$\Phi(f)$ sei eine die Menge aller stetigen Funktionen einer reellen Veränderlichen auf sich selbst eineindeutig abbildende und die Beziehung

$$\Phi(fg) = \Phi(f)\,\Phi(g)$$

erfüllende Operation. Dann ist $\Phi(f)$ von der Gestalt

$$\Phi(f) = \varphi f \varphi^{-1},$$

wo φ eine topologische Abbildung der Zahlengeraden auf sich ist.

Analoge Sätze gelten offenbar für die Unteralgebren der meßbaren oder der differenzierbaren Funktionen.

Dagegen bleibt folgende Frage offen: Sei $T(R_n)$ die Gruppe aller topologischen Abbildungen des Euklidischen n-dimensionalen Raumes R_n. Besitzt $T(R_n)$ nur innere Automorphismen?

On the derivation of additive functions of intervals in m-dimensional space.

By

A. J. Ward (Cambridge, England).

Following on some recent work of Besicovitch[1]) and the present author[2]), Saks[3]) has proved the following theorem:

If a simply additive function of rectangles, $F(R)$, satisfies

$$\underline{\lim}_{\substack{d(R)\to 0\\ R\supset(x,y)}} \frac{F(R)}{|R|} > -\infty$$

at each point (x, y) of a set E_0, then, almost everywhere in E_0,

$$\lim_{\substack{d(S)\to 0\\ S\supset(x,y)}} \frac{F(S)}{|S|}$$

exists, and is equal to

$$\underline{\lim}_{\substack{d(R)\to 0\\ R\supset(x,y)}} \frac{F(R)}{|R|}.$$

Here R denotes any rectangle, and S any square, whose sides are parallel to the co-ordinate axes.

The demonstration made use of a simple geometrical property of rectangles lying in a plane, and the question naturally arose whether the theorem held, and could be proved in the same way, for functions of intervals in space of more than two dimensions. At one time I thought that the necessary geometrical lemma was

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1) A. S. Besicovitch, Fund. Math. 25 (1935), 209—216.
2) A. J. Ward, Fund. Math. 26 (1936), 167—181.
3) S. Saks, Fund. Math. 27 (1936), 72—76.

true for m-dimensional space, but it has been shown that this is not so[4]). Now, however, I am able to give a new theorem on the derivates of an additive function of intervals in m-dimensional space, which includes the above theorem for the plane.

1. Let $F(I)$ be a simply additive function of intervals in m-dimensional Cartesian space. We denote a point $(x_1, x_2, \dots, x_m)$ of the space simply by x, and the interval $p_i \leqslant x_i \leqslant q_i$ $(i=1, 2, \dots, m)$, by $(p_1, q_1;\ p_2, q_2;\ \dots;\ p_m, q_m)$, or, if there is no risk of confusion, simply by (p_i, q_i). If I is any interval, we denote by $l(I)$ the length of its longest side, and by $s(I)$ that of its shortest. We write $r(I)$ for the ratio $s(I)/l(I)$, so that $0 < r(I) \leqslant 1$ for any non-degenerate interval.

We define

$$F^*(x) = \overline{\lim_{\substack{l(I)\to 0 \\ I \supset x}}} \frac{F(I)}{|I|},$$

and, for any a such that $0 < a \leqslant 1$,

$$\overline{F}_a(x) = \overline{\lim_{l(I)\to 0}} \frac{F(I)}{|I|},$$

considering only intervals such that $I \supset x$ and also $r(I) \geqslant a$.

It is clear that if $0 < a_1 < a_2 \leqslant 1$, then

$$\overline{F}_{a_2}(x) \leqslant \overline{F}_{a_1}(x) \leqslant F^*(x).$$

Hence, the limit

$$\overline{F}(x) = \lim_{a \to +0} \overline{F}_a(x)$$

exists, and satisfies

$$\overline{F}_a(x) \leqslant \overline{F}(x) \leqslant F^*(x) \qquad (0 < a \leqslant 1).$$

Similarly, by taking lower limits, we define the lower derivates $F_*(x)$, $\underline{F}_a(x)$, $\underline{F}(x)$. If $0 < a_1 < a_2 \leqslant 1$, then

$$F_*(x) \leqslant \underline{F}(x) \leqslant \underline{F}_{a_1}(x) \leqslant \underline{F}_{a_2}(x) \leqslant \overline{F}_{a_2}(x) \leqslant \overline{F}_{a_1}(x) \leqslant \overline{F}(x) \leqslant \overline{F}^*(x).$$

Theorem 1. *If $\underline{F}_1(x) = +\infty$ at each point of a set E_0, then* $|E_0| = 0$.

