicative.

Another fundamental result is the following

Theorem 9.3. In a countable Boolean ring \(A\) with unit every non-normal prime ideal is a barrier ideal.

Let \(p\) be a non-normal prime ideal in a countable ring. Since \(p\) is a non-principal ideal, it is a ring without unit and is therefore infinite. Hence its elements may be written in an infinite sequence \(\langle a_n \rangle\). We now construct a subsequence \(\langle b_n \rangle\) with the properties:

1. \(b_n=b_{n+1}\) for \(n=1, 2, 3, \ldots\)
2. If \(a\) is any element in \(p\), then \(a=b\) for some index \(n\). We put \(b_0=a\). If we have defined \(b_k=a_k\) for \(k=1, 2, \ldots, m\) so that \(a_k\neq a_n\) for \(k=1, 2, \ldots, m\) and \(a_n=a_{n+1}\) for \(k=1, 2, \ldots, m-1\), then we choose \(a_{m+1}\) as the first index \(n\) after \(a_m\) such that \(a_n=a_m, a_m=a_n\). This choice is possible: for if we had either \(a_m=a_n\) or \(a_m=a_n\), the element \(a_1\vee \ldots \vee a_m \vee \ldots \vee a_{m+1}=a\) would belong to \(p\) and would have the property that \(a\neq a_n\) for \(n=1, 2, 3, \ldots\); and we could then conclude that \(p\) is a principal ideal with \(a\) as its generating element. By induction, we obtain a sequence \(b_k=a_k\) for \(k=1, 2, 3, \ldots\) such that \(a_k\neq a_{n+1}\) for \(k=1, 2, 3, \ldots\). Since the relation \(a_k\neq a_{n+1}\) implies that \(a_k=a_{n+1}\), this sequence has the desired properties (1) and (2). By the construction given in the proof of Th. 5.1, we can now replace the sequence \(\langle b_n \rangle\) by the sequence \(\langle c_n \rangle\), where \(c_0=b_0\) and \(c_{m+1}=b_{m+1}+a_n\) for \(n\geq 1\), so as to secure the properties \(c_0=0, c_1=a_1 \vee c_2=0, c_2=0 \vee c_3=0, c_3=0 \vee c_4=0, c_4=0 \vee \ldots \vee c_0=0\). It is evident that \(c_k\) implies \(a_k<c_0\) for some index \(n\). Let \(s\) be the class of all elements \(c_{m+1}\) for \(m=1, 2, 3, \ldots\) and \(t\) be the class of all elements \(c_{m+2}\) for \(m=1, 2, 3, \ldots\). We shall determine the ideals \(s\) and \(t\). It is evident that \(s\cap t\neq 0\). It follows that \(s'\cap t'\neq 0\). If an element \(a\) of \(A\) is in both \(s\) and \(t\) then \(a=c_0=0\). Consequently \(a\) belongs to \(p\).

Since \(p\) is a non-normal prime ideal, we have \(p'=0\) and hence \(a=0\). Thus \(s'=0\). It follows that \(s'\cap t'\neq 0\). Combining these results with those obtained above, we see that \(s'=t', s''=t''\). The ideal \(s'\cap t'\) contains both \(s\) and \(t\), and thus contains every element \(c_0, c_1, c_2, \ldots\). Consequently, if \(a\) is any element of \(p\), we have \(a=c_0 \vee c_n\) for a suitable index \(n\) and conclude that \(a\in s'\cap t'\). The relation \(p\subseteq s'\cap t'\) is thus established.
On the other hand, let \( a \) be an element which belongs to \( s' \). Then \( a_{2m} = 0 \) for \( m = 1, 2, 3, \ldots \), or, equivalently, \( a' > c_m \) for \( m = 1, 2, 3, \ldots \). If \( a' \) were an element of \( p \), there would exist an index \( n \) such that \( a' < c_1 \ldots \ldots < c_n \). Choosing \( 2m > n \) we would therefore have \( a_m < c_1 \ldots \ldots < c_n \) and hence
\[
a_{2m} = a_{2m}(c_1 \ldots \ldots < c_n) = a_{2m} c_1 \ldots \ldots < c_n = 0.
\]
Since \( c_m \neq 0 \), we conclude that \( a' \) is not in \( p \). By R Th. 36, we reach the result that \( a a' s' \) implies \( a = p \) or, equivalently, that \( s' \subset p \). In the same way we find that \( t' \subset p \). Thus we must have \( p = s' < t' = 0 \). Since \( s' \) and \( t' \) are normal ideals by virtue of the relations \( s' = s' ', t' = t' ' \) of R Th. 20, we see that \( p \) is a barrier ideal in accordance with Def. 3.1.

We can now obtain the chief results concerning types of countable Boolean ring. We have:

**Theorem 9.1.** There exist no countable Boolean rings of any of the types \((a), (b_1), (b_2), (b_3), (b_4), (b_5), (b_6), (b_7), (b_8), (a, a, a)\). A Boolean ring of one of the three types \((b_1), (b_2), (b_3), (b_4)\) is countable if and only if in the corresponding representation as a ring of subrings of a fixed infinite class \( E \), described in Th. 6.1 and 7.1, the class \( E \) is countable.

Ths. 9.1 and 9.2 show that no ring of type \((a), (b_1), \text{ or } (b_2)\) is countable; and Th. 9.3 shows that no ring of type \((a)\) is countable. Since the rings of types \((b_1), (b_2), (b_3), (b_4), (a, a)\) contain subrings of some of the types \((b_1), (b_2), (a, a)\), none of them can be countable. Thus the only types which can contain countable rings are the three mentioned in the statement of the theorem. The condition given for a ring of any of these three types to be countable is evident.

**§ 10. The Fundamental Classification of Ideals.** In this section we shall collect some of the results of §§5—9 in such a form as to give a complete presentation of the possible relations of equality between the fundamental classes \( \mathfrak{P}, \mathfrak{P}^*, \mathfrak{S}, \mathfrak{R}, \mathfrak{Z}, \mathfrak{E} \) of ideals in a Boolean ring \( A \). We have:

**Theorem 10.1.** The possible reductions of the inclusion relations \( \mathfrak{P} \subset \mathfrak{P}^* \subset \mathfrak{S} \subset \mathfrak{R} \subset \mathfrak{Z} \subset \mathfrak{E} \) to equalities are summarized in the following table, in which the Boolean rings associated with any particular combination of equalities and inequalities are described at the right:

| \( \mathfrak{P} = \mathfrak{P}^* = \mathfrak{S} = \mathfrak{R} = \mathfrak{Z} \) | Finite rings. |
| \( \mathfrak{P} = \mathfrak{P}^* = \mathfrak{S} = \mathfrak{R} \pm \mathfrak{Z} \) | Rings of type \( (a) \). |
| \( \mathfrak{P} = \mathfrak{P}^* = \mathfrak{S} \pm \mathfrak{R} = \mathfrak{Z} \) | General rings with unit including those of types \( (\beta_1) \) and \( (\beta_2) \). |
| \( \mathfrak{P} = \mathfrak{P}^* = \mathfrak{S} = \mathfrak{R} \pm \mathfrak{Z} \) | Rings of type \( (b_1) \). |
| \( \mathfrak{P} = \mathfrak{P}^* = \mathfrak{S} \pm \mathfrak{R} = \mathfrak{Z} \) | Rings of type \( (\omega) \), other than those of type \( (b_1) \) but including those of type \( (\beta_1, \beta_2) \). |
| \( \mathfrak{P} = \mathfrak{P}^* = \mathfrak{S} \pm \mathfrak{R} \pm \mathfrak{Z} \) | Rings of type \( (b_2) \). |
| \( \mathfrak{P} = \mathfrak{P}^* = \mathfrak{S} \pm \mathfrak{R} \pm \mathfrak{Z} \) | Rings of type \( (\beta_1, \beta_2) \) including those of type \( (\beta_1, \beta_2) \). |
| \( \mathfrak{P} = \mathfrak{P}^* = \mathfrak{S} \pm \mathfrak{R} \pm \mathfrak{Z} \) | General rings without unit, including those of types \( (\beta_1, \beta_2), (\beta_1, \beta_2) \), and \( (\omega, \omega) \) other than those of type \( (\beta_1, \beta_2) \). |

