Soit maintenant $C \subset X$ une courbe simple fermée et $f \in S_1^C$ une homéomorphie. Comme $\dim X = 1$, il existe [46]) une extension $f' \in S_1^X$ de $f$, pour laquelle on a évidemment $f'$ *non* $\sim 1$ sur $C$, d'où $b_1(X) \geqslant 1$.

Soient maintenant $C_1 \subset X$ et $C_2 \subset X$ deux courbes simples fermées différentes. On trouve alors facilement deux transformations $f_1, f_2 \in S_1^{C_1+C_2}$ telles que que 1° $f_i$ transforme $C_i$ par homéomorphie $(i=1,2)$, 2° $f_1(C_2) \neq S_1 \neq f_2(C_1)$. Comme $\dim X = 1$, il existe [46]) alors des extensions $f_1', f_2' \in S_1^X$ et on a

$$f_1' \text{ non} \sim 1 \quad \text{sur } C_1, \qquad f_1' \sim 1 \quad \text{sur } C_2,$$
$$f_2' \sim 1 \quad \text{sur } C_1, \qquad f_2' \text{ non} \sim 1 \quad \text{sur } C_2.$$

Les fonctions $f_1'$ et $f_2'$ sont donc linéairement indépendantes, d'où $b_1(X) \geqslant 2$.

***Théorème 3.*** *Tout continu $X$ localement connexe, de dimension 1 et métriquement homogène est une courbe simple fermée.*

Démonstration. On a en vertu du th. 2 soit $b_1(X)=0$, soit $b_1(X)=1$. Tout continu localement connexe sans courbes simples fermées (c. à d. une dendrite) contient, comme on sait, des points qui le divisent et des points qui ne le divisent pas. Par conséquent, il n'est pas homogène. Le cas $b_1(X)=0$ est donc exclu en vertu de (1). Par conséquent $b_1(X)=1$, d'où, en vertu de (2), l'existence d'une seule courbe simple fermée $C \subset X$.

Envisageons la propriété suivante d'un point $x \in X$: $x$ appartient à une courbe simple fermée contenue dans $X$. Or, par suite de l'homogénéité de $X$, tout point $x \in X$ jouit de cette propriété, c. à d. qu'on a $x \in C$. Par conséquent $X \subset C$, d'où $X = C$.

[46]) voir p. ex. W. Hurewicz, Fund. Math. 24 (1935), p. 144.

# Concerning biconnected sets.

By

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## Introduction.

In their paper on connected sets B. Knaster and C. Kuratowski introduced the idea of a biconnected set and gave several examples of such sets [1]). Each of the biconnected sets constructed by Knaster and Kuratowski contains a dispersion point [2]), and Kuratowski raised the question [3]) whether every biconnected set contains such a point. The main object of the present paper is to prove that *if the hypothesis of the continuum is true, there exists (in a bounded portion of the euclidean plane) a biconnected set which contains no dispersion point.* The proof makes use of the axiom of Zermelo.

[1]) *Sur les ensembles connexes*, Fund. Math. II, pp. 206—255. In this paper, as well as in the present paper, a set of points is said to be connected *if it contains more than one point* and is not the sum of two non-vacuous mutually separated sets. A set of points is said to be biconnected if it is connected and is not the sum of two mutually exclusive connected sets. It is an immediate consequence of theorem XI of the Knaster-Kuratowski paper that a definition equivalent to the last in this: a set of point is biconnected if it is connected and does not *contain* two mutually exclusive connected sets. For generalizations of the idea of biconnected set see P. M. Swingle, *Generalizations of biconnected sets*, Amer. Journal of Math., LIII, no. 2, pp. 385—400.

[2]) A point $p$ of a connected set $M$ is called a dispersion point of $M$ if $M-p$ contains no connected set. For theorems on dispersion points and dispersion sets see J. R. Kline, *A theorem concerning connected point sets*, Fund. Math. III, pp. 238—239, and R. L. Wilder, *On the dispersion sets of connected point-sets*, Fund. Math. VII, pp. 214—228.