Suppose that, if possible, $|E_0| > 0$. Then we can find a subset E_1 of E_0 and a number $\delta > 0$, such that $|E_1| > 0$, and if $x \in E_1$, $I \supset x$, $a(I) = 1$, and $l(I) \leqslant \delta$, then

$$F(I) > 0. \tag{1}$$

[4]) cf. S. Saks, loc. cit., 76, footnote 4.

We take any point of outer density of E_1, and find a cube I_0 surrounding it such that $l(I_0) \leqslant \delta$ and

$$|E_1 I_0| > (1 - 2^{-(m+1)}) \cdot |I_0|. \tag{2}$$

Let $l(I_0) = l_0$, and suppose for simplicity that I_0 is the cube $(0, l_0;\ 0, l_0;\ \dots;\ 0, l_0)$. We choose a number $N > 0$ such that

$$\tfrac{1}{2} N > F(I_0)/|I_0|. \tag{3}$$

Given any point x of $E_1 I_0$, since $\underline{F}_1(x) = \infty$, all sufficiently small cubes containing x satisfy $F(I) > N \cdot |I|$. From this and from (2), it follows that we can find a sub-set E_2 of $E_1 I_0$, and an integer k_0, such that

$$|E_2| > (1 - 2^{-(m+1)}) \cdot |I_0|, \tag{4}$$

and, if I is any cube, containing a point of E_2, such that $l(I) \leqslant 2^{-k_0} \cdot l_0$, then

$$F(I) > N \cdot I. \tag{5}$$

We now divide I_0 into 2^m cubes of side $\frac{1}{2} l_0$ (by the hyperplanes $x_i = \frac{1}{2} l_0$). Each of these cubes contains a point of E_2, by (4). If any of them satisfies (5), we call it a "black cube". If any of the cubes is *not* 'black', we again divide it into 2^m cubes of half the side. If one of these cubes, in its turn, contains a point of E_2 and satisfies (5), then we call it a black cube; if however it contains no point of E_2, then we call it a "white cube" (whether it satisfies (5) or not). Now if one of these cubes of the second subdivision is neither white nor black, we again divide it, and classify its 2^m sub-cubes as white or black, and so on. From the definition of E_2, it is clear that, if we reach the k_0th stage, any cube which is not "white" must satisfy (5), and so be "black". That is, the process terminates at or before the k_0th stage, and we have divided I_0 into a finite number of non-overlapping cubes, each of which is either "white" or "black". Let the white cubes be called W_i, $i = 1, 2, \dots, n_1$, say. From (4) we see that

$$\sum_i |W_i| < 2^{-(m+1)} \cdot |I_0|.$$

Now each white cube arises from the subdivision of a cube, of twice the side, which is not white, that is, which contains a point of E_2.

It follows that all the white cubes can be covered by a finite set of cubes, say V_i, $i=1, 2, \ldots, n_2$, say, such that

$$(6) \qquad \sum_i |V_i| \leqslant 2^m \cdot \sum_i |W_i| < \tfrac{1}{2} |I_0|,$$

and each V_i contains a point of E_2 and therefore satisfies (1), since $l(V_i) \leqslant l(I_0) \leqslant \delta$. Also any V_i either contains, or does not overlap, any of the black cubes. Let B_i, $i=1, 2, \ldots, n_3$, say, be those black cubes which are not contained in any V_i; then, from (6),

$$\sum_i |B_i| > \tfrac{1}{2} |I_0|.$$

Since each B_i satisfies (5), and each V_i, (1), and since $F(I)$ is additive, we have

$$F(I_0) > N \cdot \sum_i |B_i| > \tfrac{1}{2} N \cdot |I_0|,$$

in contradiction with (3). Thus the theorem is proved.