We know from R Th. 25 that the relation \( \mathfrak{P} = \mathfrak{P}^* \) is characteristic for rings with unit and implies \( \mathfrak{P} = \mathfrak{S} \); from Ths. 3.6 and 8.1 that the relations \( \mathfrak{P} = \mathfrak{P}^* = \mathfrak{S} \) are characteristic for rings of type \( (\omega) \); from Th. 2.3 (2) that the relation \( \mathfrak{S} = \mathfrak{R} \) is characteristic for totally multiplicative rings; from Ths. 5.2 and 6.1 that the relation \( \mathfrak{R} = \mathfrak{Z} \) is characteristic for the finite rings and the rings of type \( (\beta_2) \); and from R Th. 24 that the relations \( \mathfrak{Z} \pm \mathfrak{S}, \mathfrak{R} \pm \mathfrak{S} \) both imply \( \mathfrak{Z} \pm \mathfrak{R} \). We are thus left with at most nine possible combinations of equalities and inequalities between the classes of ideal — the eight listed in the table, and the system of relations \( \mathfrak{P} = \mathfrak{P}^* = \mathfrak{S} = \mathfrak{R} = \mathfrak{Z} \) which does not appear there. This ninth system cannot actually occur since the relation \( \mathfrak{R} = \mathfrak{Z} \) characterizes rings in which either \( \mathfrak{R} = \mathfrak{P}^* = \mathfrak{S} \) — the finite case — or \( \mathfrak{P} + \mathfrak{P}^* = \mathfrak{S} \) — the case of type \( (b_1) \). Setting aside the cases where \( \mathfrak{R} = \mathfrak{Z} \) as already settled, we see that in the six remaining cases those where \( \mathfrak{S} = \mathfrak{R} \) exhaust the infinite totally multiplicative rings of types other than \( (\beta_2) \). Of these, the only ones with unit are those of type \( (\omega) \), the only ones with \( \mathfrak{P} = \mathfrak{P}^* = \mathfrak{S} = \mathfrak{R} \) are those of type \( (\beta_1) \), and the only ones with \( \mathfrak{P} = \mathfrak{P}^* = \mathfrak{S} \) are those of type \( (\beta_2) \), as we see by reference to Ths. 2.6, 3.6 and 6.4. We are thus left with the three cases where \( \mathfrak{S} = \mathfrak{R} \) and hence \( \mathfrak{R} = \mathfrak{Z} \). These cases exhaust the Boolean rings which are not totally multiplicative. Among these rings the ones with unit satisfy also the relations \( \mathfrak{P} = \mathfrak{P}^* = \mathfrak{S} \), the ones of type \( (\omega) \) the relations \( \mathfrak{P} = \mathfrak{P}^* = \mathfrak{S} \), and the
remaining ones the relations $\mathcal{P} \leftrightarrow \mathcal{P}^* + \mathcal{S}$. The distribution of the types $(\beta^*_1)$, $(\beta^*_2)$, $(\beta^*_3, \beta_1)$, $(\beta^*_3, \beta_2)$, $(\beta^*_3, \beta_3)$ and $(\alpha, \omega)$ among these three cases is settled by noting that no rings of the last four types just listed have units and that, in accordance with Th. 8.3, only type $(\beta^*_3, \beta_3)$ is included in type $(\alpha, \omega)$. The general distribution of the type $(\alpha, \omega)$ between the totally multiplicative cases and the other cases is settled by Ths. 3.7 and 8.3.

It will be observed that each row of the table contains at least one type of Boolean ring for which an explicit construction has been provided in § 2, 3, 4, 5, or 6. Hence each of the eight systems of relations is actually realized. It will be observed further that the three entries with $\mathcal{E} + R$ are of comparable degrees of generality in view of R Th. 1 and Th. 3.8, while the remaining entries characterize quite special Boolean rings.

With regard to the class $\mathcal{E}$ of prime ideals, we note the following facts:

**Theorem 10.2.** The relation $\mathcal{E} \subseteq R$ implies the relations $\mathcal{E} \subseteq \mathcal{P}^*$ and $\mathcal{E} \leq \mathcal{S}$, thus characterizing the finite Boolean rings and the Boolean rings of type $(\beta_3)$. The assertion that $\mathcal{E} \subseteq \mathcal{S}$ is a one-element class characterizes the Boolean rings of types $(\beta_3^*)$ and $(\beta_3, \beta_3)$.

From R Th. 38, we know that $\mathcal{E} \subseteq R$ implies $\mathcal{E} \subseteq \mathcal{P}^*$. R Th. 66 shows that $\mathcal{E} \subseteq R$ implies $\mathcal{E} = \mathcal{S}$: for, the ideal $e$ is semiprincipal and hence normal; and every other ideal, being the product of its prime ideal divisors, is normal in accordance with R Th. 29. The case where $\mathcal{E} \subseteq \mathcal{R}$ is a one-element class — that is, where there exists exactly one non-normal prime ideal — is settled by Th. 7.2.

By virtue of Th. 9.4 we can easily specialize the table of Th. 10.1 for the case of countable rings.

**Theorem 10.3.** The possible reductions of the inclusion relations $\mathcal{E} \subseteq \mathcal{P}^* \subseteq \mathcal{E} \subseteq \mathcal{R} \subseteq \mathcal{S}$ to equalities in the case of a countable Boolean ring are summarized in the following table:

| $\mathcal{P} \leftrightarrow \mathcal{P}^* \leftrightarrow \mathcal{P} + \mathcal{S}$ | Finite rings. |
| $\mathcal{P} \leftrightarrow \mathcal{P}^* \leftrightarrow \mathcal{S} + \mathcal{P}$ | General rings with unit, including those of type $(\beta_3^*)$. |
| $\mathcal{P} \leftrightarrow \mathcal{P}^* \leftrightarrow \mathcal{S} \leftrightarrow \mathcal{P}$ | Rings of type $(\beta_3)$. |
| $\mathcal{P} \leftrightarrow \mathcal{P}^* \leftrightarrow \mathcal{S} + \mathcal{P} + \mathcal{S}$ | General rings without unit, including those of type $(\beta_3^*, \beta_3)$. |

§ 11. Some Special Ideals. In this section we shall consider ideals constructed from the classes $\mathcal{P}^* \mathcal{P}$, $\mathcal{S} \mathcal{P}^*$, $\mathcal{R} \mathcal{S}$, $\mathcal{S} \mathcal{R}$. In particular we associate with each class the least ideal containing all its members. The associated ideal is $\omega$ in the case the class is void; otherwise it is the sum of all ideals in the class. The ideal obtained in this way from $\mathcal{P}^* \mathcal{P}$ or from $\mathcal{S} \mathcal{P}^*$ is easily seen to be either $\omega$ or $e$, the first case corresponding to the relations $\mathcal{P} = \mathcal{P}^*$, $\mathcal{P}^* = \mathcal{S}$ respectively, the second to the relations $\mathcal{P} + \mathcal{P}^* + \mathcal{S}$ respectively. Thus the classes $\mathcal{P}^* \mathcal{P}$, $\mathcal{S} \mathcal{P}^*$ do not lead to results which it is necessary to examine more carefully. On the other hand, the facts established in Ths. 7.2 and 7.4 show that the situation is different with respect to the classes $\mathcal{R} \mathcal{S}$, $\mathcal{S} \mathcal{R}$.