[3]) Fund. Math. III, p. 322.

## § 1.

## Preliminary theorems.

***Definition 1*** A family of sets will be said to possess *property B* if there exists a set which contains at least one element of each set of the family, but does not exhaust any set of the family [4]).

***Definition 2.*** If $M$ is a connected set and $C$ a continuum which separates $M$, then the set $M \cdot C$ (which of course is non-vacuous) will be called an *M-boundary*.

***Theorem 1*** [5]). *If $M$ is a biconnected set, then the associated family of $M$-boundaries fails to possess property $B$.*

Proof: Assume that the associated family of $M$-boundaries does possess property $B$. Then there is a subset $Q$ of $M$ such that both $Q$ and $M-Q$ have a point in common with every $M$-boundary. It follows easily [6]) that both $Q$ and $M-Q$ are connected. But this contradicts the hypothesis that $M$ is biconnected.

***Theorem 2.*** *If $M$ is a connected set, and every $M$-boundary has the power of the continuum, then $M$ is the sum of two mutually exclusive connected sets.*

Proof: It is well known that the set of all continua which separate a given connected set has the power of the continuum. Accordingly, the family of all $M$-boundaries consists of $\mathfrak{c}$ sets of power $\mathfrak{c}$. But Bernstein's work shows [7]) that such a family of sets possesses property $B$ Hence by theorem 1 we have that $M$ is the sum of two mutually exclusive connected sets.

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[4]) It seems that results in connection with this property were first obtained by F. Bernstein, *Zur Theorie der trigonometrischen Reihe*, Leipz. Ber. 60, 1908, pp. 325—338. For further material concerning this property see Sierpiński's book, *Hypothèse du Continu*, theorem 1, p. 113. See also E. W. Miller, *On a property of families of sets*, C. R. Soc. Sc. Varsovie 1937.

[5]) The theorems in this paper may be thought of as stated for the euclidean plane. Most of them hold true in much more general spaces.

[6]) See Knaster and Kuratowski, l. c., theorem XXXVII, and S. Mazurkiewicz, *Extension du théorème de Phragmèn-Brouwer aux ensembles non-bornés*, Fund. Math. III, pp. 20—25.

[7]) F. Bernstein, l. c. See also C. Kuratowski and W. Sierpiński, *Sur un problème de M. Fréchet concernant les dimensions des ensembles linéaires*, Fund. Math. VIII, p. 193.



***Theorem 3.*** *If $M$ is a connected set and $P$ is a finite subset of $M$, and if $M-P=M_1+M_2$, where $M_1$ and $M_2$ are mutually separated and $M_i \neq 0$ $(i=1, 2)$, then $M_i+P$ contains a connected set $(i=1, 2)$.*

Proof: The theorem will be proved by induction. In the first place, it is well known that in the case where $P$ consists of a single point $M_i+P$ $(i=1, 2)$ is connected [8]). Let us assume then that $M_i+P$ $(i=1, 2)$ contains a connected set when $P$ consists of $n$ points, and show that the same result holds when $P$ consists of $n+1$ points.

We have then that $M-P=M_1+M_2$, where $M_1$ and $M_2$ are mutually separated, $M_i \neq 0$ $(i=1, 2)$, and $P=p_1+p_2+...+p_n+p_{n+1}$. Let us denote $p_1+p_2+...+p_n$ by $Q$. We have $M-Q=M_1+p_{n+1}+M_2$. If $M_1+p_{n+1}$ is connected then $M_1+P$ certainly contains a connected set. If $M_1+p_{n+1}$ is not connected, we have $M_1+p_{n+1}=M_{11}+M_{12}$, where $M_{11}$ and $M_{12}$ are non-vacuous mutually separated sets and $M_{12} \supset p_{n+1}$. Then $M-Q=M_{11}+(M_{12}+M_2)$, where $M_{11}$ and $(M_{12}+M_2)$ are mutually separated. Now it is given that $M_{11}+Q$ contains a connected set. The same is therefore true of $M_1+P$, since $M_1+P \supset M_{11}+Q$. The same procedure of course proves that $M_2+P$ contains a connected set.

