A combinatorial condition for planar graphs[1].

By

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1. ***Introduction.*** Kuratowski[2]) has proven that a topological graph is planar, i. e., that it can be mapped in a 1—1 continuous manner on the plane, if and only if it contains no subgraph having either of two specific forms. Whitney[3]) has shown that a graph is planar if and only if it has a combinatorial "dual". This paper establishes another combinatorial condition that a graph be planar. This condition may be stated in terms of ordinary combi. natorial concepts:

Theorem I. *A combinatorial graph is planar if and only if the graph contains a complete set of circuits such that no arc appears in more than two of these circuits.*

A graph in the plane divides the plane into a number of regions, and each region is bounded by one or more circuits. These boundary circuits, with certain omissions, can be readily shown to form a complete set, and obviously no arc can appear on more than two of these boundaries. Hence the necessity of our condition is immediate (cf. § 5). The sufficiency proof is largely combinatorial in character. It is advantageous to first reduce the problem to the case of the non-separable graphs (cf. § 3) considered by Whitney. By removing a suitable arc (or arcs) from any non-separable graph we obtain a simpler non-separable graph. We first embed this simpler graph

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[1]) Presented to the American Math. Society, April 11, 1936.

[2]) C. Kuratowski, Fund. Math. Vol. XV (1930), pp. 271—283. He considers a more general point set than a graph.

[3]) H. Whitney, *Non-separable and planar graphs*, Trans. Amer. Math. Soc. 34 (1932), pp. 339—362. We refer to this paper as Whitney I.

in the plane, then show from the assumed condition that the remaining arc (or arcs) can be added in the plane (§§ 4, 5).

By strictly combinatorial means the criterion of Theorem I will be shown equivalent (Theorem II, § 6) to the existence of a dual. This is in turn known[1]) to be equivalent (combinatorially) to the condition of Kuratowski.

2. ***Definitions.*** A *combinatorial graph* G consists of a finite set of elements $a, b, c, \ldots,$ called "arcs", and a finite set of "vertices" $p, q, r, \ldots,$ such that each arc b "joins" exactly two vertices p and q. Then p and q are the *ends* of b or are *on* b, while b may be denoted by pq. We assume that each vertex is on at least one arc[2]). Any set of arcs in G, together with all the vertices on these arcs, form themselves a *subgraph* of G. Each subgraph is determined by its arcs. If $m > 1$ and if $p_1, p_2, \ldots, p_m$ denote distinct vertices, then a subgraph C with arcs $p_1p_2, p_2p_3, \ldots, p_{m-1}p_m, p_mp_1$ is a *circuit*, a subgraph D with arcs $p_1p_2, p_2p_3, \ldots, p_{m-1}p_m$ is a *chain* with *ends* p_1 and p_m, and the chain D is *suspended* if p_1 and p_m are the only vertices of D on three or more arcs of G. If A and B are subgraphs, then $A \cap B$ is the subgraph containing those arcs in both A and B, $A+B$ contains those arcs in either A or B, and $G-A$ contains all arcs of G not in A.

If a graph G has $E(G)$ edges, $V(G)$ vertices, and $P(G)$ connected pieces, then

$$R(G)=V(G)-P(G), \quad N(G)=E(G)-V(G)+P(G) \tag{1}$$

are respectively the *rank* and *nullity* of G. A *sum modulo 2* of circuits $C_1+C_2+\ldots+C_m$ is the subgraph containing all arcs present in an odd number of the C_i's. The circuits $C_1, C_2, \ldots, C_n$ form a *complete set* in G if every circuit in G can be expressed uniquely as a sum mod 2 of certain of the C_i's. Every G contains at least one complete set of $n=N(G)$ circuits.

A planar *topological graph* H consists of a finite number of arcs (1—1 bicontinuous images of line segments) in the plane intersecting, if at all, only at their endpoints. These arcs and their endpoints, considered as elements and possibly renamed, form a combinatorial graph H'. Any such combinatorial graph H' is called *planar*, and H is a *map* of H'.