Using the existence and divisibility properties of prime ideals, we find the following results:

**Theorem 11.1.** In a Boolean ring $A$, let $q$ and $r$ be the least ideals containing all ideals in the classes $\mathcal{R} \mathcal{S}$ and $\mathcal{S} \mathcal{R}$ respectively. Then $q \leq r$; and each of them is equal to $\omega$, or is a non-normal prime ideal. The possible combinations and the types of ring which they characterize are exhibited in the following table, in which the entry $* \leftrightarrow$ signifies that the ideal in question is a non-normal prime ideal:

<table>
<thead>
<tr>
<th>q</th>
<th>r</th>
<th>A</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega$</td>
<td>$\omega$</td>
<td>Finite or of type $(\beta_3)$.</td>
</tr>
<tr>
<td>$\omega$</td>
<td>$e$</td>
<td>Ring of types $(\alpha), (\beta_1), (\beta_3)$.</td>
</tr>
<tr>
<td>$* \leftrightarrow$</td>
<td>$* \leftrightarrow$</td>
<td>Ring of types $(\beta_3^*), (\beta_3, \beta_3)$.</td>
</tr>
<tr>
<td>$* \leftrightarrow$</td>
<td>$e$</td>
<td>Ring of types $(\beta_3^*, \beta_3), (\beta_3, \beta_2)$.</td>
</tr>
<tr>
<td>$e$</td>
<td>$e$</td>
<td>General ring.</td>
</tr>
</tbody>
</table>

We first prove that $q \leq r$. If $\mathcal{R} \mathcal{S}$ is void, then $q = e$ and the relation $q \leq r$ is trivial. If $\mathcal{R} \mathcal{S}$ is not void, then there exists a non-simple normal ideal $a$ and $q$ is the sum of all such ideals. Now if $\alpha$ is any such ideal, so is $\alpha';$ and the ideal $\alpha \cap \alpha'$ is not normal, by R Th. 24. Hence we have $a \mathcal{C} \alpha \cap \alpha' \mathcal{C}$ whenever $a \in \mathcal{R} \mathcal{S}$. The desired relation $q \leq r$ then follows. We may observe that, if $\mathcal{R} \mathcal{S}$ is not void, then $q = e$. It is also easily seen that, if $\mathcal{S} \mathcal{R}$ is void, $r = e$ and that otherwise $r = \omega$. Thus the relations $q = e$, $r = e$ are equivalent respectively to the relations $\mathcal{R} = \mathcal{S}$, $\mathcal{S} = \mathcal{R}$.  

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Next let us prove that, when \( r = e \), \( r = e \), then \( r \) is a non-normal prime ideal. Since \( r = e \), there exists a non-normal ideal \( a \); and Th. 24 shows that \( a \cap a^* \) is not normal. Since \( r = e \), there exists an element \( a \) not in \( r \). For any such element \( a \), the ideal \( a(a) \cap a \) is not contained in \( r \) and so must be normal. Since it is also not contained in \( q \cap r \), it must even be simple. Using the indicated properties of the ideals \( a \cap a^* \) and \( a(a) \cap a \), we have

\[
e = (a(a) \cap a) \cap (a(a) \cap a) = a(a) \cap a \cap a^* = a(a) \cap (a \cap a^*) \cap a(a) \cap r,
\]

\[
a \cap a^* \subseteq r, \quad a = a = (a \cap a^*) \cap r', \quad r' = e.
\]

Thus an ideal containing \( r \) and any element not in \( r \) must coincide with \( e \), so that \( r \) is divisorless and hence prime; and the relation \( r = e \) shows that \( r \) is normal. In a similar way, we show that, if \( q = e \) and \( q = e \), then \( q \) is a non-normal prime ideal. Let \( a \) be a non-simple normal ideal and \( a \) an element not in \( q \). Then \( a \) is a non-simple normal ideal, and \( a \cap q \subseteq a \). Also \( a(a) \cap a \) is a normal ideal by Th. 1.2; and, since it is not contained in \( q \), it must be simple. By reasoning exactly like that used above, we have \( e \subseteq a(a) \cap q \), \( q = e \). It follows, as before, that \( q \) is a non-normal prime ideal.

The possible combinations are now reduced to the five listed in the table, together with the one where \( q = e \) and \( r \) is a non-normal prime ideal. We shall see presently that this particular combination cannot arise. From the preceding results we know that \( r = e \) if and only if \( \mathfrak{S} = \mathfrak{R} \) and that \( r = e \) implies \( q = e \). We also know from Ths. 5.2 and 6.1 that the relation \( \mathfrak{S} = \mathfrak{R} \) characterizes the finite rings and the rings of type \( (\beta_2) \). If \( r \) is prime, we must have \( q = r \) or \( q = e \) by virtue of the relation \( q \subseteq r \). In either case we see that the sum of all non-simple ideals in \( A \) is the ideal \( s = q \cap r = r \). The relation \( r = e \) implies in particular that \( \mathfrak{S} = \mathfrak{S} \). Since \( s \) is prime, Th. 7.2 shows that \( A \) is a ring of type \( (\beta_2) \) or \( (\beta_1, \beta_1) \). In rings of these types we have \( \mathfrak{R} = \mathfrak{S} \) and hence \( q = e \). On the other hand, if \( A \) is a ring of either of these types, the sum \( s \) of all non-simple ideals is a prime ideal by Th. 7.3. Obviously \( s = q \cap r = r \), so that \( r \) is prime. Thus we see that the rings of types \( (\beta_2) \) and \( (\beta_1, \beta_1) \) are characterized by the statement that \( r \) is prime; and that, when \( r \) is prime, \( q \) is also prime and equal to \( r \). We now have to treat the cases where \( r = e \).

Theorems. If \( q \) is prime and \( r = e \), we have \( \mathfrak{R} = \mathfrak{S} \) and Th. 7.4 shows that the ring \( A \) is of one of the types \( (\beta_2) \), \( (\beta_1, \beta_1) \), \( (\beta_2, \beta_2) \). Conversely, if \( A \) is of one of these types, then \( q \) is prime and \( r = e \), by virtue of the same theorem. Thus the only case remaining is that where \( q = e \). This case occurs in all rings not of the types listed under the other cases, and is therefore to be regarded as the general case.

§ 12. Optimum Character of § 1. We shall now proceed to show that the general results stated in § 1 are the best possible, in a certain sense. It is of course obvious that in any ring for which relations of equality hold between some of the classes \( \mathfrak{B}, \mathfrak{P}, \mathfrak{S}, \mathfrak{R}, \mathfrak{Z} \), as listed in Th. 10.1, the results presented in the tables of § 1 can be correspondingly reduced. Thus, for example, if \( \mathfrak{R} = \mathfrak{S} \), the tables of Th. 1.2 can be simplified by writing \( \mathfrak{S} \) for \( \mathfrak{R} \) throughout; and it would also be permissible to strike out altogether the rows and columns labeled "\( \mathfrak{R} \)". What we propose to show is that, apart from the simplifications arising in this way or in connection with certain special types of ring, no reductions of the results of § 1 are possible. For convenience in comparison, we shall take up the theories of § 1 in serial order, stating for each of them a similarly numbered theorem discussing its optimum character.

**Theorem 12.1.** Under the condition \( a \subseteq b \), the ideals \( a \) and \( b \) in a Boolean ring \( A \) can be assigned arbitrarily to the classes \( \mathfrak{B}, \mathfrak{P}, \mathfrak{S}, \mathfrak{R}, \mathfrak{Z} \), with only the following exceptions:

1. the assignment is subject to the general restrictions given in Th. 1.1;
2. the assignment is subject to the limitations imposed by equalities between the classes \( \mathfrak{B}, \mathfrak{P}, \mathfrak{S}, \mathfrak{R}, \mathfrak{Z} \);
3. in a ring of type \( (\beta_2) \) or \( (\beta_1, \beta_1) \), the relation \( a \subseteq b \) is impossible when \( a \) is not normal and \( b \) is normal but not simple.

In (3) we have a new algebraic characterization of the two types concerned.