***Theorem 4.*** *If $M$ is a biconnected set which contains no dispersion point, and $P$ is a finite subset of $M$, then $M-P$ is connected.*

Proof: Let us suppose there were a finite subset $P$ of $M$ such that $M-P=M_1+M_2$, where $M_1$ and $M_2$ are non-vacuous mutually separated sets. Now since $M$ can contain no finite dispersion set [9]), either $M_1$ or $M_2$ — let us say $M_1$ — must contain a connected set [10]). But by theorem 3 we know that $M_2+P$ contains a connected set. Accordingly $M$ contains two mutually exclusive connected sets. But this is imposssible since $M$ is biconnected.

If theorem 4 is interpreted in terms of $M$-boundaries and combined with theorem 2 we obtain

***Theorem 5.*** *If $M$ is a biconnected set which contains no dispersion point, then every $M$-boundary is infinite and some $M$-boundaries have a power less than $\mathfrak{c}$* [11]).

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[8]) Knaster and Kuratowski, l. c., theorem VI, p. 210.

[9]) See R. L. Wilder, l. c., theorems 1 and 10.

[10]) See Knaster and Kuratowski, l. c., theorem III.

[11]) It is easily proved for any connected set $M$ that some $M$-boundaries will have the power of the continuum.

§ 2.

## On the existence of a biconnected set which contains no dispersion point.

***Definition.*** A set is called *widely connected* if it is connected and if every connected subset of it is everywhere dense in it [12]).

***Theorem 6.*** *If $M$ is a widely connected set whose associated family of $M$-boundaries does not possess property $B$, then $M$ is biconnected and contains no dispersion point.*

Proof: Suppose $M$ were the sum of two mutually exclusive connected sets $M_1$ and $M_2$. It follows at once that since $M_1$ is dense in $M$, there is a point of $M_1$ on every $M$-boundary. Then, since the family of $M$-boundaries does not possess property $B$, the set $M_1$ must exhaust some $M$-boundary. But the set $M_2$, since it too is dense in $M$, must have a point on this $M$-boundary. But this is impossible, since $M_1$ and $M_2$ are mutually exclusive. Accordingly $M$ is biconnected.

A widely connected set contains no cut-point [13]). It contains, a fortiori, no dispersion point.

***Theorem 7.*** *Let $K$ be an indecomposable continuum and $M$ a connected subset of $K$. Then $M$ is widely connected if no composant of $K$ contains a connected subset of $M$* [14]).

Proof: Suppose that $M$ is not widely connected. Then $M$ contains a connected set $N$ such that $\overline{N}$ (that is, $N$ together with its limit points) is a proper subcontinuum of $K$. Accordingly $\overline{N}$ lies entirely in some one composant of $K$. The same is therefore true of the connected set $N$, and the theorem is proved.

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[12]) P. M. Swingle introduced the idea of such sets and proved their existence in his paper, *Two types of connected sets*, Bull. Amer. Math. Soc. XXXVII, pp. 254—258.

[13]) See P. M. Swingle, l. c., theorem 4, p. 256.

[14]) This theorem was directly suggested by the method used by Swingle to prove the existence of widely connected sets. For the meaning of the terms *indecomposable continuum* and *composant* see Z. Janiszewski and C. Kuratowski, *Sur les continus indécomposables*, Fund. Math. I, pp. 210—222.



