On Jordan curves possessing a tangent everywhere.

By

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M. Fréchet[1]) recently raised the following question: if we know that a Jordan curve possesses a tangent everywhere, is it always possible to choose a parameter t for the curve in such a way that dx/dt, dy/dt, dz/dt exist and are not all zero (a) everywhere, (b) almost everywhere with respect to t? In this paper I give first a simple example to show that the answer to (a) is in the negative, even for rectifiable curves. In fact, there may exist a perfect set of points on the curve, at each of which it is impossible for dx/dt, dy/dt, dz/dt all to exist unless they all vanish, however we choose the parameter t. Next I give a theorem which shows that the answer to (b) is affirmative; in fact, for a suitably chosen parameter, not only is the exceptional set of values of t of measure zero, but also the corresponding set of points on the curve is of linear measure zero.

1. Suppose that we are given, in Euclidean 3-space[2]), a certain Jordan arc Γ; by saying that a given line l through a given point P_0 of Γ is the tangent at P_0, we mean, naturally, that given any double cone with P_0 as vertex and l as axis (however small its angle), all points of the curve within a sufficiently small sphere, of centre P_0, lie also within the cone. We now construct a certain arc possessing a tangent everywhere. (The construction actually requires only two-dimensional space; we may suppose that $z=0$ throughout).

[1]) Fund. Math. 26 (1936), 334.

[2]) The arguments used throughout this paper apply equally to n-dimensional space.

Given three numbers $a, b, h, b>a, h>0$, we define the arc $C_1(a, b, h)$ as being made up of the straight lines $P_1P_2, P_2P_3, \ldots, P_7P_8$, where $P_1, P_2, \ldots, P_8$ are respectively the points $(a, 0)$, $(a+l, 0)$, $(a+l, 2h)$, $(a+3l, 2h)$, $(a+3l, h)$, $(a+2l, h)$, $(a+2l, 0)$, $(b, 0)$, where $4l=b-a$. $C_2(a, b, h)$ is an arc obtained from $C_1(a, b, h)$ by rounding the corners by small circular arcs (the process is obvious and we do not define it in detail). $C_2'(a, b, h)$ is to be the mirror-image of $C_2(a, b, h)$ — that is, to be exact, the reflection in the line $x=a$ of the arc $C_2(2a-b, a, h)$. We now define the arc $K(a, b, h)$ as being made up of the points $(a, 0)$, $(b, 0)$ and all the arcs $C_2(a+l/2^n, a+l/2^{n-1}, h/4^n)$ and $C_2'(b-l/2^{n-1}, b-l/2^n, h/4^n)$, where $n=0, 1, 2, \ldots$, and again $4l=b-a$. It is easy to see that in this way we do in fact obtain a Jordan arc (even a rectifiable curve). We note that if (ξ, η) is any point of $K(a, b, h)$, we have

$$(1) \qquad 0 \leqslant \eta \leqslant \frac{8h \times \min(\xi-a, b-\xi)}{b-a}.$$

$K(a, b, h)$ is itself sufficient to show that question (a) must be answered in the negative, but we proceed further. We consider the so-called 'Cantor ternary set' F in $\langle 0,1\rangle$. For each of its complementary intervals (a, b) [(1/3, 2/3), (1/9, 2/9), (7/9, 8/9), and so on] we construct the arc $K[a, b, (b-a)^2]$. The sum of all these arcs and of the points $(x,0)$ where $x \in F$, forms a Jordan arc Γ, in fact a rectifiable curve. We say that the tangent exists at all points of Γ. This is obviously true for points (x, y) such that x is not in F. For all points $(x, 0)$ where x is in F, the line $y=0$ is the tangent to Γ. Suppose that x is in F and not an end-point of a complementary interval. Then if (ξ, η) is any point of the curve, either $\eta=0$, or ξ lies in some interval (a, b) and (ξ, η) is on $K[a, b, (b-a)^2]$. Thus

$$\begin{aligned} 0 \leqslant \eta &\leqslant 8(b-a)^2 \min(\xi-a, b-\xi)/(b-a) \qquad \text{(by (1))} \\ &\leqslant 8(b-a)\,|\xi-x|. \end{aligned}$$

Since $b-a \to 0$ as $a, b \to x$, we see that the line $y=0$ is the tangent. We omit the proof for the case when x is an end-point of a complementary interval; it proceeds on similar lines, using the definition of $K(a, b, h)$.