We shall begin with a consideration of the exception (3). Suppose we wish to find a non-normal ideal \( a \) and a non-simple normal ideal \( b \) such that \( a \subseteq b \). Obviously, we cannot do so if \( \mathfrak{R} = \mathfrak{S} \), so that we must assume the relation \( \mathfrak{R} = \mathfrak{S} \). Then we can select a non-simple normal ideal \( \mathfrak{C} \), observing that \( \mathfrak{C} \cap \mathfrak{C} \), is not normal, \( \mathfrak{C} \), normal but not simple. If we can find an ideal \( \mathfrak{C} \) such that \( \mathfrak{C} \cap \mathfrak{C} \),
is not contained in $c_1 \lor c_2$, we can prove that one of the ideals $(c_1 \lor c_2)c_1$, $(c_1 \lor c_2)c_2$ is not normal. With proper choice of notation, we may arrange that the first of them should have this property; and we can then complete our construction by putting $a=(c_1 \lor c_2)c_1$, $b=c_2$. The necessary proof runs as follows. The orthocomplement of $(c_1 \lor c_2)c_1=((c_1 \lor c_2)c_1)$ relative to $c_1$ is given by

$$(c_1 \lor c_2)c_1 \lor c_2 = (c_1 \lor c_2)c_1 = c_1 \lor c_2.$$ 

Thus if $(c_1 \lor c_2)c_1$ were normal, we would have

$$c_1 \lor c_2 = (c_1 \lor c_2)c_1 \lor c_2.$$ 

Similarly, if $(c_1 \lor c_2)c_2$ were normal, we would have $c_1 \lor c_2 = c_1 \lor c_2$. Hence if both ideals were normal we would have $c_1 \lor c_2 = c_1 \lor c_2$ against hypothesis. We now have to consider the case where the relation $c_1 \lor c_2 = c_1 \lor c_2$ holds whatever the ideal $c_2$. Here we first suppose that $c_1 \lor c_2 = c_1$ is not prime. Then there exists an element $e$ such that $c_1 e, a(c) \lor c + e$. By Th.1.2 the ideal $a(c) \lor c_1$ is normal; but the relations $(a(c) \lor c_1) \lor (a(c) \lor c_1) = a(c) \lor c_1 \lor a'(c) = a(c) \lor c_1 \lor e = e$ show that it is not simple. If we put $c_2 = a(c) \lor c_1$, we therefore have a non-prime normal ideal such that $c_1 \lor c_2$ is not contained in $c_1 \lor c_2$. Consequently the construction carried out above can be repeated with the roles of $c_1$ and $c_2$ interchanged. We are thus left to consider the case where $c_1 \lor c_2$ is a prime ideal contained in $c_1 \lor c_2$ whatever the ideal $c_2$. According to Th.7.1 this is precisely the case where $A$ is of type $(\beta \gamma)$ or $(\beta \gamma^2 \beta)$: for we have $\mathfrak{R} \perp \mathfrak{S}$ and hence $\mathfrak{S} \perp \mathfrak{S}$; and we can take $p = c_1 \lor c_2$ in Th.7.1 (1).

We complete our discussion by showing that in this case the desired construction is impossible. Let $b$ be a non-prime normal ideal and $a$ an ideal contained in $b$. Since $p \subseteq b \lor c \perp e$ we have $p = b \lor c$; and since $a \subseteq b$ we have $a' \subseteq b'$; thus the relations $p \subseteq a \lor a' \subseteq b' \subseteq c$ hold when $a \subseteq c$; and the relations $a' \lor a' = p$, $a \lor a' \subseteq p$ imply $a' \lor a' = c$. If $a$ is not contained in $p$. In one case we have $a = a' \lor a$ and hence $a = a = (a \lor a')a' = (a' \lor a')a'' = a''$, so that $a$ is normal.

Of the twenty-one examples left for us to give after the exceptions (1) and (3) are taken into account, fourteen are trivial. The five examples where $a$ and $b$ belong to the same class are found by taking $a=b$ and choosing $b$ as a non-normal ideal, a non-normal normal ideal, and so on. The four remaining examples where $b$ is semiprincipal but not principal are obtained by taking $b$ as the ideal $e$ in a ring without unit and choosing $a$ as a non-normal ideal, a non-simple normal ideal, and so on. The three remaining examples where $a$ is principal are obtained by choosing $b$ as a non-normal ideal, a non-simple normal ideal, and so on, and putting $a = a(a)$ where $a$ is an element of $b$. Of the two remaining examples where $b$ is principal, one is easily constructed. If $c$ is a non-simple normal ideal, there exists an element $b$ such that $a(b)c$ is not principal, by R Th.26. Since $a(b)c \subseteq a(b)c$, we know from Th.1.1 that $a(b)c$ cannot even be simple. On the other hand, $a(b)c$ is normal by Th.1.2. Thus on putting $a = a(b)c$ and $b = a(b)$ we obtain an example where $a$ is normal but not simple and $b$ is principal.

We begin with the two remaining examples where $b$ is not normal. We first obtain an example where $a$ is simple but not semiprincipal, and $b$ is not normal. Clearly no such example exists unless $\mathfrak{P} \perp \mathfrak{S}$ and $\mathfrak{R} \perp \mathfrak{S}$. We therefore assume these relations. Let $c_2$ be non-normal, $c_1$ simple but not semiprincipal. Then $c_1$ is also simple but not semiprincipal. If $c_2 \lor c_2$ and $c_1 \lor c_2$ were both normal, Th.1.2 would show that $c_1 = (c_1 \lor c_2)(c_1 \lor c_2)$ is also normal against hypothesis. Hence one of these ideals is not normal; and we may arrange our notation so that the first is not normal. The desired example is then obtained by putting $a = c_2$, $b = c_1 \lor c_2$. Next we assume $\mathfrak{P} \perp \mathfrak{P}$, $\mathfrak{P} \perp \mathfrak{S}$, and give an example where $a$ is semiprincipal but not principal and $b$ is not normal. We choose $c$ as a non-normal ideal and $a$ as an element in $c''$ but not in $c$. The ideal $a''(a)$ cannot be normal: for, if it were, we should have

$$c'' \lor a''(a) = (c'' \lor a''(a))'' = (a'' \lor a''(a))'' = a'' \lor a''(a) = a,$$

and hence $c \lor a = a(a) \lor a = a(a)(c'' \lor a''(a)) = a(a) c'' \lor a''(a) = a(a) = a$, contrary to hypothesis. Thus we obtain the desired example by putting $a = a''(a)$, $b = c \lor a''(a)$.
We pass next to the two remaining examples where $b$ is normal but not simple. First, we assume $\mathfrak{B}^*+\mathfrak{S}$, $\mathfrak{S}+\mathfrak{R}$, and give an example where $a$ is simple but not semiprincipal and $b$ is normal but not simple. We choose $c_1$ in $\mathfrak{S}$ and $c_2$ in $\mathfrak{R}$. The ideals $c_1\vee c_2$ and $c_1\vee c_1$ cannot both be simple; for, if they were, the ideal $c_1^n=(c_1\vee c_2)(c_1\vee c_2)$ would be simple by Th. 1.3, against hypothesis. On the other hand, both are normal by Th. 1.2. Since both $c_1$ and $c_2$ are simple but not semiprincipal, we may adjust our notation so that $c_1\vee c_2$ is normal but not simple. We then obtain the desired example by putting $a=c_1$ and $b=c_1\vee c_2$. Next we assume that $\mathfrak{B}=\mathfrak{B}^*$, $\mathfrak{S}+\mathfrak{R}$, and construct an example of an ideal which is not principal, ideal contained in a non-simple normal ideal. We choose $c$ as a non-simple normal ideal and $a$ as an element which is not in the ideal $c\vee c'=e$. Then the ideal $c\vee a'(a)$ is normal by Th. 1.2, but cannot be simple since, if it were, we should have

$$
a(a) = a\langle a \rangle '\vee a(a') \vee (c\vee a'(a))' = a\langle a \rangle ',
$$

against hypothesis. Hence we obtain the desired example by putting $a(a')=a\langle a \rangle '$.

We consider next the two remaining examples where $b$ is simple but not semiprincipal. First we assume that $\mathfrak{R}+\mathfrak{J}$, $\mathfrak{B}^*+\mathfrak{S}$, and give an example where $a$ is non-normal and $b$ is simple but not semiprincipal. If $c_1$ is a non-simple ideal and $c_2$ is a non-semiprincipal simple ideal, the argument used in the discussion of the exception (3) can be applied to show that $(c_1\vee c_2)c_2$ and $(c_1\vee c_2)c_2'$ are not both normal: for if they were we would have $c_1\vee c_2'=c_1\vee c_2''=c$ against hypothesis. Since $c_2$ is simple but not semiprincipal we may suppose our notation so chosen that $(c_1\vee c_2)c_2$ is not normal. Then we obtain the desired example by putting $a=(c_1\vee c_2)c_2$, $b=c_2$. Next, we assume that $\mathfrak{R}+\mathfrak{S}$, $\mathfrak{S}+\mathfrak{R}^*$ and give an example where $a$ is normal but not simple and $b$ is simple but not semiprincipal. We choose $c_1$ as a non-simple normal ideal and $c_2$ as a non-semiprincipal simple ideal. Then $c_1$ is also simple but not semiprincipal. The ideals $c_1c_2$, $c_2c_2$ are both normal by Th. 1.2; but they cannot both be simple since, if they were, we would have $c_1\vee c_2=c_1\vee c_2'=c_1\vee c_2\vee c_2$ so that $c_1$ would be simple contrary to hypothesis. We may suppose our notation adjusted so that $c_1c_2$ is not simple, replacing $c_1$ by $c_2$ if necessary. We then obtain the desired example by putting $a=c_1c_2$, $b=c_2$.