***Lemma.*** *On the base $AB$ of a square $ABCD$ let us take a nowhere dense perfect set $P$. At each point of $P$ erect a perpendicular to $AB$ extending to $CD$. Let us denote by $W$ the point set consisting of the points of these perpendiculars. Let $T$ denote a denumerable subset of $W$ dense in $W$. Let $H$ denote a denumerable set such that $H \cdot T = 0$. There exists a simple closed curve $J$ such that:*

1) $H \cdot J = 0$,
2) *$J$ intersects $\mathfrak{c}$ lines of $W$ and lies entirely within $ABCD$,*
3) *any line of $W$ intersects $J$ in at most two points,*
4) *$T \cdot J$ is dense in $W \cdot J$.*

Proof: Let us arrange the points of $H$ in a sequence $h_1, h_2, \dots, h_n, \dots$ In what follows, the projection on $P$ of any point of $W$ will be denoted by the Greek letter corresponding to the Roman letter used for that point.

Let $a$ and $b$ be two points of $T$ within $ABCD$ such that $\alpha$ precedes $\beta$ in the order from $A$ to $B$ and such that there are points of $P$ between $\alpha$ and $\beta$. Join $a$ to $b$ by a simple closed curve $J_1$ lying within $ABCD$ so that:

1. $h_1$ is outside $J_1$,
2. any line perpendicular to $AB$ and arising from a point within $\alpha\beta$ intersects $J_1$ in exactly two points, while any line perpendicular to $AB$ and arising from a point outside $\alpha\beta$ does not interesect $J_1$,
3. the segment within $J_1$ of any line perpendicular to $AB$ is of length $< 1$.

Now take points $t_1^{(1)} t_2^{(1)} \dots t_{k_1}^{(1)}$ of $T$ within $ABCD$ so that $\alpha$, $\tau_1^{(1)}, \tau_2^{(1)}, \dots, \tau_{k_1}^{(1)}$, $\beta$ are in that order on $AB$, so that there are points of $P$ between any two of these points and so that every point of $P$ between $\alpha$ and $\beta$ is at distance $< 1$ from some one of these points. Join $a$ to $t_1^{(1)}$ by a simple closed curve, $t_1^{(1)}$ to $t_2^{(1)}$ by a simple closed curve, etc. Each one of these simple closed curves is taken within $J_1$ so that:

1. $h_2$ is outside the entire chain of simple closed curves,
2. any line perpendicular to $AB$ intersects the chain in at most two points,
3. the segment of any line perpendicular to $AB$ within any simple closed curve of the chain is of length $< \frac{1}{2}$.

The next step, of course, is to take points $t_1^{(2)}, t_2^{(2)}, \dots, t_{k_2}^{(2)}$ within the simple closed curves of the first chain so that $\alpha, \tau_1^{(2)}, \tau_2^{(2)}, \dots, \tau_{k_2}^{(2)}, \beta$ are in that order on $AB$, so that there are points of $P$ between any two points $\tau$ so far obtained, and so that any point of $P$ between $\alpha$ and $\beta$ is at a distance $<\frac{1}{2}$ from some point $\tau$. A new chain of simple closed curves extending from $a$ to $b$ is than constructed within the first chain, having for its vertices the points $t$ so far considered and such that:

1. $h_3$ is outside the entire chain of simple closed curves,
2. (same as above).
3. (same as above, except that $\frac{1}{3}$ replaces $\frac{1}{2}$).

The indicated process is continued indefinitely and it is easy to show that the set of points common to all the chains of simple closed curves (together with their interiors) is an arc $N$ joining $a$ to $b$ and such that:

1) $H \cdot N = 0$,
2) $N$ intersects $\mathfrak{c}$ lines of $W$ and lies within $ABCD$,
3) $T \cdot N$ is dense in $W \cdot N$,
4) any line perpendicular to $AB$ intersects $N$ in at most one point.

Clearly another arc from $a$ to $b$ can be constructed by the same process so as to form with $N$ a simple closed curve $J$ satisfying the conditions mentioned in the lemma.