3. ***Non-separable graphs.*** A graph G is *separable* if it has two subgraphs F_1 and F_2 such that $F_1+F_2=G$, while F_1 and F_2 have no arcs and at most one vertex in common. If F_1 and F_2 have no common vertex, they are not connected; if they have one common vertex p, p is called a *cut vertex* of G. In either event G may be *separated* into F_1 and F_2. F_1 is either non-separable, or can itself be separated into F_3 and F_4, and likewise for F_2. Repetition of this finally yields subgraphs $G_1, G_2, \ldots, G_m$ which are no longer separable. These *non-separable components* of G are always the same, no matter

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[1]) H. Whitney, *Planar Graphs*, Fund. Math. XXI (1933), pp. 73—84 We refer to this paper as Whitney II.

[2]) This exclusion of "isolated" vertices obviously does not affect Theorem I.

how the separation is carried out [1]. On the other hand, two arcs a and b in G are *cyclically connected* if $a=b$ or if there is a circuit in G containing both a and b. It can be proven that the relation "a is cyclically connected to b" is symmetric and transitive and that it holds if and only if [2] a and b belong to the same non-separable component of G. This result gives an "invariant" definition of non-separable components, and so again proves their uniqueness.

Theorem 3.1. *If G is non-separable and has nullity greater than 1, and if R is a circuit in G, then there is* [3] *in R a chain A which is suspended in G and whose removal leaves a non-separable graph $G-A$ of nullity $N(G)-1$.*

The proof proceeds by building up G from a sequence of non-separable subgraphs $H_1 \subset H_2 \subset H_3 \subset \ldots \subset G$. As $N(G)>1$, there is an arc a_1 not in R. Pick a circuit containing a_1 and an arc of R, and call this circuit H_1. If $H_{m-1} \neq G$ has been chosen, H_m is constructed thus: First pick an arc a_m in $G-H_{m-1}$, such that a_m is not in R unless $G-H_{m-1}$ contains only arcs of R. Since G is non-separable, there is a circuit D containing a_m and an arc of H_{m-1}. Denote by A_m the piece of D containing a_m and extending in each direction from a_m to the first vertex of H_{m-1}. Then choose H_m as $H_m = H_{m-1} + A_m$.

Each subgraph H_m is non-separable, as we now show by induction. The circuit H_1 must be non-separable. If H_{m-1} is non-separable, then the arcs of A_m are cyclically connected to the rest of H_m by the circuit consisting of A_m and a chain B in H_{m-1} joining the ends of A_m, so that H_m is non-separable.

We next prove that $R \subset H_m$ implies $H_m = G$. By the construction of H_1, $R \subset H_1$ is impossible. Hence let $m>1$ be the smallest integer for which $R \subset H_m$. Then R is not contained in H_{m-1}, and there is an arc b of R not in H_{m-1}. Denote by E the piece of R which contains b and extends along R in each direction from b up to the first vertex of H_{m-1}. This chain E is in R, while $R \subset H_m$, so

[1] Whitney I, Theorem 12.

[2] A similar proof in Whitney I, Theorem 7. Cf. also Kuratowski et Whyburn, *Sur les éléments cycliques et leurs applications*, Fund. Math. XVI (1930), pp. 305—331.

[3] This includes the special case of this Theorem, which was proven by Whitney (I, Theorem 18), and which does not require $A \subset R$. Theorem 3.1 can also be proven by an induction from Whitney's Theorem.

that $E \subset H_m = H_{m-1} + A_m$. By construction, E has no arcs on H_{m-1}, so $E \subset A_m$. A_m was chosen to have its ends and no other vertices in common with H_{m-1}. E has the same property. Thus E is a subchain of A_m with the same ends as A_m, so that E must equal A_m. Now if $G-H_{m-1}$ were not contained in R, A_m would contain the arc a_m not in R, so that A_m cannot equal E, which is in R. Because of this contradiction, $G-H_{m-1}$ must be contained in R. But H_m contains all of H_{m-1} by construction and all of R by assumption, and so contains all of G. Hence $R \subset H_m$ implies $H_m = G$ and $A_m \subset R$.

