Functions of rectangles

2. If \( F(I) \) is of bounded variation, then almost everywhere either \( F^*(x, y) = -\infty \) or \( F^*_m(x, y) = F^*_m(x, y) \).

We consider only the rectangles with sides parallel to the axes. The rectangle whose lower left-hand and upper right-hand corners are the points \((a, c)\) and \((b, d)\) respectively, will be denoted by \([a, b; c, d]\). Two rectangles are said to be opposite at a corner \(p\) if \(p\) is their common corner and if they have no other points in common.

By the upper \( F^*(x, y) \) and the lower \( F^*_m(x, y) \) strong derivatives of a function of rectangles \( F(I) \) at a point \((x, y)\) we mean the upper and the lower limits of \( F(I)/|I| \) where \(I\) is an arbitrary rectangle containing \((x, y)\) of diameter \(d(I)\) tending to \(0\). If the rectangles are replaced by squares these limits are termed the upper \( F(x, y) \) and the lower \( F^*_m(x, y) \) derivatives in the ordinary sense.

If \( F(x, y) = \overline{F}(x, y) \) the function \( F(I) \) is said to be derivable (in the ordinary sense) at \((x, y)\) and we write \( F'(x, y) = \overline{F}(x, y) = \overline{F}(x, y) \). Finally, \( F^*_m(x, y) \) and \( F^*_m(x, y) \) respectively are the upper and the lower limits of \( F(S)/|S| \) as \(d(S)\) tends to \(0\), where \(S\) is any square containing the point \((x, y)\) at one corner. These notations slightly differ from those of Ward.

2. In this paper we attempt to give a generalization of the theorems stated in the preceding section. The proof given below actually rests on the idea due to Ward; it seems, however, to be slightly simpler than the latter's original argument.

**Lemma.** If \( p \) is a density point of a measurable set \( P \) and \( \epsilon > 0 \), then for any sufficiently small square \( S \) containing \( p \) there exists a square \( S^* \) such that (i) \( S^* \subseteq S \), (ii) \( d(S^*) \leq (1+\epsilon)d(S) \), (iii) all corners of \( S^* \) are points of \( P \).

**Proof.** Let \( S \) be any square containing \( p \) and let \( q_1, q_2, q_3 \) and \( q_4 \) be the corners of \( S \). Denote by \( S_1, S_2, S_3, S_4 \) four equal squares of diameter \( \frac{1}{4}\epsilon d(S) \), opposite to \( S \) at the corners \( q_1, q_2, q_3 \) and \( q_4 \) respectively. Since \( p \) is a density point of \( P \), it is readily

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1) vol. 26 (1936), pp. 167—182.
seen that there exists an \( \eta > 0 \) such that

\[
(2.1) \quad |S_i - S_i \cdot P| < \frac{\eta}{4} |S_i| \quad \text{for} \quad i = 1, 2, 3, 4 \quad \text{whenever} \quad d(S) \leq \eta.
\]

Let \( \varphi_2, \varphi_3 \) and \( \varphi_4 \) denote the parallel translations carrying the squares \( S_2, S_3 \) and \( S_4 \) respectively into \( S_1 \). On supposing \( d(S) < \eta \), in virtue of (2.1) we have \( |S_i - S_i \cdot P| < \frac{\eta}{4} |S_i| \) and \( |S_i - \varphi(S_i \cdot P)| < \frac{\eta}{4} |S_i| \) for \( i = 2, 3, 4 \); thus the product of the four sets \( S_1, \varphi_2(S_2 \cdot P), \varphi_3(S_3 \cdot P), \) and \( \varphi_4(S_4 \cdot P) \) is empty, indeed of positive measure. Let \( a_1 \) be an arbitrary point of that product, and let \( a_2, a_3 \) and \( a_4 \) denote the points which are carried into \( a_1 \) under the translations \( \varphi_2, \varphi_3 \), and \( \varphi_4 \), respectively. It is clear on a moment's consideration that \( a_1, a_2, a_3 \) and \( a_4 \) are the corners of a square \( S^* \) that satisfies conditions (i) and (ii). Since \( a_i \in S_i \cdot P \) for \( i = 1, 2, 3, 4 \), it satisfies condition (iii) also. Thus the lemma is proved.

3. Theorem. If for any additive function of rectangles \( F(I) \) there is \( F^*(x, y) \leq -\infty \) at each point of a set \( E \), then almost everywhere on \( E \) the function \( F(I) \) is derivable (in the ordinary sense) and 

\[
\bar{F}(x, y) = F^*(x, y).
\]

Proof. Suppose on the contrary that \( \bar{F}(x, y) > F^*(x, y) \geq -\infty \) on a set \( A \) of positive outer measure. We can obviously assume that \( F^*(x, y) \) is bounded below on \( A \), and even, by adding an additive function \( M[I] \) to \( F(I) \), that \( \bar{F}(x, y) > 0 \) over \( A \). Hence, there are a set \( B \subset A \) of positive outer measure, a positive number \( \sigma \) and two numbers \( \mu \) and \( \lambda \) such that

\[
(3.1) \quad F(I) > 0 \quad \text{whenever} \quad d(I) < \sigma \quad \text{and} \quad I \cdot B \neq 0,
\]

and that

\[
(3.2) \quad F(x, y) > \mu > \lambda > F^*(x, y) > 0 \quad \text{for} \quad (x, y) \in B.
\]

Let \( \mathcal{B} \) denote the set of the outer density points of \( B \), either belonging, or not, to the set \( B \). The set \( \mathcal{B} \) is apparently measurable and is introduced to make the discussion of the measurability of \( B \) superfluous.