2. *Theorem 2.* *If $F_*(x) > -\infty$ at each point of a set E_0, then, almost everywhere in E_0, $F_*(x) = \underline{F}_{\frac{1}{2}}(x)$.*

Suppose that the theorem is false. Then we can find, in succession, $\eta > 0$, a number A, and $\delta > 0$, such that, for each point x of a set $E \subset E_0$, such that $|E| > 0$, we have

$$(7) \qquad \underline{F}_{\frac{1}{2}}(x) > F_*(x) + 2.3^{m-1}\eta, \qquad A < F_*(x) \leqslant A + \eta,$$

and so

$$(8) \qquad A + 2.3^{m-1}\eta < \underline{F}_{\frac{1}{2}}(x);$$

and finally, whenever $I \supset x$ and $l(I) \leqslant \delta$, then

$$(9) \qquad F(I) > A \cdot |I|,$$

and if also $r(I) \geqslant \frac{1}{2}$, then

$$(10) \qquad F(I) > (A + 2.3^{m-1}\eta)|I|.$$

We shall show that, if I is any interval with the following properties, for some integer k, $1 \leqslant k \leqslant m$:

(a) $l(I) \leqslant \delta$;

(b) $|EI| > (1 - 3^{k-m}) \cdot |I|$;

(c) k of the sides of I, say $h_1, h_2, \ldots, h_k$, satisfy

$$(11) \qquad s(I) \leqslant h_i < 2 \cdot s(I);$$

then

$$(12) \qquad F(I) > (A + 2 \cdot 3^{k-1}\eta)\, |I|.$$

The demonstration is by downward induction with respect to k. Suppose that our statement is true for a certain $k = m'+1 \leqslant m$, and consider an interval I satisfying the conditions (a)—(c) with $k = m'$. Let us suppose for simplicity that I is the interval $(0, q_i)$, and that the m' sides of I which are known to satisfy (11) do not include the first side, of length q_1. We can find an integer N and a length h such that

$$(13) \qquad Nh = q_1$$

and

$$s(I) \leqslant h < 2s(I);$$

for example, we can take N as the integral part of $q_1/s(I)$.

Consider the N intervals I_j, namely

$$[(j-1)h, jh;\ 0, q_2;\ \ldots;\ 0, q_m], \qquad j = 1, 2, \ldots, N.$$

Such an interval clearly satisfies conditions (a) and (c) with $k = m'+1$; if it also satisfies (b), that is, if

$$|EI_j| > (1 - 3^{m'+1-m}) \cdot |I|,$$

we say that it is of class A; otherwise we say that it is of class B. Since I satisfies condition (b) with $k = m'$, we see that the number n' of intervals of class B satisfies $n' < \frac{1}{3} N$.

Now we can group the intervals I_j together into (larger) intervals, each of which contains *exactly one* interval of class A. Some of these, which we will call J_i, $i = 1, 2, \ldots, n$, say, will consist simply of one interval of class A. It is easy to see that the number, n, of such intervals satisfies

$$(14) \qquad n \geqslant N - 2n' > \tfrac{1}{3} N.$$

Each of the remaining intervals, say R_i, $i=1, 2, \ldots, n''$, say, contains an interval I_j of class A, and therefore a point of E. Since $R_i \subset I$, $l(R_i) \leqslant \delta$ and so R_i satisfies (9); that is

$$F(R_i) > A \cdot |R_i|.$$

On the other hand, by the inductive hypothesis, each interval of class A satisfies (12) with $k=m'+1$, therefore, in particular,

$$F(J_i) > (A+2.3^{m'}\eta) \cdot |J_i|.$$

Since F is additive, we obtain, from these last two inequalities,

$$F(I) > A \cdot |I| + 2.3^{m'}\eta \cdot \sum_{i=1}^{n} |J_i|;$$

that is, by (13) and (14),

$$\begin{aligned} F(I) &> (A+2.3^{m'}\eta \cdot n/N) \cdot |I| \\ &> (A+2.3^{m'-1}\eta) \cdot |I|. \end{aligned}$$

Thus we have proved our statement for $k=m'$. Now it is true for $k=m$, as we see from the relation (10); it follows that it is true for all k, $1 \leqslant k \leqslant m$. But any interval whatever satisfies condition (c) with $k=1$; so we have proved that if $l(I) \leqslant \delta$ and $|EI| > (1-3^{-m+1}) \cdot |I|$, then

$$F(I) > (A+2\eta) \cdot |I|.$$

It follows that if x is any point of outer density of E (in the strong sense), then

$$F_*(x) \geqslant A+2\eta.$$

Since almost every point of E is such a point of outer density, we have a contradiction with (7), and so the theorem is proved.

