On monotonic complete covering systems

By

Leo Zippin (Princeton).

This note was inspired by a question addressed to us by Prof. Lefschetz, which concerned the general homology theory due to Prof. Čech. The problem at issue is whether, or to what extent, one can introduce into this theory of Čech a generalisation of the projection-sequences theory of Alexandroff. One would certainly expect to abandon *countable* projection-sequences, but one might be tempted to hope that for a large class of spaces, at least, there would exist a well-ordered uncountable system of „complexes" which more and „more nearly" approached the space. Formulated in the theory of Čech, one might ask for what spaces does there exist a well-ordered *monotonic complete* family of finite coverings by open sets? The answer to this is contained in the final theorem of the paper, of which we here signalise the following special case:

In order that a Hausdorff space S admit a monotonic complete system of finite coverings by open sets, it is necessary and sufficient that S be compact metric.

The methods of this paper will be recognized as, for the greater part, classic. It is in some parts closely related to the theory of „developments" studied by E. W. Chittenden and A. D. Pitcher[1]).

[1]) Transactions Amer. Math. Soc. 20 (1919), pp. 213—233.

1. T-spaces. These shall be the ordinary topologic spaces, it being understood that we do not assume that every pair of points belong to mutually exclusive neighborhoods. Our argument will fall into several sections. We shall list the properties of our spaces, for convenience of reference as well as to exhibit their different connections with the final theorem:

1) a class of subsets is designated as open, the class including the space, itself, and the null-set (although this will not figure in our arguments). *The sum of any number of open sets shall be open.*

2) *the product of any finite number of open sets is open.*

3) *the sum of any finite number of points is closed.*

We shall understand, now, that the usual topologic notions of closedness, limit point, compactness, complete separability, have all been introduced, as customarily, on the basis of the open sets which exist in the space by definition. We shall call these spaces, T-spaces. The final theorem will be found in **6.**

A finite covering K of the space by open sets, abbreviated f. c. o. s., is said to be contained in another f. c. o. s. K', $K \subset K'$, if each of the open sets making up K belongs to some one of the open sets making up K'. A *system* of f. c. o. s. $\{K_\alpha\}$, where α is a distinguishing *ordinal number* ranging over a class of ordinals (possibly very uncountable), will be said to be *monotonic* if $K_\beta \subset K_\alpha$ whenever $\alpha \leqq \beta$ in the sense of the well-ordering of these ordinals. This system is called *complete* if, given an *arbitrary* f. c. o. s. K, there exists an ordinal α and a $K_\alpha \subset K$. These are the customary definitions.

2. The function $N(\alpha)$. Given a monotonic system $\{K_\alpha\}$ we shall define $N(\alpha)$, for each α such that there is a K_α, to be the *least integer* such that there exists in the f. c. o. s. K_α a set of $N(\alpha)$ of its subsets which cover the space. We may take our space to be not empty, so that $N(\alpha) \geqq 1$ and it is certainly finita (whenever defined). Now if $N(\alpha)$ and $N(\beta)$ are both defined and if $\alpha \leqq \beta$, then $N(\alpha) \leqq N(\beta)$. For, our system being monotonic, $K_\beta \subset K_\alpha$. Since there exist $N(\beta)$ subsets of K_β which cover space and since each of these belongs to at least one set in K_α, by the monotonicity, there must exist $N(\beta)$ sets (not necessarily distinct) in K_α which cover space. We shall need the following (known).

Lemma: If a function $N(\alpha)$ whose values are positive integers is defined for a certain class of ordinal numbers $\{\alpha\}$ so that it is *monotonic* and *unbounded*, then the class of ordinals $\{\alpha\}$ must contain a *countable* subsequence $\alpha_1 < \alpha_2 < \dots < \alpha_n < \dots$ such that this subsequence is *confinal* with the original class.

Proof. The class $\{\alpha\}$ contains at least one subsequence α_n such that $N(\alpha_n)$ is unbounded. If there existed any $\overline{\alpha}$ of this class, $\overline{\alpha} > \alpha_n$ for every n such that $N(\overline{\alpha})$ is defined, we should have $N(\alpha_n) \leqq N(\overline{\alpha})$. Since this contradicts the unboundedness, no such $\overline{\alpha}$ exists and the sequence α_n is confinal with the original class by definition of *confinality*.