Now let t be any parameter for the curve Γ such that, as t varies from 0 to 1, Γ is described steadily[3]) in the direction from

[3]) The argument applies even if the curve is not described steadily as t increases.

(0,0) to (1,0). Let t_0 be a value of the parameter such that $x_0=x(t_0)$ lies in F. We say that if dx/dt, dy/dt both exist at t_0, they must both vanish. We give the proof for the case when x_0 is not an end-point of a complementary interval of F, and leave the other case to the reader. We can find a sequence of complementary intervals (a_n, b_n), such that

(i) $a_n>x_0$, $a_n\to x_0$ as $n\to\infty$, and so $b_n-a_n\to 0$;

(ii) all complementary intervals of F between x and a_n have length less than b_n-a_n.

Let $l_n=(b_n-a_n)/4$, $h_n=(b_n-a_n)^2$. Consider the following points $(x_1^{(n)}, y_1^{(n)})$ and $(x_2^{(n)}, y_2^{(n)})$ of $C_2(a_n+l_n, a_n+2l_n, h_n)$, namely $(a_n+7l_n/4, 3h_n/2)$, $(a_n+3l_n/2, h_n/2)$. Let $t_1^{(n)}$, $t_2^{(n)}$ respectively be their parameters as points of Γ, then we can see from the definition of $C_2(a, b, h)$ that

$$t_2^{(n)} > t_1^{(n)}. \tag{2}$$

On the other hand, it follows from (ii) that $a_n-x_0<4l_n$, and so

$$x_2^{(n)}-x_0 < 22\,(x_1^{(n)}-x_0)/23. \tag{3}$$

Since $t_2^{(n)}$, $t_1^{(n)}$ must tend to t_0 as $n\to\infty$, we see from (2) and (3) that if dx/dt exists at t_0, it must vanish. Again, using (1), we have as before, for all points (x, y) of Γ (replacing $b-a$ by its greatest value)

$$0\leqslant y\leqslant 8\,|x-x_0|/3.$$

Hence if $dx/dt=0$ at t_0, $dy/dt=0$ also.

2. We now suppose that we are given a Jordan arc Γ (in three-dimensional space), expressed by given continuous functions $x(t)$, $y(t)$, $z(t)$ of a variable t, $0\leqslant t\leqslant 1$, in such a way that to each point of Γ corresponds a unique value of t. We denote any point of space by P, and any direction by λ, the opposite direction being denoted by $-\lambda$. By $S(P_0, \lambda, \eta)$ we mean the set of points P, other than P_0, such that the line P_0P, in that sense, makes an angle less than η with the direction λ (that is, an open single cone). For convenience we write $f(t)$ for the point with co-ordinates $x(t)$, $y(t)$, $z(t)$. Γ is represented by $P=f(t)$. If dx/dt, dy/dt, dz/dt all exist at t_0, we say that df/dt exists there; if they are all zero we say that df/dt is zero. If H is a set of points in $\langle 0,1\rangle$, $f(H)$ is the set of points $f(t)$ on Γ such that $t\epsilon H$; conversely if H is a set of points on Γ, $f^{-1}(H)$ is the set of values of t such that $f(t)\epsilon H$.

Lemma[4]. *If at each point t of a set e, of finite outer measure, all the derivates of $x(t)$, $y(t)$, $z(t)$ are less than k in modulus, then*[5] $L^*[f(e)]\leqslant K\sqrt{3}\, m_e(e)$.

Given $\varepsilon>0$, let e_n be the set of points t of e such that, if $|h|\leqslant 1/n$,

$$|x(t+h)-x(t)|\leqslant (k+\varepsilon)\,|h|,$$

and the similar inequalities hold for $y(t)$, $z(t)$. Since $e_n\subset e_{n+1}$ and $\sum e_n=e$, we can find n so large that $L^*[f(e_n)]>L^*[f(e)]-\varepsilon$. Given any $\varrho>0$, we can cover e_n by intervals I_s, $s=1, 2, \ldots$, such that $mI_s<\min[\varrho, 1/n]$, all s, and also $\sum mI_s<m_e(e)+\varepsilon$. Then if t_1, t_2 are two points of e_nI_s we have

$$|x(t_1)-x(t_2)|\leqslant (k+\varepsilon)\,|t_1-t_2|,$$

and similar inequalities for y, z. Hence

$$\operatorname{diam}[f(e_nI_s)]\leqslant\sqrt{3}\,(k+\varepsilon)\,mI_s\leqslant\sqrt{3}\,(k+\varepsilon)\,\varrho,$$

and

$$\sum_s \operatorname{diam}[f(e_nI_s)]\leqslant\sqrt{3}\,(k+\varepsilon)\,[m_e(e)+\varepsilon].$$

Since ϱ is arbitrary small,

$$L^*[f(e)]<L^*[f(e_n)]+\varepsilon\leqslant\sqrt{3}\,(k+\varepsilon)\,[m_e(e)+\varepsilon]+\varepsilon.$$

Since ε is arbitrary small, we have the result.