***Theorem 8.*** *If the hypothesis of the continuum is true, there exists in a bounded portion of the euclidean plane a biconnected set which contains no dispersion point.*

Proof: We shall begin with an indecomposable continuum $K$ and two squares $EFGH$ and $ABCD$ such that,

a) $K$ lies entirely within and upon $EFGH$,
b) $ABCD$ lies entirely within $EFGH$,
c) $K \cdot (ABCD + \text{its interior})$ is a set $W$ related to $ABCD$ in the way specified in the statement of our lemma [15].

[15]) For an indecomposable continuum for which it can easily be seen that such squares exist one may refer to Z. Janiszewski, *Sur les continus irréductibles entre deux points*, Journal de l'Ecole Polytechnique, II Série 16-ème Cahier, 1912, example 6, p. 114.



The non-dense perfect subset of $AB$, from whose points the perpendiculars in the set of points $W$ arise, will again be designated by $P$.

Now the set of all different composants of an indecomposable continuum has the power of the continuum [16]). Let us then denote by $\Omega_{\mathfrak{c}}$ the first transfinite ordinal to correspond to the cardinal number of the continuum, and arrange the different composants of $K$ in a well ordered series of type $\Omega_{\mathfrak{c}}$:

$$C_1, C_2, \dots, C_\alpha, \dots \qquad \alpha < \Omega_{\mathfrak{c}}.$$

Let us likewise arrange in a well ordered series of type $\Omega_{\mathfrak{c}}$ all continua which separate $K$:

$$B_1, B_2, \dots, B_\alpha, \dots \qquad \alpha < \Omega_{\mathfrak{c}}.$$

Now let $\Delta$ be a denumerable subset of $W$ dense in $W$. Consider all subsets $\Delta'$ of $\Delta$ such that $\Delta'$ is dense in some $W$-region [17]). There are $\mathfrak{c}$ such subsets and we shall arrange all of them in a well ordered series of type $\Omega_{\mathfrak{c}}$:

$$\Delta_1, \Delta_2, \dots, \Delta_\alpha, \dots \qquad \alpha < \Omega_{\mathfrak{c}}.$$

We shall proceed to define for every $\alpha < \Omega_{\mathfrak{c}}$ subsets $M_\alpha$ of $K$ and simple closed curves $J_\alpha$ with the following properties:

I. $M_\alpha = 0$ if $B_\alpha \cdot \Delta \neq 0$,

II. $M_\alpha = p_\alpha \in B_\alpha \cdot K$ if $B_\alpha \cdot \Delta = 0$,

III. If $M_\mu \neq 0$ and $M_\nu \neq 0$, where $\mu_\nu < \Omega_{\mathfrak{c}}$, $\nu < \Omega_{\mathfrak{c}}$ and $\mu \neq \nu$, then $M_\mu$ and $M_\nu$ belong to different composants of $K$,

IV. $J_\alpha$ separates $K$,

V. $J_\alpha \sum_{\mu<\Omega_{\mathfrak{c}}} M_\mu = 0$ and $J_\alpha \cdot (\Delta - \Delta_\alpha) = 0$.

Let us show first that if we succeed in constructing the sets $M_\alpha$ and $J_\alpha$ so that conditions I—V are satisfied, then $M = \Delta + \sum_{\mu<\Omega_{\mathfrak{c}}} M_\mu$ is biconnected and contains no dispersion point.

We may notice at the outset that $M$ is a subset of $K$. Then from II we have at once that $M$ is connected.

[16]) See S. Mazurkiewicz, *Sur les continus indécomposables*, Fund. Math. X, pp. 305—310.

[17]) Let $S$ be any square which contains no point in the exterior of $ABCD$ and whose sides are parallel to the sides of $ABCD$. If $S$ contains a point of $W$ in its interior, then the set of all points of $W$ in the interior of $S$ will be called the *W-region* corresponding to $S$.