Each A_m is a suspended chain in H_m, so that its addition to H_{m-1} increases the number of arcs by 1 more than the number of vertices and so increases the nullity by 1 (cf. (1)). Thus $N(H_m)=m$. The construction process finally stops with an $H_n = G$, and this n must be $N(G)$. The last added chain $A = A_n$ is a suspended chain in $H_n = G$ and is contained in R (see the paragraph above), while $G-A$ has nullity $n-1$, just as required in the Theorem.

Separable graphs can also be built up in a standard fashion:

Theorem 3.2. *If G is separable, then there is a non-separable component H of G such that H and $G-H$ have at most one vertex in common.*

Proof: Pick out any component H_1 of G. If H_1 does not have the desired property, there is a component H_2 in $G-H_1$ with a vertex p_1 in common with H_1. If H_2 does not have the desired property, there is a component $H_3 \neq H_2$ containing a vertex $p_2 \neq p_1$ in common with H_2. Were $H_3 = H_1$, then chains of H_2 and H_1 joining p_1 to p_2 would form a circuit contained in no one component, an impossibility. Hence H_1, H_2 and H_3 are distinct. If H_3 does not have the desired property, we find a new component H_4, etc. The graph is finite, so the process must end with a component H_m with but one vertex in common with $G-H_m$, as required.

4. *The Induction Process.* Consider a *combinatorial graph G which satisfies the condition of Theorem I.* That is, assume that G contains a complete set of circuits

(2) $$C_1, C_2, \ldots, C_n; \qquad n=N(G),$$

which contain no arc more than twice. We call such a set (2) a *2-fold complete set*. The C_i are independent, so that the sum

(3) $$R = C_1 + C_2 + \ldots + C_n \pmod 2$$

is not zero. We call R the *rim* of G. It has the following property:

Lemma 4.1. *For a non-separable G, the rim (3) is a circuit.*

Suppose instead that R is not a circuit. It is then a cycle (i. e., each vertex is on an even number of arcs of R) and so contains a proper subcircuit D. The representation of D in terms of the complete set (2) may be written, if the C_i's are suitably renumbered, as

$$D = C_1 + C_2 + \dots + C_m \qquad \text{(mod 2)}$$

As $D \neq R$, n exceeds m. Hence neither of the graphs

$$F_1 = C_1 + C_2 + \dots + C_m, \qquad F_2 = C_{m+1} + C_{m+2} + \dots + C_n$$

is void. We shall show that G separates into F_1 and F_2. An arc b of $F_1 \cap F_2$ must be in one of the first m C's and in one of the last $(n-m)$ C's. Since (2) is a 2-fold set, b is in no more of the C's. Thus b is in just one summand of D and in two summands of R, mod 2, so that b is in D and not in R, although $D \subset R$. This contradiction shows that F_1 and F_2 have no arcs in common.

As G is non-separable, there is a circuit E containing an arc of F_1 and one of F_2. The representation of E in the complete set is

$$E = \sum{}' C_i + \sum{}'' C_j \qquad \text{(mod 2)},$$

where the first sum runs over some of the indices from 1 to m and the second sum over some of the remaining indices. Since E contains edges of both F_1 and F_2, neither sum is void. The first sum $E' = \sum{}' C_i$ (mod 2) is thus not equal to E. But E' is contained in F_1, so that none of its arcs can be contained in the circuits $C_{m+1}, \dots, C_n$ of F_2. Therefore $E' \subset E$. The circuit E has a proper subcycle E', an impossibility. Hence R is necessarily a circuit.

In a non-separable planar graph there must be a region boundary C_i which abuts on the outside boundary R only along a single chain. The corresponding combinatorial result can be stated thus:

Lemma 4.2. *If G is non-separable and has nullity greater than 1, then there is a suspended chain A in G such that*