Corollary 1. *If $df/dt=0$ at each point of e, and $m_e(e)<\infty$, then* $L[f(e)]=0$.

Corollary 2. *If the derivates of $x(t)$, $y(t)$, $z(t)$ are all finite at each point of e, and $m(e)=0$, then $L[f(e)]=0$.*

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[4]) The principle of this lemma is well known; see S. Saks, *Théorie de l'Intégrale*, Monografie Matematyczne, Warszawa 1933, 156, 174. However, it seems advisable to give the proof here, since we are now concerned with linear measure on a curve.

[5]) L^* denotes Carathéodory outer linear measure. C. Carathéodory, Göttinger Nachrichten (Math. Phys.), 1914, 404—426.

3. Theorem. *Suppose that with each point $P_0=f(t_0)$ of a certain set $\mathcal{E}$ on Γ, such that $L(\Gamma-\mathcal{E})=0$, is associated a direction λ and an angle $\eta<\frac{1}{2}\pi$, such that all points of Γ sufficiently near to P_0 lie in one of the cones $S(P_0, \lambda, \eta)$, $S(P_0, -\lambda, \eta)$* [6]. *Then we can express the curve in terms of a new parameter as*

$$P=F(\nu) \quad [x=X(\nu),\ y=Y(\nu),\ z=Z(\nu)],$$

in such a way that $dF/d\nu$ exists and is not zero, except at the points of a set N such that $m(N)=L[F(N)]=0$.

Further, the co-ordinates can be expressed as Perron-Stieltjes integrals [7]:

$$X(\nu_1)=X(0)+(PS)\int_0^{\nu_1}\{g(\nu)\,dX(\nu)+(dX/d\nu)_{E-N}\,d\nu\},$$

where $g(\nu)=0$ for ν in $E=F^{-1}(\mathcal{E})$ and $g(\nu)=1$ elsewhere, and $(dX/d\nu)_{E-N}=dX/d\nu$ for ν in $E-N$ and $(dX/d\nu)_{E-N}=0$ elsewhere. Similar expressions hold for $Y(\nu)$, $Z(\nu)$.

Corollary. *If the tangent exists at all points of Γ, then we can express Γ by $P=F(\nu)$ in such a way that $dF/d\nu$ exists, and is not zero, except at the points of a set N such that $m(N)=L[F(N)=0$. The co-ordinates can be expressed by (ordinary) Perron integrals such as*

$$X(\nu_1)=X(0)+\int_0^{\nu_1}(dX/d\nu)\,d\nu,$$

where $dX/d\nu$ is to be replaced by zero at those points where it does not exist.

We say first that $x(t)$, $y(t)$, $z(t)$ are VBG^* [8]) on the set $e=f^{-1}(\mathcal{E})$. We observe that if $P_0=f(t_0)$ is in $\mathcal{E}$, then all points with $t-t_0$ sufficiently small and positive must fall into only one of the cones $S(P_0, \lambda, \eta)$, $S(P_0, -\lambda, \eta)$, since the curve passes through P_0 only once. On the other hand, it is impossible [9]) for $f(t)$ to fall into one cone $S(P_0, \lambda, \eta)$ for all sufficiently small $t-t_0$, of either sign, except for an enumerable set, say e_0, of values of t_0. Thus, in general, $f(t)$ must fall into one cone for small positive $t-t_0$, and in the other for small negative $t-t_0$. Let $\lambda_1, \lambda_2, \dots, \lambda_n, \dots,$ be an enumerable everywhere dense set of directions (for example, the set of directions whose direction-ratios can be expressed rationally).