3. *Lemma.* *If I is any interval, and α a fixed number such that $0<\alpha<1$, we can divide I into a finite set of non-overlapping intervals I_n such that, for each n,*

$$r(I_n) \geqslant \alpha \quad \textit{and} \quad s(I_n) \geqslant (1-\alpha) \cdot s(I).$$

Let I be the interval (p_i, q_i), and let $q_i - p_i = h_i$, $i=1, 2, \ldots, m$. Define the integers a_i such that

$$a_i(1-\alpha) \cdot s(I) \leqslant h_i < (a_i+1)(1-\alpha) \cdot s(I). \tag{15}$$

Since $h_i \geqslant s(I)$, we have, for all $i=1, 2, \ldots, m$, from (15), $a_i+1 > 1/(1-\alpha)$ and so $a_i/(a_i+1) > \alpha$. Hence, from (15),

$$(1-\alpha) \cdot s(I) \leqslant h_i/a_i < \alpha^{-1}(1-\alpha) \cdot s(I). \tag{16}$$

We can divide I into $a_1 a_2 \ldots a_m$ equal intervals of sides h_i/a_i, and it follows from (16) that each of these satisfies the required conditions.

Theorem 3. *If $0<\alpha<1$, $0<\beta<1$, and $\underline{F}_\alpha(x) > -\infty$ at each point of a set E_0, then, almost everywhere in E_0,*

$$\overline{F}_\beta(x) = \underline{F}_\alpha(x).$$

Suppose that the theorem is false. We write for convenience

$$K = \alpha^{-m}\beta^{-m}2^{5m+3}. \tag{17}$$

We can find, in succession, $\eta>0$, a number A, and $\delta>0$, such that, for each point x of a set $E \subset E_0$, such that $|E|>0$, we have

$$\begin{aligned} &\overline{F}_\beta(x) > \underline{F}_\alpha(x) + K\eta, \\ &A < \underline{F}_\alpha(x) \leqslant A+\eta, \end{aligned} \tag{18}$$

and so

$$A + K\eta < \overline{F}_\beta(x); \tag{19}$$

and finally, whenever $I \supset x$ and

$$l(I) \leqslant \delta \quad \text{and} \quad r(I) \geqslant \alpha, \tag{20}$$

then

$$F(I) > A \cdot |I|. \tag{21}$$

We shall show that if I is any interval satisfying (20), and such that

$$|EI| > (1-2^{-(2m+2)}) \cdot |I|, \tag{22}$$

then

$$F(I) \geqslant (A+2\eta) \cdot |I|.$$

It will follow that, at any point of outer density of E,

$$\underline{F}_\alpha(x) \geqslant A+2\eta.$$

Since almost every point of E is a point of outer density of E, we shall have a contradiction with (18), and the theorem will be proved.

We consider any one fixed interval I_0 satisfying (20) and (22), and we may suppose for convenience, by a trivial change of co-ordinates, that I_0 is the interval $(0, h_i)$, $i=1,2,\dots,m$. Let $s(I_0)=s_0$, $l(I_0)=l_0$.

Let E_1 be the set of points of E, interior to I_0, where E has outer density 1. Then

(23) $$|E_1| = |EI_0| > (1-2^{-(2m+2)})\cdot|I_0|.$$

Let x' be any point of E_1. By (19) and the definition of E_1, we can find an interval I_1 containing x', as small as we please, such that

(24) $$r(I_1) \geqslant \beta,$$

(25) $$F(I_1) > (A+K\eta)\cdot|I_1|,$$

and also, if I is any interval containing x', such that

$$l(I) \leqslant 16\,\alpha^{-1}\cdot l(I_1),$$

then

(26) $$|EI| > (1-\epsilon)\cdot|I|,$$

where $\epsilon=\alpha^m(1-\alpha)^m\beta^m 2^{-4m}$.