(i) *$G - A$ is non-separable of nullity $N(G) - 1$;*

(ii) *A is contained in R and in one and only one C_i, say in C_n,*

(iii) *$C_1, C_2, \dots, C_{n-1}$ form a 2-fold complete set for $G - A$,*

(iv) *The ends p and q of A are on R', where*

(4) $$R' = C_1 + C_2 + \dots + C_{n-1} \qquad \text{(mod 2)},$$

(v) *R' consists of two chains $R - A$ and $C_n - A$ joining p to q.*

Proof: Pick a suspended chain $A \subset R$ as in Theorem 3.1. Then $G - A$ is non-separable, as in (i). Since G is non-separable, each arc of A, and hence all of A, is contained in some circuit. Any circuit is a sum, mod 2, of the C's, so that A is contained in some C_i. By renumbering we can make $A \subset C_n$. Since A is in R and at most two C's, A can be in no other $C_i \neq C_n$, as asserted in (ii). The circuits $C_1, C_2, \dots, C_{n-1}$ are thus in $G - A$, and they are independent in $G - A$ as in G. Since there are at most $n - 1 = N(G - A)$ independent circuits in $G - A$, these $n - 1$ circuits form a 2-fold complete set in $G - A$, as stated in (iii).

We next show $A = C_n \cap R$. For suppose b were an arc in $C_n \cap R$ but not in A. Then b is in $G - A$, hence is in a circuit D of $G - A$. D is representable, by (iii), as a sum mod 2 of some of $C_1, C_2, \dots, C_{n-1}$. Yet b is in C_n and R, hence can be in no other C_i, and so is not in the representation of D, contrary to $b \in D$. No such edge b is possible, so that $A = C_n \cap R$.

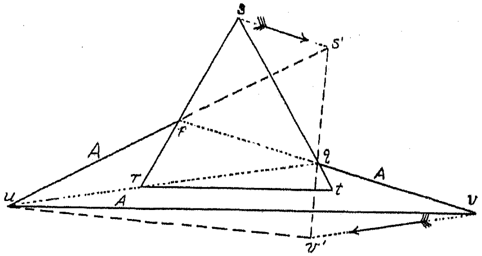
By (4), $R' = R + C_n$ (mod 2), so that R' consists of all arcs in R or in C_n but not in both. Those in both are in $R \cap C_n = A$, so that R' must consist of the arcs of $R - A$ and of $C_n - A$. As R is a circuit, $R - A$ (and also $C_n - A$) is a chain joining the ends p and q of A. $R - A$ is in R', so that its ends p and q are in R'. This gives the last conclusions (iv) and (v) of the Lemma.

5. *The Sufficiency Proof.*

Theorem 5.1. *A non-separable graph G with a 2-fold complete set (2) can be mapped on the plane in such a way that each C_i becomes the boundary of one of the finite regions into which G divides the plane, while R becomes the boundary of the exterior region.*

The proof will be by induction on the nullity $N(G)$. To avoid irrelevant topology, we shall show more explicitly that G can be embedded in such a way that *each arc of G becomes a broken line segment, while R becomes an equilateral triangle.* In the first case, if $N(G) = 1$, G is non-separable and so is simply a circuit (cf. Whitney I, Theorem 10); hence it can be mapped on the plane. If $N(G) > 1$, then Lemma 4.2 yields a suspended chain A such that $G - A$ is non-separable, has a 2-fold complete set, and has a smaller nullity. By the induction assumption, $G - A$ can be mapped on the interior and boundary of a triangle, with R' on the boundary. By Lemma 4.2, (iv), the ends p and q of A are already on the boundary R', and by (v) the remainder $C_n - A$ of C_n is one of the arcs of this boundary joining p to q. Hence we can add a new broken line segment A^* outside the triangle and with ends p and q in such a way that A^* and $C_n - A$ together form the boundary of the new finite region. Then by (v) the boundary of the new exterior region is $R = A^* + (R - A)$. The induction proof will be complete if we show that R can be made an equilateral triangle.

To do this, let $G - A$ be mapped on the triangle with vertices r, s, and t in the figure. Consider first the case when the arc



$C_n - A$ on the edge of this triangle contains two vertices r and t of the triangle. Map A as the broken line $puvq$ in the figure. To make the rim $spuvqs$ a triangle, first shear[1] the half plane which contains s and has the edge pq until the new position ps' of ps is a prolongation of up. Then shear the half plane with the edge uq until $s'qv$ is a straight line $s'qv'$. The new rim is now a triangle $us'v'$. It may be made isosceles by a shear with edge $v's'$, then equilateral by a compression toward this edge. As the shears used carry straight lines into broken lines, we do obtain a broken-line map of G on an equilateral triangle. The other cases, when $C_n - A$ contains other vertices r, s, or t of the map $G - A$, may be similarly treated.