Let e_{mn} be the set of points t_0 with the property:

$$f(t) \text{ lies in } S[f(t_0),\ \lambda_m,\ \tfrac{1}{2}\pi-1/n] \text{ if } 0<t-t_0\leqslant 1/n,$$
$$f(t) \text{ lies in } S[f(t_0),\ -\lambda_m,\ \tfrac{1}{2}\pi-1/n] \text{ if } 0>t-t_0\geqslant -1/n.$$

It is clear that $\sum_{m,n} e_{mn}$ covers $e-e_0$. We shall show that $x(t)$ is VB^* on each of the sets $e_{mnp}=e_{mn}[p/n\leqslant t\leqslant(p+1)/n]$, and since e_0 is enumerable this will show that $x(t)$ is VBG^* on e. The proof for $y(t)$, $z(t)$ is exactly the same. Let $t_1, t_2, t_1<t_2$, be any two points of e_{mnp}. Since $0<t_2-t_1\leqslant 1/n$, $f(t_2)$ lies in $S[f(t_1), \lambda_m, \frac{1}{2}\pi-1/n]$. For $t_1<t<t_2$, $f(t)$ lies in $S[f(t_1), \lambda_m, \frac{1}{2}\pi-1/n]$ and also in $S[f(t_2), -\lambda_m, \frac{1}{2}\pi-1/n]$. Let $d(t,t')$ be the length of the straight line from $f(t)$ to $f(t')$, and $d_m(t,t')$ its projection on the direction λ_m (with regard to sense). Then the remark just made shows that, for $t_1<t<t_2$, $d_m(t_1,t)$ and $d_m(t,t_2)$ are positive and at least equal to $d(t_1,t)\sin(1/n)$, $d(t,t_2)\sin(1/n)$ respectively. Hence

$$\begin{aligned} d(t_1,t) &\leqslant d_m(t_1,t)\operatorname{cosec}(1/n) \\ &\leqslant \{d_m(t_1,t)+d_m(t,t_2)\}\operatorname{cosec}(1/n) \\ &= d_m(t_1,t_2)\operatorname{cosec}(1/n), \end{aligned}$$

and so we see that $\omega[x(t), \langle t_1,t_2\rangle]$ [the oscillation of $x(t)$] is at most $2d_m(t_1,t_2)\operatorname{cosec}(1/n)$.

Now let $t_0<t_1<\dots<t_N$ be any finite set of points of e_{mnp}. Then we have

$$\begin{aligned} \sum_{i=1}^{N}\omega[x(t), \langle t_{i-1},t_i\rangle] &\leqslant \sum_{i=1}^{N} 2d_m(t_{i-1},t_i)\operatorname{cosec}(1/n) \\ &= 2d_m(t_0,t_N)\operatorname{cosec}(1/n) \end{aligned}$$

which is bounded. That is, $x(t)$ is VB^* on e_{mnp}.

[6]) According to a recent result of Roger (*Comptes rendus*, 200 (1935), 2050, see also S. Saks, Fund. Math. 27 (1936), 151—152), it follows at once that the tangent exists except in a set of linear measure zero. We do not use Roger's theorem in our proof.

[7]) A. J. Ward, Math. Zeitschrift 41 (1936), 578—604.

[8]) Saks, loc. cit., 158 ff.

[9]) G. Durand, Acta Math. 56 (1931), 363—369.

Since $x(t), y(t), z(t)$ are continuous and VBG^* on e, we can find a strictly increasing function $\chi(t)$ such that

$$(4) \qquad \overline{\lim_{t\to t_0}} \left|\frac{x(t)-x(t_0)}{\chi(t)-\chi(t_0)}\right| < \infty$$

[and similarly for $y(t)$, $z(t)$] at each point of e [10]).

Define the function $P=\varphi(\tau)$, $[x=\xi(\tau),\ y=\eta(\tau),\ z=\zeta(\tau)]$, by the equation

$$\varphi(\tau)=f(t) \qquad \text{if} \qquad \chi(t-0)\leqslant\tau\leqslant\chi(t+0).$$

Since χ is strictly increasing, $\xi(\tau)$, $\eta(\tau)$, $\zeta(\tau)$ are one-valued continuous functions. By (4), their derivates are all finite at each point of the set H (say) $=\varphi^{-1}(\mathcal{E})$, except at for most an enumerable set, D. As τ varies from $\chi(0)$ to $\chi(1)$, $\varphi(\tau)$ describes Γ "steadily in the wide sense"; that is, we may have $\varphi(\tau_1)=\varphi(\tau_2)$ for $\tau_1<\tau_2$, but if so, then $\varphi(\tau)=\varphi(\tau_1)$ whenever $\tau_1\leqslant\tau\leqslant\tau_2$.