Lemma: If a subsystem $\{K_\beta\}$ of a complete monotonic f. c. o. s. system $\{K_\alpha\}$ is confinal with it, then $\{K_\beta\}$, also, is complete.

Proof. Let K be an arbitrary f. c. o. s. of space. There exists an ordinal $\overline{\alpha}$ such that $K_{\overline{\alpha}} \subset K$. Since the class $\{\beta\}$ is confinal with the containing class $\{\alpha\}$ there exists at least one $\overline{\beta} \geqq \overline{\alpha}$. By the monotonicy $K_{\overline{\beta}} \subset K_{\overline{\alpha}}$, therefore $K_{\overline{\beta}} \subset K$.

3. ***Theorem:*** A compact, completely separable T-space always possesses a countable monotic f. c. o. s. system.

Proof. Since our space, let us call it T, is completely separable, it possesses (by definition) a countable set $U_1, U_2, \dots$, of open subsets such that if x is any point of T and U any open set containing x then, for some integer n, $x \subset U_n \subset U$. Now, let us observe that *any* collection of the (U_n) which covers T possesses a *finite* subcollection which covers T. To prove this, let U_{n_j}, $j=1, 2, \dots$, denote the members of any „covering“ collection. Then there must exist, for each j, a point x_j of T such that $x_j \subset U_{n_i}$ if $i \leqq j$. The set (x_j) of points so defined has at least one limit point x (by the compactness of T) and there is at least one integer j such that $x \subset U_{n_j}$, because our collection covers T. The set of points $\sum_1^j x_i$ is closed (by condition 3 on T, **1**). Then there is an open set $U \supset x$, $U \supset x_i$ if $i=1, 2, \dots, j$, by definition. The intersection $U \cdot U_{n_j}$ is open (by condition 2) and contains x, *therefore* at least one point x_m, say, of (x_j) by definition of limit point. But now $m > j$ contrary to our choise of x_m.

We observe that in virtue of our first condition on T (see **1**) there is at least one *covering* of T by open sets, namely T. It follows that our fundamental sequence (U_n) of open sets also covers T. For if we have *any* covering of T, each point belongs to some open set of this covering, by definition, and *therefore* to at least one U_n. From what we proved above, it follows that the fundamental sequence contains *finite* coverings of T. Since there are only a countable number of different *finite* combinations of the U_n, $n=1, 2, \dots$, there cannot be more than a countable system of finite coverings of T by these sets. Let $H_1, H_2, \dots$, represent the distinct f. c. o. s. which we obtain in this way, arranged in some convenient order.

For a moment only, suppose that U and U' are open sets chosen at random from among those making up an arbitrary f. c. o. s. H and f. c. o. s. H', respectively. The set $U'' = U \cdot U'$ is open, by condition 2) of 1. As U and U' vary in H and H' we obtain a finite set of the U'', and we shall denote the collection of them by H''. We may express this symbolically: $H'' = H \cdot H'$. It is clear from the definition of H'' that it is an f. c. o. s. and that $H'' \subset H, \subset H'$.

Let K_0 denote T. Let $K_1 = K_0 \cdot H_1$, $K_2 = K_1 \cdot H_2$, and generally, $K_{n+1} = K_n \cdot H_{n+1}$. It is obvious, from the remark above, that $\{K_n\}$ is a monotonic f. c. o. s. system. We shall have proved our theorem when we show that it is complete. To this and, let K denote an arbitrary f. c. o. s. We have already observed that there must exist a collection of our original U_n, $n=1, 2, \dots$, which cover T and each of them contained in some set of K. We proved, further, that any such covering contained at least one f. c. o. s. which we may denote by K^*. It is clear that $K^* \subset K$. But there must be some integer n such that $K^* \equiv H_n$, by the definition of the system $\{H_n\}$. Finally, by construction, $K_n \subset H_n \equiv K^* \subset K$ and the system $\{K_n\}$ is complete.