Theorem 5.2. *A graph G with a 2-fold complete set of circuits can be mapped on the plane in such a way that each arc becomes a broken line-segment.*

If G is separable, it can be reduced to non-separable components $H_1, H_2, \ldots, H_m$. Each component H_i which is not a single

[1]) A shear in a half-plane moves each point P parallel to the edge a distance which is a constant times the distance of P from that edge.

arc contains one or more of the given complete set of circuits (2), say the circuits $C_1, C_2, \ldots, C_k$. These circuits form a complete set for H_i: In the first place, they are independent (mod 2); secondly, any circuit D in H_i is expressible in terms of all the C's in the form $D = \sum' C_j + \sum'' C_j$ (mod 2), where the first sum runs over certain indices $j \leqq k$, the second sum over certain indices $j > k$. Then $D - \sum' C_j$ has arcs only in H_i and is equal to $\sum'' C_j$, which has arcs only in other components, and so must be void. $D = \sum' C_j$ (mod 2) is thus represented in terms of $C_1, C_2, \ldots, C_k$, and these circuits are a 2-fold complete set for H_i. Therefore H_i is planar, by Theorem 5.1. Furthermore it will suffice to prove the Theorem for a connected graph G.

The proof proceeds by induction on the number of components. Choose one component, say H_m, so that H_m and $G - H_m$ have but one vertex p in common (Theorem 3.2), and make the induction assumption that $G - H_m$, with components $H_1, H_2, \ldots, H_{m-1}$, has already been mapped on the plane as a graph F. If H_m has nullity zero, it is but a single arc and can be readily added to F in the plane. If $N(H_m) > 0$, then the vertex p appears in at least one of the, circuits C_j of the remaining component H_m. If p is not already on the rim of H_m, we can make this the case by using a new 2-fold complete set for H_m like the old one except that one C_j containing p is replaced by the rim of H_m. Then p is on the new rim C_j. By Theorem 5.1, map H_m on the plane as the graph H_m^*, so that p becomes a point p_1^* on the outside boundary. But p also appears in the map of $G - H_m$ as point p_2^*; it remains to fit these maps together so that p_1^* and p_2^* will coalesce.

To do this, cut the plane of H_m^* into two halfplanes by a line through p_1^* and shear each half plane until the boundary of H_m^* makes at p_1^* an angle α smaller than the angle β between two of the adjacent (straight line) arcs which meet in the corresponding vertex p_2^* of the map of $G - H_m$. Then shrink H_m^* until it can fit inside the region which contains the angle β, place H_m^* in this region and let the points p_1^* and p_2^* coalesce, thus forming the desired cut vertex p^* in a planar map[1]) of $G = (G - H_m) + H_m$. All the transformations involved map the arcs of G into straight or broken line segments.

[1]) Another proof, using more topology, is indicated in Whitney I, Theorem 27.

This establishes the sufficiency of the criterion of Theorem I. It remains to verify its necessity. If each component of G has a 2-fold complete set of circuits, all these circuits together form a 2-fold complete set for G; hence it suffices to consider the non-separable case.

Theorem 5.3. *If G is a non-separable planar graph with $N(G) > 0$, then each finite region into which G divides the plane has as boundary a circuit of G, and these circuits together form a 2-fold complete set, while their sum, mod 2, is the boundary of the external region.*

For $N(G)=1$ this is the Jordan curve Theorem: in general, it follows by a simple induction on $N(G)$, using Theorem 3.1 and the fact that a cross-cut in the interior (exterior) of a Jordan curve cuts the interior (exterior) into two regions with suitable boundaries. This insures that each added arc is on the boundary of at most two regions.

6. *A Combinatorial Construction of Duals.* Whitney's condition for a planar graph, the existence of a combinatorial dual, must be equivalent to the existence of a 2-fold complete set of circuits. We shall establish this equivalence by combinatorial arguments, thus giving another proof for the criteria of Whitney and Kuratowski for planar graphs.