Let H_1 be a measurable set including H and of the same outer measure; then as the derivates of $\xi(\tau)$, $\eta(\tau)$, $\zeta(\tau)$ are finite for τ in $H-D$, and are measurable functions, they are finite for almost all τ in H_1. By a well-known theorem, it follows that $d\xi/d\tau$, $d\eta/d\tau$, $d\zeta/d\tau$ all exist almost everywhere in H_1. Let H_2 be the sub-set of H_1 for which $d\varphi/d\tau$ is zero, and H_3 the subset of H_1 for which $d\varphi/d\tau$ exists but is not zero. Define the function $\nu(\tau)$ by

$$\nu(\tau_1)=m\{H_3[\chi(0)\leqslant\tau\leqslant\tau_1]\}=m\{[H_1-H_2][\chi(0)\leqslant\tau\leqslant\tau_1]\}.$$

$\nu(\tau)$ will be an increasing function, but not in general a strictly increasing function, of τ. Suppose that $\nu(\tau_1)=\nu(\tau_2)$, $\tau_1<\tau_2$. The points of the interval $\langle\tau_1,\tau_2\rangle$ may be divided into three sets as follows.

(i) The set $\langle\tau_1,\tau_2\rangle-H$. Since $\varphi(\langle\tau_1,\tau_2\rangle-H)\subset\Gamma-E$, we have $L[\varphi(\langle\tau_1,\tau_2\rangle-H]=0$, and so $m[\xi(\langle\tau_1,\tau_2\rangle-H)]=0$.

(ii) The set $HH_2\langle\tau_1,\tau_2\rangle$. Since $d\xi/d\tau=0$ for each point of the set, $m[\xi(HH_2\langle\tau_1,\tau_2\rangle)]=0$ [11]).

(iii) The set $(H-H_2)\langle\tau_1,\tau_2\rangle$. Since the derivates of $\xi(\tau)$ are finite on $H-D$, and $m[(H-H_2)\langle\tau_1,\tau_2\rangle]=0$ since $\nu(\tau_1)=\nu(\tau_2)$, we see that $m[\xi\{(H-H_2)\langle\tau_1,\tau_2\rangle\}]=0$.

[10]) Ward, loc. cit. lemma 6. Since $x(t)$ is continuous, it is easily seen that we can do away with the enumerable set of exceptional points in that lemma. By taking functions χ_1,χ_2,χ_3 as in the lemma for $x(t), y(t), z(t)$ respectively, we obtain by addition our required function $\chi(t)$.

[11]) We are using an obvious analogue of the corollaries to our lemma.

These three results show that $m[\xi(\langle\tau_1,\tau_2\rangle)]=0$. Since $\xi(\tau)$ is continuous, this means that $\xi(\tau_1)=\xi(\tau_2)$. Similarly $\eta(\tau_1)=\eta(\tau_2)$, $\zeta(\tau_1)=\zeta(\tau_2)$; that is, $\varphi(\tau_1)=\varphi(\tau_2)$. Conversely, if $\varphi(\tau_1)=\varphi(\tau_2)$, then $\varphi(\tau)$ is constant in the interval $\langle\tau_1,\tau_2\rangle$, and therefore $\nu(\tau)$ is constant. Thus we see that $\nu(\tau)$ is constant in exactly those intervals where $\varphi(\tau)$ is constant. Thus if we define

$$F(\nu)=\varphi(\tau) \qquad \text{if} \qquad \nu=\nu(\tau),$$

we have a one-valued function $F(\nu)$, and further, $F(\nu)$ uniquely defines ν. The representation of Γ by $P=F(\nu)$ (say $x=X(\nu)$, $y=Y(\nu)$, $z=Z(\nu)$) is the required parametrisation. Since $\nu(\tau)$ is an increasing function of τ, not constant except where $\varphi(\tau)$ is constant, it follows that $X(\nu)$, $Y(\nu)$, $Z(\nu)$ are continuous functions of ν. Now $d\nu/d\tau=1$ for almost all τ in H_3, say for all τ in a set $H_4\subset H_3$. Since $d\varphi/d\tau$ exists and is not zero for τ in H_3, $dF/d\nu$ must exist, and be not zero, for all ν in $\nu(H_4)$.