Finally, we come to the one remaining case where $b$ is principal. We assume that $\mathfrak{J}+\mathfrak{R}$ and give an example where $a$ is normal and $b$ is principal. If $c$ is a non-normal ideal, then $c\vee c'=e$ since $c$ is certainly not simple. If $c$ is an element not in $c\vee c'$, then the ideal $(c\vee c')\langle a \rangle$ cannot be normal: for, if it were, we would have $c\vee c'\supset (c\vee c')\langle a \rangle = a\langle a \rangle$, contrary to hypothesis, by virtue of the relations $(c\vee c')\langle a \rangle = (c\vee c')\langle a \rangle ' = (c\vee c')\langle a \rangle = a\langle a \rangle$. Hence we obtain the desired example by putting $a=(c\vee c')\langle a \rangle$ and $b=a\langle a \rangle$.

**Theorem 12.2.** The tables of Th. 1.2 give best possible results with the following exceptions:

1. the tables are to be reduced whenever there is any relation of equality between the classes $\mathfrak{B}$, $\mathfrak{B}^*$, $\mathfrak{S}$, $\mathfrak{R}$, $\mathfrak{J}$;
2. in any ring of type ($\beta^*$) or type ($\beta$, $\beta^*$), $ab$ is normal whenever $a$ is; and $a'$ is simple whenever $a$ is not normal.

In (2) we have further algebraic characterizations of the two types ($\beta^*$), ($\beta$, $\beta^*$).

The elementary properties of the three classes $\mathfrak{B}$, $\mathfrak{B}^*$, $\mathfrak{S}$ show that the first three entries of the first table can be improved only by assuming some equality between these classes. If $a$ is normal but not simple, then $a'$ is a' since $a'\vee a''=a\vee a'=e$. Hence the last two entries can be improved, in view of the inclusion $\mathfrak{R}+\mathfrak{J}$, only by assuming an equality between the classes $\mathfrak{R}$ and $\mathfrak{S}$. In a ring of type ($\beta^*$) or type ($\beta$, $\beta^*$), the relation $a\subset a'$ is known to be normal, shows in accordance with Th. 12.2.3 (that, if $a$ is not normal, then $a'$, and hence also $a'$, is simple. On the other hand a ring with $\mathfrak{R}+\mathfrak{S}$ in which $a\subset a\subset a\subset a'$ implies $a\subset a\subset a'$ is necessarily of type ($\beta^*$) or type ($\beta$, $\beta^*$). This we prove by showing that in such a ring there is exactly one non-normal prime ideal and then applying Th. 7.2. By Th. 11.1, the sum $q$ of all non-simple normal ideals is either prime or equal to $e$. Hence if the ring contains two distinct non-normal prime ideals $p_1$, $p_2$, their product cannot contain $q$. Hence one of them, let us say $p_1$, does not contain $q$; and there must exist a non-simple normal ideal $a$ which is not contained in $p_1$. The ideal $a\mathfrak{p}_1$ is contained in $a$ so that $(a\mathfrak{p}_1)\subset a'$. On the other hand the relation $(a\mathfrak{p}_1)a\mathfrak{p}_1=0$ implies $(a\mathfrak{p}_1)a\mathfrak{p}_1=0$ and hence $(a\mathfrak{p}_1)a'$. We thus find that $(a\mathfrak{p}_1)'=a'$, $(a\mathfrak{p}_1)'=a'=a\mathfrak{p}_1$. Thus the ideal $a\mathfrak{p}_1$ is not normal and the ideal $(a\mathfrak{p}_1)'=a'$ is normal but not simple.
Since this result contradicts our hypothesis, we conclude that there exists at most one non-normal prime ideal. The relation \( \mathfrak{R} \subseteq \mathfrak{S} \) together with Th. 10.2 shows that at least one non-normal prime ideal exists. This completes the discussion.

To show that the entries in the third table are the best possible we apply Th. 12.1. Since \( ab = ba \) we may confine our attention to the entries on or below the principal diagonal. We take \( \alpha \subseteq \beta \) so that \( ab = \alpha \) and assign \( \alpha \) and \( \beta \) to such classes as are called for by the various entries in the table. Thus, unless we encounter one of the special exceptions noted under Th. 12.1, we obtain precisely the entries given in the second table and conclude that all these entries, except possibly the three entries "\( \mathfrak{R} \)" which have to be made in the first column because of the condition \( \alpha \subseteq \beta \), are the best possible. Even in the general case where \( \alpha \) is not assumed to be contained in \( \beta \), an entry "\( \mathfrak{R} \)" is obviously the best possible. Hence the general table gives best possible results unless there are relations of equality between some of the class \( \mathfrak{P} \), \( \mathfrak{P}' \), \( \mathfrak{S} \), \( \mathfrak{R} \), \( \mathfrak{S} \) or, possibly, unless the ring considered is of type \( (\mathfrak{P}_{*}) \) or type \( (\mathfrak{P}_{*}, \mathfrak{P}) \). That an improvement can be made in the latter case is shown as follows: if \( \alpha \) is normal and \( \beta \) arbitrary, then \( ab \subseteq \alpha \) implies that \( ab \) must be normal by Th. 12.1. (3).

We proceed similarly in the case of the second table. Since \( a \lor b = b \lor a \), we may confine our attention to entries on or above the principal diagonal. If we take \( \alpha \subseteq \beta \), we have \( a \lor b = b \). On assigning \( \alpha \) and \( \beta \) to the various classes called for by the various entries in the table, we obtain all entries as given in Th. 1.2 except the entry \( \mathfrak{S} \) for the case where \( \alpha \) and \( \beta \) are normal; and the only possibility of improvement occurs when there is a relation of equality between some of the classes \( \mathfrak{P} \), \( \mathfrak{P}' \), \( \mathfrak{S} \), \( \mathfrak{R} \), \( \mathfrak{S} \). Now if \( \alpha \) is normal but not simple, \( \alpha' \) has the same properties and \( a \lor \alpha' \) is not normal since \( (a \lor a') = (a'a') = a = a \lor a' \). Thus the exceptional entry is also the best possible. It should be observed that the exceptional case (3) of Th. 12.1 does not cause trouble, since we do not work below the diagonal.

In Ths. 12.1 and 12.2 we arranged the proofs so that we could ascertain the precise effect of each possible equality between any two of the classes \( \mathfrak{P} \), \( \mathfrak{P}' \), \( \mathfrak{S} \), \( \mathfrak{R} \), \( \mathfrak{S} \). In the succeeding theorems our analysis will not be quite so detailed.

**Theorem 12.3.** The tables of Th. 1.3 give best possible results with the following exceptions:

1. the tables are to be reduced whenever there is any relation of equality between the classes \( \mathfrak{P} \), \( \mathfrak{P}' \), \( \mathfrak{S} \), \( \mathfrak{R} \), \( \mathfrak{S} \);
2. in a ring of type \( (\mathfrak{P}_{*}) \), the product \( ab \) of a principal ideal \( a \) with an arbitrary ideal \( b \) is semi-principal relative to \( b \);
3. in a ring of type \( (\mathfrak{P}_{*}) \), the product \( ab \) of a principal ideal \( a \) with a normal ideal \( b \) is semi-principal relative to \( b \); but the product \( ab \) of a principal ideal \( a \) with a non-normal ideal \( b \) may be non-semi-principal relative to \( b \).

In (2) and (3) we have further algebraic characterizations of the rings of the respective types \( (\mathfrak{P}_{*}) \) and \( (\mathfrak{P}_{*}) \).

