On derivates of functions of rectangles.

By

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1. In a very interesting paper published in these „Fundamenta"[1]) A. J. Ward has established a series of theorems on the strong derivability of additive functions of rectangles. The principal results of his paper are as follows:

Th. 1. If for an additive function $F(I)$ we have

$$-\infty < \underline{F}^*(x, y) \leqslant \overline{F}^*(x, y) < +\infty$$

at each point (x, y) of a set E, then almost everywhere in E the function $F(I)$ is strongly derivable, i. e.

$$\underline{F}^*(x, y) = \overline{F}^*(x, y) = F^*(x, y).$$

Th. 3. If $-\infty < \underline{F}^*(x, y) \leqslant \overline{F}^*(x, y) < +\infty$ at each point of E, then $\underline{F}^*(x, y) = \underline{F}_{\mathrm{m}}(x, y) = \overline{F}_{\mathrm{m}}(x, y)$ almost everywhere in E.

As immediate corollaries of his Theorems 1 and 2 Ward mentions:

1. If at each point of E we have $\underline{F}^*(x, y) > -\infty$, then $\underline{F}^*(x, y) = \underline{F}_{\mathrm{m}}(x, y)$ almost everywhere in E.

[1]) vol. **26** (1936), pp. 167—182.

2. If $F(I)$ is of bounded variation, then almost everywhere either $\underline{F}^*(x, y) = -\infty$ or $\underline{F}^*(x, y) = F'_{\mathrm{m}}(x, y)$[2]).

We consider only the rectangles with sides parallel to the axes. The rectangle whose lower left-hand and upper right-hand corners are the points (a, c) and (b, d) respectively, will be denoted by $[a, b; c, d]$. Two rectangles are said to be opposite at a corner p, if p is their common corner and if they have no other points in common.

By the upper $\overline{F}^*(x, y)$ and the lower $\underline{F}^*(x, y)$ strong derivates of a function of rectangles $F(I)$ at a point (x, y) we mean the upper and the lower limits of[3]) $F(I)/|I|$ where I is an arbitrary rectangle containing (x, y) of diameter $\mathrm{d}(I)$ tending to 0. If the rectangles are replaced by squares these limits are termed the upper $\overline{F}(x, y)$ and the lower $\underline{F}(x, y)$ derivates (in the ordinary sense).

If $\underline{F}(x, y) = \overline{F}(x, y)$ the function $F(I)$ is said to be derivable (in the ordinary sense) at (x, y) and we write $F'(x, y) = \overline{F}(x, y) = \underline{F}(x, y)$. Finally, $\overline{F}_{\mathrm{m}}(x, y)$ and $\underline{F}_{\mathrm{m}}(x, y)$ respectively are the upper and the lower limits of $F(S)/|S|$ as $\mathrm{d}(S) \to 0$, where S is any square containing the point (x, y) at one corner. These notations slightly differ from those of Ward.

2. In this paper we attempt to give a generalization of the theorems stated in the preceding section. The proof given below actually rests on the idea due to Ward; it seems, however, to be slightly simpler than the latter's original argument.

Lemma. *If p is a density point of a measurable set P and $\varepsilon > 0$, then for any sufficiently small square S containing p there exists a square S^* such that* (i) $S \subset S^*$, (ii) $\mathrm{d}(S^*) \leqslant (1+\varepsilon)\cdot \mathrm{d}(S)$, (iii) *all corners of S^* are points of P.*

Proof. Let S be any square containing p and let q_1, q_2, q_3 and q_4 be the corners of S. Denote by S_1, S_2, S_3 and S_4 four equal squares of diameter $\frac{1}{2}\varepsilon\cdot\mathrm{d}(S)$, opposite to S at the corners q_1, q_2, q_3 and q_4 respectively. Since p is a density point of P, it is readily

[2]) This theorem was established by Besicovitch (these Fundamenta, **25** (1935), pp, 209—216) for the case when $F(I)$ is an indefinite Lebesgue integral i. e. an absolutely continuous function. Ward actually assumes less than the bounded variation of $F(I)$ and in his proof uses the fact only that $\overline{F}_{\mathrm{m}}(x, y) = \underline{F}_{\mathrm{m}}(x, y)$ almost everywhere. As it follows from the theorem of § 3 of this paper the result holds for any additive function of rectangles and the derivative $F'_{\mathrm{m}}(x, y)$ may be replaced by the ordinary derivative $F'(x, y)$.