We wish to show that the complement of $\nu(H_4)$ (in the interval $\langle 0,\nu[\chi(1)]\rangle$), say E_5, satisfies $mE_5=0$, $LF(E_5)=0$. Now $E_5\subset\nu(\text{comp}\,H_3)+\nu(H_3-H_4)$. Now $d\nu/d\tau=0$ almost everywhere in $(\text{comp}\,H_3)$, and the derivates of $\nu(\tau)$ all lie between 0 and 1 everywhere. Hence, by the same argument as before, we see that $mE_5=0$. Again

$$\begin{aligned}F(E_5)&=\Gamma-F[\nu(H_4)]=\Gamma-\varphi(H_4)\\&\subset\Gamma-\varphi(H)+\varphi(HH_2)+\varphi(H-H_2-H_3)+\varphi[H(H_3-H_4)].\end{aligned}$$

Now we know that $L(\Gamma-\varphi(H)]=L(\Gamma-\mathcal{E})=0$. Also $d\varphi/d\tau=0$ on HH_2, and all the derivates of $\xi(\tau),\eta(\tau),\zeta(\tau)$ are finite at each point of $H-D$. Finally, $m(H-H_2-H_3)=m[H(H_3-H_4)]=0$. By the corollaries to our lemma, this shows that $L[F(E_5)]=0$. If we write N for the set of all points where $dF/d\nu$ does not exist, or exists and vanishes, then $N\subset E_5$ and so $m(N)=L[F(N)]=0$.

Since $dX/d\nu$ exists on $E-N$, we have, for all ν_1,

$$X(\nu_1)=X(0)+(PS)\int_0^{\nu_1}\{g_1(\nu)\,dX(\nu)+(dX/d\nu)_{E-N}\,d\nu\},$$

where $g_1(\nu)=0$ for ν in $E-N$ and $g_1(\nu)=1$ elsewhere [12]). To prove

[12]) Ward, loc. cit., Theorem 14.

the required formula we have only to show that $\int_0^{\nu_1}\{g_1(\nu)-g(\nu)\}\,dX(\nu)$ vanishes for all ν_1. This is true [13]), since $g_1(\nu)-g(\nu)$ vanishes except on the set EN, of measure zero, and $X(\nu)$ is VBG^* on $E\supset EN$. (For ν, t are different parameters for the same curve Γ, and $x(t)$ is VBG^* on $e=f^{-1}[F(E)]$.)

The corollary follows at once. E reduces to the whole interval, so that $g(\nu)$ vanishes, and the PS-integral reduces to an ordinary Perron integral.

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[13]) loc. cit., Theorem 10.

Warszawa, 1936.

Ultraconvergence et espace fonctionnel.

Par

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1. Cette note contient un théorème général sur l'existence de séries de puissances *ultraconvergentes* [1]), basé sur l'étude d'un espace fonctionnel.

2. Désignons par R_2 le plan de la variable complexe z. G étant un domaine *simplement connexe*, désignons par $\mathfrak{A}(G)$ l'ensemble de toutes les fonctions holomorphes dans G. Nous définirons dans $\mathfrak{A}(G)$, considéré comme un espace fonctionnel, une *distance* par une méthode due en principe à M. Fréchet. Choisissons dans G un point arbitraire z', posons $\lambda'=\varrho(z', R_2-G)$, enfin désignons pour $0<\lambda<\lambda'$ par $G^*(\lambda)$ l'ensemble des $z\epsilon G$ tels que

(1) $$\varrho(z, R_2-G)>\lambda; \qquad |z-z'|<\frac{1}{\lambda}.$$

Soit $G(\lambda)$ le composant de $G^*(\lambda)$ contenant z'. $G(\lambda)$ est borné, simplement connexe, on a $\overline{G(\lambda_1)}\subset G(\lambda_2)\subset G$ pour $\lambda_1>\lambda_2$ et, pour une suite $\{\lambda_j\}$, la condition $\lambda_j\to 0$ entraîne $\sum_{j=1}^{\infty} G(\lambda_j)=G$.

Posons pour $f, g\,\epsilon\,\mathfrak{A}(G)$:

(2) $$\sigma_G(f,0)=\operatorname*{Inf}_{\lambda}\,(\lambda+\operatorname*{Sup}_{z\epsilon G(\lambda)}|f(z)|),$$

(3) $$\sigma_G(f,g)=\sigma_G(f-g,0).$$

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[1]) Une série de puissances S est dite *ultraconvergente* dans un domaine U contenant le cercle de convergence de S, si une suite de sommes partielles de S converge dans U, la convergence étant uniforme dans tout sous-ensemble fermé et borné de U. L'ultraconvergence a été étudiée par M. M. Jentsch, Ostrowski et Bourion.