Now, let \( \alpha \) be a positive number such that

\[
(3.3) \quad \alpha < 1 \quad \text{and} \quad \mu \cdot (1 - 3\alpha) > \lambda.
\]

We shall first prove that

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(M) With any point \( p \) in \( \mathcal{B} \cdot B \) a sequence of squares \( \{S_n\} \) may be associated so that (i) all corners of any \( S_n \) belong to \( \mathcal{B} \), (ii) \( p \in S_n \), (iii) \( F(S_n) > \mu \cdot (1 - 2\alpha) \cdot |S_n| \) and (iv) \( d(S_n) \to 0 \) as \( n \to \infty \).

Indeed, in virtue of (3.2), there is a sequence of squares \( \{S_n\} \) such that

\[
(3.4) \quad d(S_n) < \frac{\sigma}{2} \quad \text{for} \quad p \in S_n \quad F(S_n) > \mu \cdot |S_n| \quad \text{and} \quad d(S_n) \to 0.
\]

In virtue of the lemma of § 2, on supposing all squares \( S_n \) sufficiently small, we can attach a square \( S_n^* \) to any \( S_n \) so as to satisfy condition (i) in (M) and to have

\[
(3.5) \quad d(S_n^*) < (1 + \alpha) \cdot d(S_n) \quad \text{and} \quad S_n \subset S_n^*.
\]

Then it directly follows from (3.4) that \( S_n^* \) satisfies conditions (ii) and (iv) in (M). In order to establish the remaining condition (iii), subdivide the area \( \mathcal{B} - S_n \) into four rectangles, \( I_1^o, I_2^o, I_3^o \) and \( I_4^o \), say, so that any of them should contain a corner of \( S_n^* \). Hence, each of these rectangles contains a point of \( \mathcal{B} \), and consequently a point of \( B \) also. Thus, it results from (3.1) that

\[
F(I_k^o) > 0 \quad \text{for} \quad k = 1, 2, 3, 4, \quad \text{as} \quad k \to \infty.
\]

Since \( I_k^o \) are orthogonal, and \( \mathcal{B} \) is measurable, we have

\[
F(S_n^*) \geq F(S_n) > \mu \cdot |S_n| > \mu \cdot |S_n|/(1 + \alpha) > \mu \cdot (1 - 2\alpha) \cdot |S_n|,
\]

which is condition (iii) in (M).

Now, since almost all points of the set \( B \) are its density points in the strong sense, it follows from (3.2) that there is a rectangle \( R \) such that

\[
(3.6) \quad F(R) < \mu \cdot |R|, \quad |B - R| = |B \cdot B \cdot R| > (1 - \alpha) \cdot |R|, \quad \text{and} \quad d(R) < \sigma.
\]

Hence, by (M) and by the well-known Vitali Lemma there exist in \( R \) a finite set of not overlapping squares \( J_1, J_2, ..., J_p \) such that

\[
(3.7) \quad \sum_j |J_k| \geq (1 - \alpha) \cdot |R| \quad \text{and} \quad F(J_k) \geq \mu \cdot (1 - 2\alpha) \cdot |J_k| \quad \text{for} \quad k = 1, 2, ..., p.
\]
the corners of each $J_a$ belonging to $B$. Further, as it is easily seen, the area $B - \sum J_a$ may be subdivided into a finite number of not overlapping rectangles each of which contains one at least of the corners of the squares $J_a$. Hence, each of them contains points of $B$, and, consequently, of $B$. Since $d(E) < a$, it results from (3.1) that the function $F(I)$ is positive for each of these rectangles, and so, by (3.6) and (3.7)

$$\lambda |E| > F(R) > \sum F(J_a) \geq \mu \cdot (1 - 2a) (1 - a) \cdot |E| \geq \mu (1 - 3a) \cdot |E|.$$

This, however, is contradictory to (3.3) and concludes the proof.

4) This is directly obvious for the plane, but is not true for the space as seen from a simple example kindly communicated to the author by Mr. O. Nikodym. The problem whether the theorem itself holds for the space seems to be open, and the same remark applies to the results of Besicovitch, l. c. 4).

Ensembles dont les dimensions modulaires de Alexandroff coïncident avec la dimension de Menger-Urysohn 1).

Par

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Dans la théorie homologique de la dimension, due à M. P. Alexandroff 2), on est conduit d'une manière naturelle à considérer une infinité d'invariants topologiques qui méritent — au moins du point de vue d'homologie — d'être appelés "dimensions" (modulaires). Toutes ces "dimensions", différentes pour les ensembles compacts arbitraires, se montrent identiques avec la dimension au sens de Menger-Urysohn pour les ensembles dont la structure topologique est peu compliquée (en particulier pour tous les polyèdres). Dans le domaine de ces derniers ensembles, la théorie de la dimension prend une forme particulièrement simple, naturelle et intuitive. Ainsi p. ex. se trouve réalisée pour ces ensembles "l'hypothèse du produit" qui est en défaut — d'après M. L. Pontrjagin 3) — dans le domaine des ensembles compacts arbitraires.

Le but de cette Note est de définir par des notions de la topologie générale une classe d'ensembles (comprenant en particulier tous les polyèdres) pour lesquels toutes les dimensions modulaires coïncident avec la dimension au sens de Menger-Urysohn.