Now from I and II we have that $M_\alpha$ is either vacuous or else consists of a single point. Accordingly, since $\Delta$ is denumerable we have from III that no composant of $K$ contains a connected subset of $M$. It follows then by theorem 7 that $M$ is widely connected.

Now let $Q$ be any set which contains a point of each $M$-boundary. Then, from IV and from the fact that $M$ is dense in $K$, it follows that $Q$ has a point in common with every set $M\cdot J_\alpha$. Let $Q^*=Q\cdot\sum_{\alpha<\Omega_c} MJ_\alpha$. Now from V and the way in which the sets $\Delta_\alpha$ were defined, it follows easily that $Q^*$ is dense in $W$. But from the first part of V we have at once that $Q^*\subset\Delta$. Hence there is an ordinal $\beta$ such that $Q^*\supset\Delta_\beta$. But from V we have that $M\cdot J_\beta=\Delta_\beta\cdot J_\beta$. Therefore $Q^*$ and, a fortiori, $Q$ itself exhausts $M\cdot J_\beta$. We have shown then that the family of $M$-boundaries fails to possess property $B$. Applying theorem 6 we have that $M$ is biconnected and contains no dispersion point.

Our object now is to show that the sets $M_\alpha$ and $J_\alpha$ can be defined so that conditions I—V hold. Let us begin with $B_1$. If $B_1\cdot\Delta\neq 0$ we shall put $M_1=0$. If $B_1\cdot\Delta=0$ we shall put[18] $M_1=p_1\epsilon B_1\cdot C_1$. Now let $J_1$ be a simple closed curve such that:

a) $J_1$ intersects $\mathfrak{c}$ lines of $W$ and any line of $W$ intersects $J_1$ in at most two points,
b) $J_1\cdot M_1=0$ and $J_1\cdot(\Delta-\Delta_1)=0$,
c) $J_1$ lies within $ABCD$,
d) $J_1\cdot\Delta_1$ is dense in $J_1\cdot W$.

The existence of such a simple closed curve is assured by our lemma. We have shown then that if $\beta=2$ we have:

I′. $M_\alpha=0$ if $B_\alpha\cdot\Delta\neq 0$ $(\alpha<\beta)$,
II′. $M_\alpha=p_\alpha\epsilon B_\alpha\cdot K$ if $B_\alpha\cdot\Delta=0$ $(\alpha<\beta)$,
III′. If $M_\mu\neq 0$ and $M_\nu\neq 0$ where $\mu<\beta$, $\nu<\beta$ and $\mu\neq\nu$, then $M_\mu$ and $M_\nu$ belong to different composants of $K$,
IV′. $J_\alpha$ separates $K$ $(\alpha<\beta)$,
V′. $J_\alpha\cdot\sum_{\mu<\beta}M_\mu=0$ and $J_\alpha\cdot(\Delta-\Delta_\alpha)=0$ $(\alpha<\beta)$,
VI. $J_\alpha$ intersects $\mathfrak{c}$ lines of $W$ and any line of $W$ intersects $J_\alpha$ in at most two points $(\alpha<\beta)$,
VII. $J_\alpha$ lies within $ABCD$ $(\alpha<\beta)$,
VIII. $J_\alpha\cdot\Delta_\alpha$ is dense in $J_\alpha\cdot W$ $(\alpha<\beta)$.

[18]) The set $B_1\cdot C_1$ is non-vacuous since any composant of an indecomposable continuum $K$ is dense in $K$. See Z. Janiszewski and C. Kuratowski, l. c., theorem 8, p. 221.



Now let $\beta$ be any ordinal $<\Omega_c$ such that there exist $M_\alpha$ and $J_\alpha$ fulfilling for all $\alpha<\beta$ conditions I′—V′ and VI, VII, and VIII. We will show that $M_\beta$ and $J_\beta$ can be defined so that these conditions hold for $\alpha\leqslant\beta$ [19]). (More precisely in III′ we will have $\mu\leqslant\beta$, $\nu\leqslant\beta$ and $\mu\neq\nu$, and in V′ we will have $\mu\leqslant\beta$ as well as $\alpha\leqslant\beta$).