Let I_1 be the interval (p_i, q_i), and let $s(I_1)=s_1$, $l(I_1)=l_1$. Define the integers k and a_i, $i=1,2,\dots,m$, such that

(27) $$2^{-(k+3)}s_0 < l_1 \leqslant 2^{-(k+2)}s_0,$$

(28) $$a_i\cdot 2^{-(k+1)}h_i \leqslant x_i' < (a_i+1)\cdot 2^{-(k+1)}h_i.$$

We remark also that

(29) $$p_i \geqslant x_i' - l_1, \qquad q_i \leqslant x_i' + l_1.$$

Let

$$a_i' = \tfrac{1}{2}a_i \qquad \text{if} \quad a_i \quad \text{is even},$$
$$a_i' = \tfrac{1}{2}(a_i+1) \qquad \text{if} \quad a_i \quad \text{is odd}.$$

Consider the interval $I_2=(c_i, d_i)$, say, namely

(30) $$\{(a_i'-1)\cdot 2^{-k}h_i,\ (a_i'+1)\cdot 2^{-k}h_i\}.$$

By taking I_1 sufficiently small, we may suppose that $l(I_2)\leqslant\delta$.

We see from (27), (28) and (29), since $h_i\geqslant s_0$, for all i, that

(31) $$p_i \geqslant c_i + 2^{-(k+2)}s_0,$$

(32) $$q_i \leqslant d_i - 2^{-(k+2)}s_0.$$

Now we can divide I_2-I_1 into the $2m$ non-overlapping intervals:

$$
\begin{array}{lllll}
(c_1, p_1; & c_2, d_2; & \dots\dots\dots; & & c_m, d_m),\\
(q_1, d_1; & c_2, d_2; & \dots\dots\dots; & & c_m, d_m),\\
(p_1, q_1; & c_2, p_2; & c_3, d_3; \dots; & & c_m, d_m),\\
(p_1, q_1; & q_2, d_2; & \dots\dots\dots; & & c_m, d_m),\\
\dots & \dots & \dots\dots\dots & & \dots,\\
(p_1, q_1; & \dots; & p_{m-1}, & q_{m-1}; & q_m, d_m).
\end{array}
$$

Each of these, by (24), (27), (31), and (32), satisfies

$$s(I) \geqslant \min[2^{-(k+2)}s_0,\ s_1]$$
$$> \beta\, s_0\, 2^{-(k+3)}.$$

Using the lemma, we see that these intervals can be further divided into a finite set of non-overlapping intervals, say R_i, $i=1,2,\dots,N$, such that, for each i,

$$r(R_i) \geqslant \alpha,$$
$$s(R_i) \geqslant (1-\alpha)\,\beta\cdot 2^{-(k+3)}s_0.$$

But

$$|I_2| = 2^{-m(k-1)}\cdot|I_0|$$
$$\leqslant 2^{-m(k-1)}l_0^m$$

(33) $$\leqslant 2^{-m(k-1)}\alpha^{-m}s_0^m,$$

since $r(I_0)\geqslant\alpha$.

Hence

$$|R_i| \geqslant [s(R_i)]^m$$
$$\geqslant (1-\alpha)^m\beta^m 2^{-m(k+3)}s_0^m$$

(34) $$\geqslant \alpha^m(1-\alpha)^m\beta^m 2^{-4m}\cdot|I_2|.$$

Again,

$$l(I_2) = 2^{-(k+1)}l_0$$
$$\leqslant \alpha^{-1}\cdot 2^{-(k+1)}s_0$$
$$\leqslant 16\,\alpha^{-1}\cdot l_1, \qquad \text{by (27)}.$$