So far we have proved the *sufficiently* part only of our final theorem. We heve used all of the conditions on T which we enumerated in the first section. We now address ourselves to considerations of *necessity*.

4. ***Lemma:*** If a T-space possesses an m. c.-system (read „monotonic complete f. c. o. s. system“) $\{K_\alpha\}$ and if for some integer $s > 0$ it contains a set of s distinct points $x_1, x_2, \dots, x_s$ such that every subset of $X = \Sigma x_i$ is *closed*, then there exists an ordinal α such that $N(\alpha) \geqq s$.

Proof. Since, for every $1 \leqq i \leqq s$, $X - x_i$ is closed, there must exist an open set U_i such that $U_i \supset x_j$ if and only if $i=j$. Furthermore, $U_0 = T - X$ is open. It is clear that the collection of sets $(U_0, U_1, \ldots, U_s)$ which we may denote by K is an f. c. o. s. Therefore, for some ordinal α, $K_\alpha \subset K$. Now if $N(\alpha) < s$, there must exist at least two points x_i and x_j, $i \neq j$, which belong to the same open set in K_α, therefore to some *one* open set in K. This contradicts the construction of K.

Corollary: If a T-space contains infinitely many points such that every finite subset of them is closed and if it possesses an m. c.-system, then it possesses a *countable* m. c.-system.

For, by the lemma above, the function $N(\alpha)$ must be unbounded. By the two lemmas of **2**, the given m. c.-system must contain a *countable complete* subsystem, necessarily monotonic.

Theorem: If a T-space possesses an m. c.-system, it is completely separable.

Proof. It T consist of a finite set of points it has an at most finite number of subsets and therefore at most a finite number of open sets. Then it is certainly completely separable, since it contains at least one open subset (by property 1) of **1**).

It T contains infinitely many distinct points, than we shall *now* invoke property 3) of a T-space and apply the corollary above. That the countable set of open sets contained in a countable m. c.--system is a *fundamental* set for the space is obviously the case, whenever each individual point x is closed; i. e., whenever $T - x$ is open for every x.

5. ***Theorem:*** If a T-space possesses an m. c.-system, it must be compact.

This theorem is essentially set-theoretic and uses only the first property of a T-space.

The proof will proceed by contradiction. Suppose that T is not compact. Then it contains, by definition, an infinite countable set of distinct points $x_1, x_2, \ldots$, such that *every* subset of $X = \Sigma x_n$ is *closed*. Therefore, by the preceding *corollary*, it possesses a countable m. c.-system which we shall denote by $\{K_j\}$, $j = 1, 2, \ldots$.

With each f. c. o. s. K_j we shall associate a certain *partition* of X as follows. We fix on a definite ordering of the open sets in K_j. Let N_j denote the number of them. We assign to each point of X a coordinate with N_j ordered components, where the s-th component, $1 \leqq s \leqq N_j$, is *one* if the point belongs to the s-th set and *zero* if it does not. There are at most a finite number of different coordinates. We shall say that two points of X belong to the same class if and only if they have the *same* coordinate. In this way we obtain a separation of the points of X *into a finite number of mutually exclusive classes such that if two points of X belong to the same class then every subset of K_j contains both or neither of them.* We shall designate this *partition* by P_j.

Now let F denote an arbitrary *proper* subset of X, and let $F' = X - F$. Since both F and F' are closed subsets of T, it is clear that $T - F$ and $T - F'$ define a f. c. o. s. of T. Therefore there is at least one j such that K_j is contained in this f. c. o. s. We see at once that each *class* in P_j must belong entirely to F or entirely to F'. We shall say that the *disjunction* (F, F') of X contains the partition P_j: symbolically $(F, F') \supset P_j$. We have proved that *every* (F, F') disjunction of X contains a partition P_j.

We shall now show that a *fixed* partition P_j cannot be contained in more than a *finite* number of distinct (F, F') disjunctions. In fact, if the number of classes in P_j is N, then the number of distinct disjunctions which can contain it is precisely $2^N - 2$. This is trivail. *For* if we arrange the *classes* in P_j into two sets, call them R and L for the moment, such that each of these contains at least one class and no class belongs to both, then the totality, F, of points of X which fall into R in this division form a proper subset. The set of points in L is precisely its complement $X - F = F'$. Each *arrangement*, R and L, determines the corresponding (F, F') disjunction uniquely.