[3]) If A is a point set, $|A|$ denotes its outer measure.

seen that there exists an $\eta > 0$ such that

(2.1) $|S_i - S_i \cdot P| < \frac{1}{4}|S_i|$ *for* $i=1,2,3,4$ *whenever* $d(S) \leqslant \eta$.

Let φ_2, φ_3 and φ_4 denote the parallel translations carrying the squares S_2, S_3 and S_4 respectively into S_1. On supposing $d(S) < \eta$, in virtue of (2.1) we have $|S_1 - S_1 \cdot P| < \frac{1}{4}|S_1|$ and $|S_1 - \varphi_i(S_i \cdot P)| < \frac{1}{4}|S_1|$ for $i=2,3,4$; thus the product of the four sets $S_1 \cdot P, \varphi_2(S_2 \cdot P), \varphi_3(S_3 \cdot P)$, and $\varphi_4(S_4 \cdot P)$ is not empty, indeed of positive measure. Let a_1 be an arbitrary point of that product, and let a_2, a_3 and a_4 denote the points which are carried into a_1 under the translations φ_2, φ_3 and φ_4, respectively. It is clear on a moment's consideration that a_1, a_2, a_3 and a_4 are the corners of a square S^* that satisfies conditions (i) and (ii). Since $a_i \in P \cdot S_i$ for $i = 1, 2, 3, 4$, it satisfies condition (iii) also. Thus the lemma is proved.

3. Theorem. *If for any additive function of rectangles* $F(I)$ *there is* $\underline{F}^*(x,y) > -\infty$ *at each point of a set* E, *then almost everywhere on* E *the function* $F(I)$ *is derivable (in the ordinary sense) and* $\underline{F}^*(x,y) = F'(x,y)$.

Proof. Suppose on the contrary that $\overline{F}(x,y) > \underline{F}^*(x,y) > -\infty$ on a set A of positive outer measure. We can obviously assume that $\underline{F}^*(x,y)$ is bounded below on A, and even, by adding an additive function $M|I|$ to $F(I)$, that $\underline{F}^*(x,y) > 0$ over A. Hence, there are a set $B \subset A$ of positive outer measure, a positive number σ and two numbers μ and λ such that

(3.1) $F(I) > 0$ *whenever* $d(I) < \sigma$ *and* $I \cdot B \neq 0$,

and that

(3.2) $\overline{F}(x,y) > \mu > \lambda > \underline{F}^*(x,y) > 0$ *for* (x,y) *in* B.

Let $\widetilde{B}$ denote the set of the outer density points of B, either belonging, or not, to the set B. The set $\widetilde{B}$ is apparently measurable and is introduced to make the discussion of the measurability of B superfluous.

Now, let α be a positive number such that

(3.3) $\alpha < 1$ *and* $\mu \cdot (1 - 3\alpha) > \lambda$.

We shall first prove that

(M) *With any point* p *in* $\widetilde{B} \cdot B$ *a sequence of squares* $\{S_n^*\}$ *may be associated so that* (i) *all corners of any* S_n^* *belong to* $\widetilde{B}$, (ii) $p \in S_n^*$, (iii) $F(S_n^*) > \mu \cdot (1 - 2\alpha) \cdot |S_n^*|$ *and* (iv) $d(S_n^*) \to 0$ *as* $n \to \infty$.

Indeed, in virtue of (3.2), there is a sequence of squares $\{S_n\}$ such that

(3.4) $d(S_n) < \frac{\sigma}{2}$, $p \in S_n$, $F(S_n) > \mu \cdot |S_n|$ *and* $d(S_n) \to 0$.

In virtue of the lemma of § 2, on supposing all squares S_n sufficiently small, we can attach a square S_n^* to any S_n so as to satisfy condition (i) in (M) and to have

(3.5) $d(S_n^*) < (1+\alpha) \cdot d(S_n)$ *and* $S_n \subset S_n^*$.