It follows that I_2 satisfies (26); therefore, by (34), each R_i contains a point of E, and so, since $l(R_i)\leqslant l(I_0)\leqslant\delta$ and $r(R_i)\geqslant\alpha$, satisfies (21). Using (25), we obtain by addition

$$F(I_2) > A\cdot|I_2| + K\eta\cdot|I_1|.$$

But, by (24), (27), and (33),

$$
\begin{aligned}
|I_1| &\geqslant s_1^m \geqslant \beta^m l_1^m \\
&> 2^{-m(k+3)} \beta^m s_0^m \\
&> 2^{-4m} \alpha^m \beta^m |I_2|.
\end{aligned}
$$

Hence, by (17),

$$F(I_2) > (A+2^{m+3}\eta)\cdot|I_2|. \tag{35}$$

Since the interval I_1 can be as small as we like, it follows that we can find a decreasing sequence of intervals, containing x', of the form (30), satisfying (26) and (35), whose diameters tend to zero. This is true for each x' of E_1. We now apply Vitali's theorem, remembering (23). We obtain a finite set of non-overlapping intervals, say J_i, $i=1, 2, \ldots, n$, such that

$$\sum_{i=1}^{n} |J_i| > \tfrac{1}{2}|I_0|,$$

each of which satisfies (26) and (35) and is of the form (30). Now if we divide each J_i into 2^m similar intervals of half the side, by hyperplanes through its middle point, then each of these, by (26), contains a point of E, and at least one (for each J_i) must satisfy (35). Hence there exists a finite set of non-overlapping intervals, say S_j, $j=1, 2, \ldots, n$, each of which is of the form

$$(b_i\cdot 2^{-k} h_i,\ (b_i+1)\, 2^{-k} h_i), \tag{36}$$

satisfies

$$F(S_j) > (A+2^{m+3}\eta)\cdot|S_j|, \tag{37}$$

and contains a point of E; and

$$\sum_{1}^{n} |S_j| > 2^{-(m+1)}\cdot|I_0|.$$

Now we take I_0 and divide it up into similar intervals of the form (36), just as we divided the cube in Theorem 1. The "black intervals" are those which coincide with an interval S_j, and the "white intervals" are those which do not contain or overlap any black interval. Just as before, the process, in a finite number of steps, divides I_0 into non-overlapping black and white intervals. Let W_j' be those white intervals which contain no point of E_1.

Then, by (23),

$$\sum |W_i'| < 2^{-(2m+2)} |I_0|.$$

Just as before, we can cover all the W_j' by a finite number of similar intervals V_j, $j=1, 2, \ldots, n_1$, say, satisfying

$$\sum |V_j| < 2^m\cdot\sum |W_j'| < 2^{-(m+2)} |I_0|,$$

such that each V_j contains a point of E_1 and therefore, since $r(V_j)=r(I_0)\geqslant\alpha$, satisfies (21). $I_0-\sum V_j$ will consist of a certain number of white intervals each containing a point of E and satisfying (21), say W_j, $j=1, 2, \ldots, n_2$, and a certain number of black intervals, say B_j, $j=1, 2, \ldots, n_3$, say. We have

$$
\begin{aligned}
\sum_j |B_j| &\geqslant \sum_j |S_j| - \sum_j |V_j| \\
&> 2^{-(m+2)} |I_0|.
\end{aligned}
$$

Since each V_j or W_j satisfies (21) and each B_j satisfies (37), we obtain by addition

$$
\begin{aligned}
F(I_0) &> A\cdot|I_0| + 2^{m+3}\eta\cdot\sum |B_j| \\
&> (A+2\eta)\cdot|I_0|,
\end{aligned}
$$

which is the required inequality. Thus the theorem is proved.

4. For each of the theorems 1—3, there exists a symmetrical theorem in which upper and lower derivates are interchanged, and the signs of infinities and of inequalities are reversed. Now suppose that, for some fixed α, $0<\alpha<1$, $\underline{F}_\alpha(x)>-\infty$ at each point of a set E. Neglecting sets of measure zero, we have the following results at each point of E.

By theorem 1, $\underline{F}_1(x)<\infty$, *a fortiori* $\underline{F}_\alpha(x)<\infty$.

By theorem 3, $\overline{F}_{1/n}(x)=\underline{F}_\alpha(x)$, all n, and so $\overline{F}(x)=\overline{F}_\alpha(x)=\underline{F}_\alpha(x)<\infty$.