A graph G' is a *dual* of a graph G if there is a 1—1 correspondence between the arcs of G and those of G' such that, if H is any subgraph of G and H' the subgraph containing the corresponding arcs of G', then $R(G'-H')=R(G')-N(H)$; (cf. (1)). A set S of arcs in G is a *cut set* if $G-S$ has either more connected pieces or fewer vertices [1]) than G, and if neither of these results would hold were S replaced by a proper subset of S (Whitney II, p. 76). A graph has a dual if and only if each of its components has a dual (Whitney I, Theorems 23 and 25), and has a 2-fold complete set of circuits if and only if its components have such sets, by the arguments of the preceding section. Hence we restrict our attention to the non-separable case.

Theorem II. *A non-separable graph G has a combinatorial dual if and only if it has a 2-fold complete set of circuits.*

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[1]) Practically, this s cond case means that $G-S$ is disconnected and has one piece which is an isolat d vertex. We have excluded such vertices.

The trivial case when G is a single arc will be omitted. First, let G have the 2-fold complete set (2), with $n>0$, and denote the rim R in (3) by C_{n+1}. Since every arc is on at least one circuit, and on at most two circuits of the complete set, each arc is on exactly two of $C_1, \ldots, C_{n+1}$. Construct a new graph G' with vertices $p_1, \ldots, p_{n+1}$ corresponding to the circuits C_i and with arcs $b'_1, b'_2, \ldots, b'_t$ in 1—1 correspondence with the arcs of G, such that an arc b'_j has the ends p_i and p_k if the corresponding arc b_j in G is on the circuits C_i and C_k. We shall prove that this "circuit graph" G' is in fact a dual of G, by means of a combinatorial analog of the Jordan curve Theorem:

Lemma 6.1. *If D is a circuit in G, then the corresponding subgraph D' in G' is a cut set in the circuit graph G'.*

Proof: D has a representation in terms of the complete set of C's. Renumber the C's so that the first k appear in this representation, and then use the definition (3) of C_{n+1} to obtain

$$D=C_1+C_2+\ldots+C_k=C_{k+1}+C_{k+2}+\ldots+C_{n+1} \pmod 2. \tag{5}$$

There is a corresponding subdivision of the vertices of G' into two sets $p_1, \ldots, p_k$ and $p_{k+1}, \ldots, p_{n+1}$. By the representation (5), D' consists of all the arcs of G' which have one end in the first set of vertices and the other end in the second. Thus in $G'-D'$ no vertex of the first set is connected to any vertex of the second. However, the vertices of the first set are all connected to each other in $G'-D'$. For were some of the vertices, say $p_1, p_2, \ldots, p_j$, connected by no arc of $G'-D'$ to the remaining vertices, then the sum $E=C_1+ +\ldots+C_j \pmod 2$ of the corresponding circuits of G would contain no arc also contained in $C_{j+1}, \ldots, C_k$ and so would contain only arcs in D. Thus E would be a proper subcycle of the circuit D, a contradiction. Similarly the vertices of the second set are all connected to each other in $G'-D'$. The addition of any single arc of D' connects the first set of vertices to the second, so that D' is a cut set, as asserted in the Lemma.

To show G' a dual of G it suffices [1]) (Whitney II, Theorem 2) to show that a circuit in G always corresponds to a cut set of arcs in G', and conversely. The first half is the above Lemma. For the

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[1]) At this point we could alternatively parallel the argument of Whitney I, Theorem 29, that a planar graph has a dual, replacing the Jordan curve Theorem by the above Lemma.

converse, let S' be a cut set in G'. Since by the proof of the Lemma the cut set D' cuts G' into just two pieces, G' itself must be connected. Therefore S' cuts G' into two „pieces". Let the first contain the (re-numbered) vertices $p_1, \ldots, p_k$; the second, the vertices $p_{k+1}, \ldots, p_{n+1}$. Then the corresponding circuits in G give a cycle

$$D = C_1 + C_2 + \ldots + C_k = C_{k+1} + C_{k+2} + \ldots + C_{n+1} \qquad (\text{mod } 2)$$

Every arc on D belongs to one of the first k C's and to one of the remaining C's, and so corresponds to an arc in G' connecting the first set of vertices to the second set. This arc must belong to the cut set, so that $D \subset S$. But the cycle D must contain a subset D_1 which is a circuit. The corresponding set D_1' is, by Lemma 6.1, a cut set in G', although $D_1' \subset S'$. By definition a cut set has no proper subset which is a cut set, so that S must be identical with the circuit D_1, and cut sets do correspond to circuits. This shows G' to be a dual of G.