We observe that, because of the equivalence of the relations \( a \lor b = b \lor a \), \( ab = a \), the first table is a special case of the two others. Hence an entry in the second or third table which is the same as the corresponding entry in the first can be improved only in case the latter can be improved. Thus we see that it is sufficient for us to consider the first table, the last two entries in the first row of the second, and the second and third entries in the first column of the third.

The entries "\( \mathfrak{R} \)" in the first table are obviously the best possible. Since \( a \) is principal relative to \( b \) if and only if it is principal, Ths. 1.1 and 12.1 show that no further entries "\( \mathfrak{R} \)" can be made in the first table. Thus the entries "\( \mathfrak{R} \)" in the first table are the best possible also. The entries "\( \mathfrak{S} \)" in the second and third places of the first column of the third table are also the best possible, since, on taking \( \alpha \) as a non-principal ideal which is either semi-principal or simple and \( b \) as a principal ideal contained in \( \alpha \), we see that \( \alpha \) is not principal in \( a \lor b = a \). In view of the inclusion relations \( \mathfrak{P} \subseteq \mathfrak{P} \subseteq \mathfrak{S} \subseteq \mathfrak{S} \), no improvement in the last row of the first table can be made without improving the first entry. That this entry is the best possible we prove as follows. By Th. 12.1, we choose \( \beta \) as a principal ideal and \( a \) as a non-normal ideal contained in \( b \). If \( a \) were normal relative to \( b \), we would have \( a = a \lor b \) and would conclude that \( a \), as the product of the normal ideals \( a' \) and \( b \), is normal.

We show similarly that the entries in the fourth row are the best possible, by examination of the first. Taking \( b \) as a principal ideal
and $a$ as a non-simple normal ideal contained in $b$, we see immediately that $a$ cannot be simple relative to $b$: for, if it were, we would have $a'/a''=a'/a''b'\cap a''b''=b'\cap a''b''=b'/b''=e$ since $a'b$ implies $a'/a''$. We show similarly that the last four entries in the third row are the best possible, by treating the first six of them. We take $b$ as a non-principal semiprincipal ideal and $a$ as a non-semiprincipal simple ideal contained in $b$. Then $a'$ is simple but not semiprincipal and $b'$ is principal. If $a$ were semiprincipal relative to $b$, then $a'b$ would be principal since $a$ is not; and the fact that $b'$ is principal would show that $a'b'=a'/a''b'$ are principal. We thus reach a contradiction, since $a'$ is not principal. Finally we show that the last two entries in the second row are the best possible, by treating the first of them. We take $b$ as a non-simple normal ideal and $a$ as a non-principal semiprincipal ideal contained in $b$. If $a$ were semiprincipal relative to $b$, then $a'b'$ would be principal; and it would follow that $b=b(a'/a')=a'/a''b'$, as the sum of semiprincipal ideals, is semiprincipal. We are thus left to consider the exceptional entries in the second table.

We discuss next the construction of an example of a principal ideal $a(a)$ and a non-simple normal ideal $b$ such that $a'b$ is not semiprincipal relative to $b$. Assuming $\mathfrak{M}=\mathfrak{G}$, we select a non-simple normal ideal $b$. First, let us suppose that this can be done in such a way that $b\cap b'$ is not prime. Then there exists an element $a$ not in $b\cap b'$ such that $a(a)\cap b\cap b'=e$. It follows at once that $a(a)\cap b\cap b'=e$: for otherwise we would have

$$a'(a)\cap b\cap b'=a'(a)\cap b\cap b'=a'(a)\cap b\cap b'=a(a)\cap b\cap b'=b\cap b',$$

contrary to our choice of $a$. If $a(a)$ were principal, then the ideal $a(a)\cap b\cap b'=a'(a)\cap b\cap b'=a'(a)\cap b\cap b'=a(a)\cap b\cap b'=e$. If $a(a)$ were principal we would have $a(a)\cap b\cap b'=e$ in the same way. Thus we can obtain the desired example by putting $a=a(a)$ and using the ideal $b$ originally selected. If now we find two non-simple normal ideals $b_1$ and $b_2$ such that $b_1\cap b_2$ and $b_1\cap b_2'$ are distinct but both prime, we can construct a non-simple normal ideal $b$ such that $b\cap b'$ is not prime. Since $b_1$, $b_2$, $b_1'$, and $b_2'$ are all non-simple normal ideals, since $p_1=p_2\cap b_2'$ cannot contain both $b_1$ and $b_2$, and since $p_1=p_2\cap b_1'$ cannot contain both $b_1$ and $b_2$, we can choose our notation so that $p_1\cap b_2=e$, $p_2\cap b_1'=e$. The ideal $b=b_1b_2$ is normal. We can show that $(b_1b_2)'$ is contained in $p_1p_2$. By symmetry, it is sufficient to prove that $(b_1b_2)'$ is contained in $p_1p_2$. Let $c$ be an element of $(b_1b_2)'$. Then $c$, as an element of $p_1\cap b_2=e$, can be expressed in the form $c=a\cap b$, where $a\in p_1$, $b\in b_2$. Since $b\subset (b_2b_2)'$, we have $b (b_2b_2)'$; and since $a(b_2b_2) \subset b_2$, we have $a(b_2b_2)=a(b_2b_2)=a\subset b_2$. We see therefore that $b\subset p_1$, $c\subset p_1$, and $(b_1b_2)' \subset p_1$. The ideal $b=b_1b_2$ thus satisfies the relation $b\cap b'=b'$, for $b$ is not simple and $b\cap b'=e$. We have thus seen all cases except that where there exists a prime ideal $p$ such that $b\cap b'=p$ for every non-simple normal ideal $b$. Even in this case we can still obtain the desired example if the given ring has no unit. Let $a$ be an element not in $p$. Then the ideal $a(a)$ is not principal and is contained in $p$. Hence if $b$ is a non-principal normal ideal, the ideals $a'(a)\cap b'$ and $a'(a)\cap b'$ cannot both be principal: for, if they were, $a(a)\cap b'=a'(a)\cap b\cap b'=a'(a)\cap b'=a(a)\cap b$, would be principal. We may suppose that $a(a)$ is not principal. If $b(a)$ were principal it would be principal relative to $a(a)$. Thus we would find that its orthocomplement $a(a)\cap b'$ relative to the ring $a(a)$ with unit is principal. We could then conclude that $a(a)\cap b'=a(a)\cap b'$ is principal. On the other hand, since $p'=b\cap b'=b\cap b'=a\cap b$, and $p$ is not normal and Th. 1.6 (2) shows that $a(a)\cap b'$ is not normal. Hence we see that neither $a(a)\cap b'$ nor $a(a)\cap b'$ is principal. We can therefore obtain the desired example by putting $a=a(a)$ and using the ideal $b$. Hence the only possibility of improving the fourth entry in the first row of the second table occurs in the case of a ring with unit in which there is a prime ideal $p$ such that $p=b\cap b'$ for every non-simple normal ideal $b$. By reference to Ths. 7.1-7.4 and 11.1, we see that such a ring must be of type $(7\ell)$ or type $(\ell\ell)$. We conclude by showing that simplifications of the second table occur for rings of these types. Every non-simple ideal in a ring of type $(7\ell)$ satisfies the relation $b=b\cap b'$, where $b$ is the prime ideal of Th. 7.1 (1). If $a$ is an arbitrary element, then one of the elements $a$ and $a'$ is in $p$; hence either $a(a)$ or $a'(a)$ is principal relative to $p$. In particular, if $a$ is not in $p$, then $a(a)$ is semiprincipal but not principal relative to $p$ by Th. 1.8. Considered as an ideal in $p$, the ideal $b$ is simple. Hence the ideal $a(a)\cap b'=a(a)\cap b'$ is semiprincipal relative to $b$ by an application of Th. 1.3 within the ring $p$. Every non-simple normal ideal $b$ in a ring of type $(\ell\ell)$ satisfies the relation $b=b\cap b'$, where $p$ is the prime ideal of Th. 7.3 (1). Hence the argument used above shows that $a(a)$ is semiprincipal relative to $b$ for every $a$ and every such $b$. If we apply the preceding work to the ring $p$, for which $\mathfrak{P}=\mathfrak{P}^*=\mathfrak{G}=\mathfrak{R}$, in both cases, we find that $a$ and $b$ can
be chosen so that $a(a)b$ is not principal: for every non-principal semiprincipal ideal in $p$ is representable in the form $a(a)p$, $a \in p$, by proper choice of $a'$ in $p$. In a ring of type $(\beta^+)$, there exist a principal ideal $a(a)$ and a proper ideal $b$ such that $a(a)b$ is not semiprincipal relative to $b$. By Th. 7.2, there is a non-normal prime ideal $q$ distinct from $p$. If we take $a$ in $q$ but not in $p$, we obtain the desired example by putting $a=a(a), b=pq$. We then know that $a(a)b$ and $a(a')b=a(a)b$ are simple relative to $b$. Considering $a(a)$ and $b$ as ideals in $q$, we know that $a(a)$ is not contained in $b=pq$ and that $b$ is a non-normal prime ideal relative to $q$ by Th. 1.7. Hence $a(a)b$ cannot be principal, by virtue of Th. 1.8. Since $a'$ is in $p$ but not in $q$, a similar argument shows that $a(a')b$ cannot be principal. Hence $a(a)b$ is not semiprincipal relative to $b$. It follows that $b$ is not normal, a fact which can be verified directly also.