By the theorem symmetrical to theorem 3, since $\overline{F}_\alpha(x)<\infty$, $\underline{F}_{1/n}(x)=\overline{F}_\alpha(x)$, all n, and so

$$\underline{F}(x)=\overline{F}_\alpha(x)=\overline{F}(x).$$

Hence, remembering theorem 2, we easily derive the final result.

Theorem 4. If $F(I)$ is any additive function of intervals, then, at almost all points, the derivates satisfy one of the following conditions:

(i) $F^*(x) = \overline{F}(x) = \underline{F}(x) = F_*(x)$, *finite*;

(ii) $F^*(x) = +\infty$, $\overline{F}(x) = \underline{F}(x) = F_*(x)$, *finite*;

(iii) $F^*(x) = \overline{F}(x) = \underline{F}(x)$, *finite*; $F_*(x) = -\infty$;

(iv) $F^*(x) = +\infty$, $\overline{F}(x) = \underline{F}(x)$, *finite*; $F_*(x) = -\infty$;

(v) *For all* $\alpha < 1$, $\overline{F}_\alpha(x) = F^*(x) = +\infty$, $\underline{F}_\alpha(x) = F_*(x) = -\infty$.

The values of $\overline{F}_1(x)$, $\underline{F}_1(x)$, in the last case, are not yet determined.

5. There is, however, at least one case in which the above theorem holds also for $\alpha = 1$; namely, when $m = 2$ and $F(I)$ is continuous. This follows as a particular case of a theorem which we will now prove.

Theorem 5. If $F(R)$ is an additive function of rectangles and, at each point (x, y) of a set E_0,

$$\overline{F}_1(x, y) < \infty,$$

and also, for some fixed α such that $\frac{2}{3} \leqslant \alpha < 1$,

$$\overline{\lim_{\substack{l(R)\to 0\\ R\supset(x,y)\\ r(R)\geqslant\alpha}}} F(R) = 0,$$

then, almost everywhere in E_0,

$$\overline{F}_\alpha(x, y) < \infty.$$

If the theorem is not true, we can find $N > 0$ so large, and $\delta_0 > 0$ so small, that, at each point (x, y) of a set E, $|E| > 0$, we have

(38) $$F(R) \leqslant N \cdot |R|$$

whenever $r(R) = 1$, $l(R) \leqslant \delta_0$, and R contains (x, y); further,

(39) $$\overline{\lim_{\substack{l(R)\to 0\\ R\supset(x,y)\\ r(R)\geqslant\alpha}}} F(R) = 0,$$

but

(40) $$\overline{F}_\alpha(x, y) = \infty.$$

Let (x_0, y_0) be a point of outer density of E. We can choose δ_1, $0 < \delta_1 \leqslant \delta_0$, such that, if $R \supset (x_0, y_0)$ and $l(R) \leqslant \delta_1$, then

(41) $$|ER| > [1 - \tfrac{1}{256}\alpha(1-\alpha)^2] \cdot |R|.$$

Now let I be any interval containing (x_0, y_0), such that $l(I) \leqslant \delta_1$ and $r(I) \geqslant \alpha \geqslant \frac{2}{3}$. We say that there exists an interval J containing (x_0, y_0), such that

(42) $$r(J) \geqslant \alpha,$$

(43) $$l(J) \leqslant \tfrac{7}{8} s(I)$$

and $$F(J) - N \cdot |J| \geqslant F(I) - N \cdot |I|.$$

Suppose that I is the interval $(x_0 - h_1, x_0 + h_2; y_0 - k_1, y_0 + k_2)$, and write for convenience

$$h_1 + h_2 = h, \qquad k_1 + k_2 = k, \qquad s(I) = s.$$

Since $r(I) \geqslant \frac{2}{3}$, we have $s \leqslant h \leqslant \frac{3}{2} s$. We shall suppose [5], for definiteness, that $h_1 \geqslant h_2$, $k_1 \geqslant k_2$, so that

$$h_2 \leqslant \tfrac{1}{2} h \leqslant \tfrac{3}{4} s,$$

and similarly

$$k_2 \leqslant \tfrac{3}{4} s.$$

Since $k \geqslant s$, we can find an integer n_1 such that

(44) $$\tfrac{1}{16}(1-\alpha)s \leqslant k/n_1 \leqslant \tfrac{1}{8}(1-\alpha)s.$$