Now this implies at once that the number of distinct (F, F') disjunctions is countable, since each of them contains some P_j and at most a finite number can contain the same one. On the other hand, the number of these is precisely the „number" of proper subsets of X. Since X is infinite, this number cannot be countable, by the Cantor Theorem. The contradiction completes our proof.

6. It should be clear that we have already proved this final

Theorem: In order that a T-space possess a monotonic, complete, finite-covering-by-open-sets-system, it is necessary and sufficient that it be completely separable and compact.

We shall conclude with the following remarks. It is well-known that a completely separable and compact T-space need not be, metrizable, but *if it is not* then it must also fail to be *regular*. *Further*, in a completely separable, compact T-space, *regularity* is implied by the weaker separation property (Hausdorff) that to each pair of points x and y there exist *mutually exclusive* open sets U_x and U_y, containing x and y respectively. Therefore, if a T-space posseses an m.c.-system and is *not* metrizable, it must contain at least one pair of points which cannot belong to mutually exclusive open sets. Therefore, the theorem with we opened this paper is a special case of our final one.

For completeness sake, it is perhaps worth while to give the argument upon which our last remarks are based. Suppose that X is a closed subset of a T-space, and y a point of that space. Suppose, further, to to each point x of X we may associate a pair of mutually exclusive neighborhoods U_x and U_y^x, $U_x \supset x$, $U_y^x \supset y$. Now, if our T-space is completely separable and compact, then, first of all, we may suppose that the open sets are drawn from a countable fundamental set and, secondly, we may apply the Heine-Borel theorem which we proved for these spaces, in **3.** It will be clear that our proof applies to closed subsets of the space, also. Then we conclude that there exists a finite set of open sets U_{x_i}, $i=1, 2, ..., N$, whose *sum* covers X. The *product* of the open sets $U_y^{x_i}$, $i=1, ..., N$, is an open set containing y, which has no point in common with that *sum*. This is regularity.

Institute for Advanced Study
Princeton, New Jersey.

Symmetrical Cut Sets.

By

Wayne Dancer (Ann Arbor).

I. Introduction.

After a survey of characterizations of the simple closed curve and simple closed surface [1]) it has occurred to us that the principle of symmetry, which has been used to only a limited extent in analysis situs [2]), might be advantageous in forming such characterizations.

To this end we shall say that a set S is a *symmetrical cut set* of a set M if $M-S$ can be expressed as the sum of two mutually separated sets M_1 and M_2 which are such that there exists a continuous (1—1) correspondance Δ having the properties that, $\Delta(M_1+S)=M_2+S$, and $\Delta(S)=S$. If S consists of a finite number of points it will be called a *permutable symmetrical cut set* provided that $\Delta(P_i)=P_{i+1}$ $(i=1, 2, ..., n-1)$, and $\Delta(P_n)=P_1$. The set S will be called a *strong symmetrical cut set* of M if, in addition to being a symmetrical cut set of M as defined above, $\Delta(P)=P$, for every point P of S. Hereafter the sets M_1 and M_2, defined above, will be referred to as *symmetric separates* of M with respect to S.

It is easy to see that every pair of distinct points of a simple closed curve is a symmetrical cut set of the curve; likewise it has been shown that every simple closed curve of a simple closed surface is a strong symmetrical cut set of the latter [3]). On the other hand,

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[1]) A *simple closed surface* is the homeomorph of the unit sphere $x^2+y^2+z^2=1$ in cartesian 3-space.

[2]) H. M. Gehman, *Centers of symmetry in analysis situs*, Amer. Jour. Math. 52 (1930), pp. 543—547.

[3]) A. Schoenflies, *Beiträge zur Theorie der Punktmengen*, III, Math. Ann. 62 (1906), p. 324, and J. R. Kline, *A new proof of a theorem due to Schoenflies*, Proc. Nat. Acad. Sc., 6 (1920), pp. 529—531.