**Theorem 12.4.** The table of Th. 1.4 gives best possible results except when reductions are made possible by equalities between some of the classes $\mathfrak{F}, \mathfrak{F}^*, \mathfrak{S}, \mathfrak{R}, \mathfrak{F}$.

The entries "*" in the table are obviously the best possible. If we take $b$ as a non-principal ideal in any of the four classes $\mathfrak{F}^*, \mathfrak{S}, \mathfrak{R}, \mathfrak{F}$ and put $a=b$, we see that $a$ is semiprincipal but not principal relative to $b$ and belongs in $A$ to the same class as $b$. Hence the entries in the second row of the table are the best possible; and so also the last three entries in the last column. Since $a$ is normal (simple) relative to $b$ if it is normal (simple) in $A$, by Th. 1.3, it is evident that the first four entries in the fourth and fifth rows are the best possible. Since the class of simple ideals relative to $b$ contains the class of semiprincipal ideals relative to $b$ no improvement in the third and fourth entries of the third row can be made without violations of the corresponding entries in the second row; but the latter are already known to be the best possible. To complete our discussion we have only to study the second entry in the third row. Using Th. 12.1, we choose $b$ as a non-principal semiprincipal ideal and $a$ as a non-semiprincipal simple ideal contained in $b$. Then $a$ is simple but not principal relative to $b$. In order to show that $a$ is not semiprincipal relative to $b$, we prove that $a'b$ is not principal: if were we would have $a'=a'b \lor a'b'$, where $a'b'$ is principal since $a'$ is simple and $b'$ principal, and hence $a'$ would be principal against hypothesis. Thus $a$ and $b$ furnish the desired example.

**Theorem 12.5.** Under the condition $a \subseteq b$, the ideal $b$ can be assigned to an arbitrary class, and the ideal $a$ to an arbitrary class relative to $b$, with only the following exceptions:

1. the assignment is subject to the general restrictions given in Th. 1.5;
2. the assignment is subject to the limitations imposed by equalities between the classes $\mathfrak{F}, \mathfrak{F}^*, \mathfrak{S}, \mathfrak{R}, \mathfrak{F}$ in the given ring;
3. in a ring of type $(\beta^+)$ or type $(\beta_1, \beta_2)$ the choice of $a$ to be not simple relative to $b$ and of $b$ to be not normal is impossible;
4. in a ring of type $(\beta_1)$, type $(\beta_1, \beta_2)$ or type $(\beta_2, \beta_2)$, the choice of $a$ to be normal but not simple relative to $b$ and of $b$ to be normal but not simple is impossible.
5. in a ring of type $(\omega, \omega)$ the choice of $a$ to be simple but not semiprincipal relative to $b$ and of $b$ to be simple but not semiprincipal is impossible.

In (3) and (4), we have further algebraic characterizations of the various types $(\beta_1), (\beta_1, \beta_1), (\beta_1^*, \beta_2)$ and $(\beta_2, \beta_2)$.

Any relation of equality between the classes of ideals in the given ring will have an effect on the classes of ideals $a$ considered relative to $b$ containing ideal $b$. It is our purpose to show that apart from this and the various other exceptions noted above, $b$ can be assigned to any of the five classes at pleasure and, independently, $a$ to any of the five classes relative to $b$.

We begin with the exceptional cases mentioned in (3), (4), (5). Assuming that $A \not\subseteq \mathfrak{S}$ and hence that $A \not\subseteq \mathfrak{S}$, we construct an example where $b$ is not normal and $a$ is not normal relative to $b$. In a ring with $A \not\subseteq \mathfrak{S}$, there exist ideals $c$ such that $c' \rightarrow c$, $c \leftarrow c' \rightarrow c'$: if $a$ is a non-simple ideal then $c:=a \lor a'$ has these properties. Let $b$ and $c$ be two such non-normal ideals with $b \not\sim c$. By proper choice of notation we may suppose that $bc \not\sim b$. Considering $a:=bc=b$ as an ideal in $b$ we have $(bc)'=b=bc'=b$ so that $a$ is not normal relative to the non-normal ideal $b$. Thus we have the desired example unless there is only one non-normal ideal $b$ with $b=\alpha$. This ideal $b$ must be prime: for $a \subseteq b$ implies $a \subseteq b' \alpha=\alpha$, $a' \alpha=\alpha$ so that either $a$ is non-normal and $a=\beta$ or $a$ is normal and $a=a' \alpha=\alpha$. Now if $a$ is a non-arbitrary ideal then either $a \lor a'=\alpha$ or $a \lor a'$ is a non-normal ideal with $(a \lor a')'=\alpha$ so that $a \lor a'=\alpha$. Hence $b$ has the properties demanded under (1) of Th. 7.1; and our exceptional case proves to fall under the case where the given ring is of type $(\beta^+)$ or type $(\beta_1, \beta_2)$. That all such
rings are actually exceptional is easily proved. In fact we can prove that in such a ring the relation $a/Cb$ implies that $a$ is simple relative to $b$ whenever $b$ is not simple. If $p$ is the prime ideal of Th. 7.1 (1) and if $b$ is not simple we have $b\lor b'=pC_{a\lor a'}$. Hence we obtain the relations $a\lor a'=a\lor b=(a\lor a')\lor p=0, b$, so that $a\lor a'=b$ and $a$ is simple relative to $b$.

We next assume that $A=\mathfrak{U}$ and hence that $A=\mathfrak{F}$, and construct an example where $a$ is normal but not simple relative to $b$ and $b$ is not normal. We choose a normal ideal $c_1$ and a non-normal ideal $c_2$. Let us suppose that we can so choose them that $c_1\lor c_2$ is not contained in $c_1\lor c_2$. We then obtain the desired example by putting $a=c_1\lor c_2, b=c_2\lor c_2$; for $a$ is normal relative to $b$ by virtue of the fact that $c_1$ is normal but $a$ is not simple relative to $b$ because of the relations $a\lor a'=(c_1\lor c_2)\lor c_2=(c_1\lor c_2)\lor c_2+c_2\lor c_2=b_2$. We now examine the conditions under which such a choice of $c_1$ and $c_2$ is impossible. Clearly, if every normal ideal is simple we shall have $c_1\lor c_1=c_1\lor c_1$. In this case we see that the given ring is totally multiplicative by Th. 2.3 (2). Hence any ideal $b$ is totally multiplicative, and an ideal $a$ normal relative to $b$ is necessarily simple relative to $b$, again by Th. 2.3 (3). Now let us assume that $A=\mathfrak{U}$. If we cannot effect the previous construction using non-normal normal ideals $c_1$ and $c_2$ it can only be because for all such ideals $c_1\lor c_1=c_1\lor c_1$. Even in this situation we can effect the previous construction using a non-normal normal ideal $c_1$ and a non-normal ideal $c_2$ unless $c_1\lor c_2\subset c_1\lor c_2$ for every non-normal ideal $c_2$ and every non-normal normal ideal $c_1$. Thus the only possible exceptional case, as we see by reference to Th. 11.1, is that where, first, the sum of all non-normal normal ideals is a prime ideal $c$, given here by $q=c_1\lor c_1=c_1\lor c_1+e$ when $c_1$ and $c_2$ are such ideals, and, second, the sum of all non-normal ideals, being contained in $c$, coincides with it. By Th. 11.1 this case occurs only for rings of type $(\beta_2)$ or $(\beta_1, \beta_2)$. From the preceding work we know that such rings actually constitute exceptional cases here.