Conversely, let G have a dual G'. To find a 2-fold complete set of circuits, note first that the non-separability of G implies [1]) that of G'. By the definition of a dual, the nullity n of G is the same as the rank $V(G') - 1$ of G' (cf. (1)), so that G' has $n+1$ vertices $p_1, p_2, \ldots, p_{n+1}$. The set D_i' of all arcs on any one vertex p_i is a cut set in G', for its removal deletes p_i. Consequently the corresponding set of arcs D_i in G must be a circuit, by the Theorem of Whitney quoted previously. These circuits $D_1, D_2, \ldots, D_{n+1}$ contain each arc of G exactly twice, so that their sum is zero (mod 2). No other relation mod 2 is possible. For suppose instead that

$$D_1 + D_2 + \ldots + D_m = 0 \quad (\text{mod } 2); \qquad m < n+1.$$

Then any arc of G on one of these D's is on another one, so that any arc of G' with a first end on one of $p_1, \ldots, p_m$ has a second end on another one of these vertices, and $p_1, \ldots, p_m$ are not connected to the remainder of G'. This contradicts the non-separability of G'. Thus $D_1, \ldots, D_n$ are independent (mod 2), are $n = N(G)$ in number, and so form a 2-fold complete set of circuits. Theorem II is established.

[1]) Whitney I, Theorem 26. The exclusion of isolated vertices is essential here.

Freie Überdeckungen und freie Abbildungen.

Von

Heinz Hopf (Zürich).

Einleitung.

1. Den Ausgangspunkt für unsere Betrachtungen bildet die folgende Eigenschaft der n-dimensionalen Sphären, die zuerst von L. Lusternik und L. Schnirelmann, und dann noch einmal von K. Borsuk entdeckt und bewiesen worden ist [1]):

Satz A_n. *Ist die n-dimensionale Sphäre S^n mit $n+1$ abgeschlossenen Mengen überdeckt, so enthält wenigstens eine dieser Mengen ein antipodisches Punktepaar der Sphäre.*

Die isolierte Stellung dieses interessanten Satzes reizt zu dem Versuch, ihn in ein System allgemeinerer Überdeckungssätze einzuordnen. Beginnt man bei einem solchen Versuch mit der Analyse des Falles $n=1$, so sieht man sofort, daß der Satz A_1 nur ein Korollar des folgenden viel allgemeineren Satzes ist:

Satz A_1^*. *Bilden die abgeschlossenen Mengen F_1 und F_2 eine Überdeckung des zusammenhängenden topologischen Raumes [2]) R, und ist f irgend eine stetige Abbildung von R in sich, so enthält wenigstens eine der beiden Mengen ein Punktepaar $\{x, f(x)\}$.*

Denn da R zusammenhängend ist, gibt es einen Punkt $x \in F_1 \cdot F_2$, und das Punktepaar $\{x, f(x)\}$ gehört der Menge F_i an, wenn diese den Punkt $f(x)$ enthält.

[1]) L. Lusternik et L. Schnirelmann, *Méthodes topologiques dans les problèmes variationnels*, Moskau 1930 (in russischer Sprache), S. 26, Lemma 1. K. Borsuk, *Drei Sätze über die n-dimensionale Sphäre*, Fund. Math. XX (1933), S. 177. Man vergl. auch P. Alexandroff und H. Hopf, *Topologie I* (Berlin 1936), S. 486—487. Dieses Buch wird im folgenden als A.-H. zitiert.

[2]) Unter einem *topologischen Raum* soll immer ein Raum verstanden werden, der die Kuratowskischen Axiome erfüllt; s. A.-H. (cf. Fußnote [1])), S. 37 ff.