Then it directly follows from (3.4) that S_n^* satisfy conditions (ii) and (iv) in (M). In order to establish the remaining condition (iii), subdivide the area $\overline{S_n^* - S_n}$ into four rectangles, $I_1^{(n)}, I_2^{(n)}, I_3^{(n)}$ and $I_4^{(n)}$, say, so that any of them should contain a corner of S_n^*. Hence, each of these rectangles contains a point of $\widetilde{B}$, and consequently a point of B also. Thus, it results from (3.1) that $F(I_k^{(n)}) > 0$ for $k=1,2,3,4$, as by the first inequalities in (3.4) and (3.5), respectively, the diameters of S_n^*, and consequently those of $I_k^{(n)}$, are less than σ. Hence, by (3.5) and the third relation in (3.4), we have

$$F(S_n^*) \geqslant F(S_n) > \mu \cdot |S_n| > \mu \cdot |S_n^*|/(1+\alpha)^2 > \mu \cdot (1 - 2\alpha) \cdot |S_n^*|.$$

which is condition (iii) in (M).

Now, since almost all points of the set B are its density points in the strong sense, it follows from (3.2) that there is a rectangle R such that

(3.6) $F(R) < \lambda \cdot |R|$, $|B \cdot R| = |B \cdot \widetilde{B} \cdot R| > (1 - \alpha) \cdot |R|$, $d(R) < \sigma$.

Hence, by (M) and by the well-known Vitali Lemma there exist in R a finite set of not overlapping squares $J_1, J_2, \ldots, J_p$ such that

(3.7) $\sum_k |J_k| \geqslant (1 - \alpha) \cdot |R|$ *and* $F(J_k) \geqslant \mu \cdot (1 - 2\alpha) \cdot |J_k|$ *for* $k=1, 2, \ldots p$, the corners of each J_k belonging to $\widetilde{B}$. Further, as it is easily seen, the area $\overline{R-\sum_k J_k}$ may be subdivided into a finite number of not overlapping rectangles each of which contains one at least of the corners of the squares [4]) J_k. Hence, each of them contains points of $\widetilde{B}$, and, consequently, of B. Since $d(R)<\sigma$, it results from (3.1) that the function $F(I)$ is positive for each of these rectangles, and so, by (3.6) and (3.7)

$$\lambda\cdot|R|>F(R)>\sum_k F(J_k)\geqslant\mu\cdot(1-2\alpha)\,(1-\alpha)\cdot|R|\geqslant\mu\,(1-3\alpha)\cdot|R|.$$

This, however, is contradictory to (3.3) and concludes the proof.

[4]) This is directly obvious for the plane, but is not true for the space as seen from a simple example kindly communicated to the author by Mr. O. Nikodym. The problem whether the theorem itself holds for the space seems to be open, and the same remark applies to the results of Besicovitch, l. c. [2]).

Ensembles dont les dimensions modulaires de Alexandroff coïncident avec la dimension de Menger-Urysohn [1]).

Par

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Dans la théorie homologique de la dimension, due à M. P. Alexandroff [2]), on est conduit d'une manière naturelle à considérer une infinité d'invariants topologiques qui méritent — au moins du point de vue d'homologie — d'être appelés „dimensions“ (modulaires). Toutes ces „dimensions“, différentes pour les ensembles compacts arbitraires, se montrent identique avec la dimension au sens de Menger-Urysohn pour les ensembles dont la structure topologique est peu compliquée (en particulier pour tous les polyèdres). Dans le domaine de ces derniers ensembles, la théorie de la dimension prend une forme particulièrement simple, naturelle et intuitive. Ainsi p. ex. se trouve réalisée pour ces ensembles *„l'hypothèse du produit“* qui est en défaut — d'après M. L. Pontrjagin [3])— dans le domaine des ensembles compacts arbitraires.

Le but de cette Note est de définir par des notions de la topologie générale une classe d'ensembles (comprenant en particulier tous les polyèdres) pour lesquels toutes les dimensions modulaires coïncident avec la dimension au sens de Menger-Urysohn.

[1]) Les résultats principaux de cet ouvrage ont été signalés (sans démonstrations rigoureuses) dans les C. R. 201 (1935), p. 1086—7, séance du 2 décembre et C. R. 202 (1936), p. 187—189, séance du 20 janvier.

[2]) Cf. P. Alexandroff, *Dimensionstheorie*, Math. Ann. 106 (1932), p. 161—238.

[3]) L. Pontrjagin, C. R. 190 (1930), p. 1105—7, séance du 12 mai.