Consider $B_\beta$. If $B_\beta\cdot\Delta\neq 0$ we put $M_\beta=0$. Now, *by the hypothesis of the continuum*, $\sum_{\mu\leqslant\beta}M_\mu$ *is denumerable*. Also from the fact that $M_\beta=0$ and from I′ and II′ we have $\Delta\cdot\sum_{\mu\leqslant\beta}M_\mu=0$. We may accordingly apply our lemma to obtain a simple closed curve $J_\beta$ such that conditions IV′, V′ (with $\mu\leqslant\beta$), VI, VII, and VIII hold for all $\alpha\leqslant\beta$. Conditions I′ and II′ will clearly be fulfilled for $\alpha\leqslant\beta$ and III′ for $\mu\leqslant\beta$, $\nu\leqslant\beta$ and $\mu\neq\nu$.

Assume now that $B_\beta\cdot\Delta=0$. We will show that there exists a point $p_\beta$ of $B_\beta$ such that:

1) $p_\beta$ is not a point of any $J_\alpha$ with $\alpha<\beta$,
2) $p_\beta$ is in a composant of $K$ which contains no point $\sum_{\mu<\beta}M_\mu$.

Let us denote by $(W)$ the subset of $W$ actually within $ABCD$. Suppose there are $\mathfrak{c}$ composants which intersect $B_\beta$ in points not in $(W)$. Now, each $J_\alpha$ lies entirely within $ABCD$, and the set $\sum_{\mu<\beta}M_\mu$ is denumerable in virtue of the hypothesis of the continuum. There is accordingly no difficulty in obtaining a point $p_\beta$ satisfying 1) and 2).

If the set of composants which intersect $B_\beta$ in points not in $(W)$ has a power $<\mathfrak{c}$, then there are certainly $\mathfrak{c}$ composants which intersect $B_\beta$ only in points of $(W)$. In fact, since any line of $(W)$ lies entirely in exactly one composant of $K$, there will be $\mathfrak{c}$ composants which intersect $B_\beta$ only in points which lie on lines of $(W)$ arising from interior points of the perfect set $P$. From each of these composants let us select a point $x$ of $B_\beta\cdot(W)$ and let us associate with each point $x$ a sub-continuum $R_x$ of $B_\beta$ which contains the point $x$ and lies entirely within $ABCD$ [20]). Two cases arise.

[19]) We are of course mainly interseted in the possibility of defining $M_\beta$ and $J_\beta$ so that I′—V′ hold for $\alpha\leqslant\beta$. This will be seen however to depend in part upon the realization of VI, VII, and VIIII for $\alpha<\beta$.

[20]) A theorem due to Janiszewski assures the existence of such continua. See Z. Janiszewski, l. c., theorem IV, p. 100.

Case 1. Each continuum $R_x$ is a segment of the line of $W$ which contains the point $x$.

Since $\sum_{\mu<\beta} M_\mu$ is denumerable, there exists a composant $C$ of the sort just described which contains no point of $\sum_{\mu<\beta} M_\mu$. Let us denote the point $x$ selected from $C$ by $z$. The segment $R_z$ is clearly a subset of $C$. Now, by VI, any $J_\alpha$ (with $\alpha<\beta$) has at most two points in common with $R_z$. Furthermore, there are at most a denumerable infinity of simple closed curves $J_\alpha$ for $\alpha<\beta$. Obviously then, there is a point on $R_z$ which is not a point of any $J_\alpha$ for $\alpha<\beta$. Such a point may be taken as our point $p_\beta$.

Case 2. There is a continuum $R_x=R$ which contains points of two different lines of $W$.