We consider next the construction of an example where $b$ is normal but not simple and $a$ is normal but not simple relative to $b$. We assume that $A=\mathfrak{U}$, since no such example could be obtained otherwise. As in the preceding paragraph, we consider the case where non-normal normal ideals $c_1$ and $c_2$ can be obtained such that $c_1\lor c_1$ does not contain $c_1\lor c_2$. The ideals $c_1$, $c_2$ are normal in the given ring and also normal relative to $c_2$ and $c_2$, respectively. However, $c_1\lor c_2$ and $c_2\lor c_2$ cannot both be simple relative to $c_1$ and $c_2$ respectively; for, if they were, we should have $c_1\lor c_2\lor c_2=\mathfrak{V}, c_1\lor c_2\lor c_2=\mathfrak{V}$ and hence $c_1\lor c_1\lor c_2\lor c_2=\mathfrak{V}$, contrary to hypothesis. We then obtain the desired example by putting $a=c_1, b=c_2$ or $a=c_2, b=c_2$. We see therefore that our construction breaks down only in the case where the sum of all non-simple normal ideals is a prime ideal $q$. According to Th. 11.1 this situation occurs only for rings of one of the types $(\beta_2)$, $(\beta_1, \beta_2)$, $(\beta_2, \beta_3)$, $(\beta_1, \beta_2, \beta_3)$. We know already that the first two of these types provide actual exceptional cases. We can easily show that the three remaining types do likewise. Let $p$ be the prime ideal of Th. 7.3 (1). Then if $b$ is normal but not simple, we have $p\subset b''\lor b''=b\lor b''=e$ and hence $p=b\lor b''$. Thus if $a$ is an arbitrary ideal normal relative to $b$, we have $a=a''b, a\lor a''b=\mathfrak{V}b\lor b''=b$ and hence $a\lor a''b=b$, so that $a$ is simple relative to $b$. This completes the discussion.

The exception (5) has already been established in Th. 8.2.

Of the nineteen examples which we still have to give, seventeen offer no difficulty. The five cases where $a$ is principal relative to $b$ are treated by assigning $b$ to any of the five classes $\mathfrak{B}, \mathfrak{B}^*, \mathfrak{U}, \mathfrak{E}, \mathfrak{Z}$ at pleasure and taking $a$ as the principal ideal generated by an element of $b$. The four remaining cases where $a$ is semiprincipal relative to $b$ but not principal are treated by assigning $b$ to any of the four classes $\mathfrak{B}, \mathfrak{E}, \mathfrak{Z}, \mathfrak{U}$ at pleasure and putting $a=b$. The four remaining cases where $a$ is not normal relative to $b$ are treated by assigning $b$ to one of the four classes $\mathfrak{B}, \mathfrak{B}^*, \mathfrak{E}, \mathfrak{Z}$ at pleasure and choosing $a$ as a non-normal normal ideal contained in $b$: for according to Th. 1.4, $a$ cannot be normal relative to $b$ under these circumstances unless it is normal in the given ring. The three remaining cases where $a$ is normal but not simple relative to $b$ are treated by assigning $b$ to any of the classes $\mathfrak{B}, \mathfrak{B}^*, \mathfrak{E}, \mathfrak{Z}$ at pleasure and choosing $a$ as a non-normal normal ideal contained in $b$: for $a$ is then normal relative to $b$ by Th. 1.3 but, by virtue of Th. 1.4, cannot be simple relative to $b$. In a ring for which $\mathfrak{B}^*\neq\mathfrak{E}$, we take $a$ as a non-semiprincipal simple ideal and put $b'=e$, since $e$ is semiprincipal but not principal, in order to obtain an example where $a$ is simple but not semiprincipal relative to $b$ and $b$ is semiprincipal but not principal.

There are still two cases to be studied. First we consider the construction of an ideal $a$ which is simple but not semiprincipal relative to a non-simple normal ideal $b$. We choose a non-simple
normal ideal $c_i$ and a non-semiprincipal simple ideal $c_j$ such that $c_iC_{C_i}$. Since $c_i$ is simple, $c_i \lor c_j$ is normal. On the other hand $c_i \lor c_j$ is not simple: for, if it were, we would have

$$e = (c_i \lor c_j) \lor (c_i \lor c_j) = c_i \lor c_j \lor c_i \lor c_j$$

since $c_iC_{C_i}$ implies $c_i \supseteq c_j$ and we would thus have $e = c_i \lor c_j$ against hypothesis. Now $c_i = c_i(c_i \lor c_i)$ is simple relative to $c_i \lor c_i$ but neither $c_i$ nor $c_i = c_i(c_i \lor c_i) = c_i(c_i \lor c_i)$ can be principal. Hence we obtain the desired example by putting $a = c_i$ and $b = c_j \lor c_i$.

Finally we consider the construction of an ideal $a$ which is simple but not semiprincipal relative to a non-normal ideal $b$. We choose a non-normal ideal $c_i$ and a non-semiprincipal simple ideal $c_j$ such that $c_iC_{C_i}$. The ideal $c_i \lor c_j$ is not normal: for, if it were, $a = c_j(c_i \lor c_j)$ would be normal against hypothesis. Now the ideal $c_i = c_i(c_i \lor c_j)$ is simple relative to $c_i \lor c_j$ but neither $c_i$ nor $c_i = c_i(c_i \lor c_j) = c_j(c_i \lor c_j)$ can be principal. Hence we obtain the desired example by putting $a = c_i$, $b = c_j \lor c_i$. This completes the proof of the theorem.

It is evident that Ths. 1.6, 1.7, and 1.8 give complete information on the topics discussed. The only question left open is that of the existence of a prime ideal of given class either containing or not containing an ideal $a$ of given class. In this connection we have:

**Theorem 12.6.** Let $a$ be an ideal in a Boolean ring $A$ with $a \neq e$.

In order that a have a normal prime divisor it is necessary and sufficient that $a$ contain an atomic element; and in order that every prime ideal divisor of $a$ be normal it is necessary that $a = a'$, where $a$ is an atomic system. In order that there exist a normal prime ideal which is not a divisor of $a$ it is necessary and sufficient that $a$ contain an atomic element; and in order that every prime ideal which is not a divisor of $a$ be normal it is necessary that $s \subseteq a \subseteq s'$ where $s$ is an atomic system.

By R Th 38 we know that an ideal $p$ is prime and normal if and only if $p = a'(a)$ where $a$ is an atomic element. If $a \subseteq p = a'(a)$, then $a' \cap a' = a$ so that $a \in a'$; and, if $a \in a'$, then $a(a) \subseteq a'$, $p = a'(a) \subseteq a'$ so that $p \subseteq a$. If $p$ does not contain $a$, then $a \cap p = e$ so that $a(a) = (a)(a \cap p) = a(a) \cap (a) = a(a) \subseteq a$ so that $a \in a$; if $a \in a$ then $p = a'(a)$ does not contain $a$ or $a$. If $p \subseteq a$, where $p$ is prime, implies that $p$ is normal, then $a$, as the product of its prime ideal divisors, is normal; and $a'$ contains an atomic system $s$. Since $b \subseteq s'$ implies $b \subseteq a'(a)$ for every $a$ in $s$, we see that $s \subseteq a$. On the other hand $s \subseteq a'$ implies $s \subseteq a' = a$. Hence we have $a = a'$.

We shall prove that $a = a'$. If $a \subseteq s'$, then $a \subseteq b = a'(a)$ for every $a$ in $s$ and hence $b \subseteq a'(a)$ for $a \subseteq s$; and also $b \subseteq p$ whenever $p$ is a prime ideal containing $a$. Thus $b$ is contained in every prime ideal and must be the element $0$. Hence $a = a'$ at which we wished to prove. The relations $s \subseteq a$ and $a = a'$ show that $s \subseteq a \subseteq s'$.

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