We define the integer p_1 such that

$$h - p_1 k/n_1 \leqslant \tfrac{7}{8} s < h - (p_1 - 1) k/n_1.$$

Since $h \geqslant s$, $p_1 > 0$. Write $h - p_1 k/n_1 = h'$. It follows at once from (44) that

$$\tfrac{7}{8} S - \tfrac{1}{8}(1-\alpha)s < h',$$

whence

(45) $$\tfrac{7}{8}\alpha s < h' \leqslant \tfrac{7}{8} s,$$

and also

$$h_2 \leqslant \tfrac{3}{4} s \leqslant h'.$$

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[4]) It will be easily seen that this supposition does not essentially affect the argument.

Again, since $h' > \frac{1}{2}s$, we can find an integer n_2 such that

(46) $$\tfrac{1}{16}(1-\alpha)s \leqslant h'/n_2 \leqslant \tfrac{1}{8}(1-\alpha)s.$$

We define the integer $p_2 > 0$ such that

$$k - p_2 h'/n_2 \leqslant \tfrac{7}{8}s < k - (p_2-1)h'/n_2,$$

and write $k - p_2 h'/n_2 = k'$. Just as before, we have

(47) $$\tfrac{7}{8}\alpha s < k' \leqslant \tfrac{7}{8}s$$

and also

$$k_2 \leqslant k'.$$

Hence the following interval, which we define as the interval J, namely

$$(x_0+h_2-h',\ x_0+h_2;\quad y_0+k_2-k',\ y_0+k_2).$$

contains (x_0, y_0). By (45) and (47), it satisfies (42) and (43). Now $I-J$ can be divided into $n_1 p_1$ squares of side k/n_1, namely

$$\left[x_0-h_1+\frac{(r-1)k}{n_1},\ x_0-h_1+\frac{rk}{n_1};\qquad y_0-k_1+\frac{(s-1)k}{n_1},\ y_0-k_1+\frac{sk}{n_1}\right],$$

$1 \leqslant r \leqslant p_1$, $1 \leqslant s \leqslant n_1$, and $n_2 p_2$ squares of side h'/n_2, namely

$$\left[x_0+h_2-h'+\frac{(r-1)h'}{n_2},\ x_0+h_2-h'+\frac{rh'}{n_2};\quad y_0-k_1+\frac{(s-1)h'}{n_2},\ y_0-k_1+\frac{sh'}{n_2}\right],$$

$1 \leqslant r \leqslant n_2$, $1 \leqslant s \leqslant p_2$. By (41), (44), and (46), each of these squares contains a point of E (since $s^2 \geqslant \alpha\cdot|I|$), and therefore satisfies (38). Hence, if we enumerate all these squares as S_i, $i=1, 2, \ldots, n$, say, we have, since F is additive,

$$\begin{aligned} F(J) &= F(I) - \sum_{i=1}^{n} F(S_i) \\ &\geqslant F(I) - N\sum_{i=1}^{n}|S_i| \\ &= F(I) - N\cdot|I| + N|J|, \end{aligned}$$

that is

$$F(J) - N\cdot|J| \geqslant F(I) - N\cdot|I|.$$

Now by (40) we can find an interval I_1, containing (x_0, y_0), such that $F(I_1) \geqslant \alpha$, $l(I_1) \leqslant \delta_1$, and

$$F(I_1) \geqslant 2N\cdot|I_1|,$$

that is,

$$F(I_1) - N\cdot|I_1| \geqslant N\cdot|I_1|.$$

By applying the result we have just proved, we obtain from I_1 a sequence of intervals I_n containing (x_0, y_0), $n=1, 2, \ldots$, such that, for each n,

$$l(I_{n+1}) \leqslant \tfrac{7}{8}s(I_n),$$

$$r(I_n) \geqslant \alpha,$$

$$F(I_n) - N\cdot|I_n| \geqslant N\cdot|I_1|.$$

This clearly contradicts (39), and so our theorem is proved.

Warszawa, October 1936.