In this case since $R$ contains the point $x$, and $x$ lies on a line of $W$ arising from an interior point of $P$, it is clear that the projection on $P$ of the continuum $R$ is an "interval" (i. e. a perfect set segment) $I_1$ of $P$.

We will first show that the projection on $I_1$ of $J_\alpha \cdot R$ for any $\alpha<\beta$ is nowhere dense on $I_1$. Let $I_2$ be any sub-"interval" of $I_1$ and let us suppose that there is a point $q$ of $J_\alpha \cdot R$ which projects into a point of $I_2$. From VIII we have that $q$ is a limit point of $J_\alpha \cdot \Delta$. Then, since $B_\beta \cdot \Delta = 0$, we can find a point $r$ of $J_\alpha \cdot \Delta$ and an arc $A$ of $J_\alpha$ which contains $r$, is free of points of the closed set $B_\beta$ (and is therefore free of points of $R$) and whose projection on $I_2$ is a sub-"interval" $I_3$. It is clear that the arc $A$ may be chosen so that its end points are points of $W$. Now, from the second part of VI, it is clear that the two lines of $W$ which contain the end points of $A$ determine a second arc of $J_\alpha$. If this arc contains a point of $R$, by a repetition of the process just described we obtain a sub-arc of it which contains no point of $R$ and whose projection on $I_3$ is a sub-"interval" $I_4$. It is clear that $I_4$ contains no point which is projected from $J_\alpha \cdot R$. It has been shown, then, that the projection on $I_1$ of $J_\alpha \cdot R$ is nowhere dense on $I_1$.

Now any composant $C$ of $K$ is an $F_\sigma$ [21]). Accordingly, we may write $C \cdot R = \sum_{n=1}^{\infty} F_n$ where $F_n$ is closed. Now consider the projection of $F_n$ on $I_1$. It cannot exhaust an entire sub-"interval" of $I_1$, for in that case all of the lines of $W$ arising from that sub-"interval" would have to be contained in $C$. This, of course, is impossible, since *every* composant of $K$ is dense in $K$. Since the projection of $F_n$ on $I_1$ is a closed set and does not exhaust any sub-"interval" of $I_1$, it must be nowhere dense on $I_1$. Therefore the projection on $I_1$ of $CR$ is of the first category with respect to $I_1$. Now again, in virtue of the hypothesis of the continuum, there are only a denumerable infinity of composants of $K$ which contain a point of $\sum_{\mu<\beta} M_\mu$. The product of their sum with $R$ will project into a set of the first category with respect to the perfect set $I_1$. Since the projection of $\sum_{\alpha<\beta} J_\alpha \cdot R$ is also a set of the first category with respect to $I_1$, it is clear that there is a point $y$ of $I_1$ which is not a point of either of these sets. It follows that we may take any point of $R$ on the line of $W$ arising from y as our point $p_\beta$.

To complete the induction, it remains merely to mention that if we put $M_\beta = p_\beta$, then in the first place I′ and II′ are clearly satisfied for all $\alpha \leqslant \beta$ and III′ is satisfied for $\mu \leqslant \beta$, $\nu \leqslant \beta$ and $\mu \neq \nu$. In the second place, we have $J_\alpha \cdot \sum_{\mu \leqslant \beta} M_\mu = 0$ for $\alpha<\beta$. Furthermore $\sum_{\mu \leqslant \beta} M_\mu$ is denumerable and $\Delta \cdot \sum_{\mu \leqslant \beta} M_\mu = 0$. We can accordingly apply our lemma to prove the existence of a simple closed curve $J_\beta$ such that IV′, V′, VI, VII, and VIII hold for all $\alpha \leqslant \beta$. We have shown now that for every $\alpha < \Omega_c$ there exist sets $M_\alpha$ and $J_\alpha$ such that conditions I—V hold, and our theorem is proved.

[21]) See S. Mazurkiewicz, *Un théorème sur les continus indécomposables*, Fund. Math. I, pp. 35—39.