A lemma on the topological index [1]).

By

Tibor Radó (Columbus, Ohio).

Introduction.

Let the closed continuous curve C be given by equations $x=f(t)$, $y=g(t)$, $0 \leqq t \leqq 1$, $f(0)=f(1)$, $g(0)=g(1)$, where $f(t)$, $g(t)$ are continuous in $0 \leqq t \leqq 1$. If the point (x, y) is not on C, let $n(x, y)$ denote the topological index of (x, y) with respect to C [2]). If (x, y) is on C, then put $n(x, y)=0$. In connection with investigations in various fields (theory of the area, Calculus of Variations, transformation of double integrals) the study of $\int\int n(x, y)\, dx\, dy$ in its dependence upon the curve C became of fundamental importance in recent years. Let C_j be a sequence of closed continuous curves approximating C in the sense of Fréchet [3]). Let $n_j(x, y)$ have the same meaning with respect to C_j as that of $n(x, y)$ with respect to C. In most of the investigations referred to above, the scope of the result was determined by the character of the additional assumptions used by the respective authors to secure the relation $\int\int n_j \to \int\int n$. The purpose of this paper is to discuss this relation under conditions less restrictive than those considered in the literature, as far as it is known to the author. Our result is as follows. Let $T[g]$, $T[g_j]$ denote the total variations of the y-coordinates $g(t)$, $g_j(t)$ of the closed continuous curves C, C_j. If $C_j \to C$ *in the sense of Fréchet, and if* $T[g_j] \to T[g]$, *then* $\int\int |n_j - n| \to 0$ [3a]). Note that we obtain, instead of $\int\int n_j \to \int\int n$, the much stronger relation $\int\int |n_j - n| \to 0$. On the

[1]) Presented to the American Mathematical Society at the meeting in Chicago, April 1936.

[2]) For the properties of the topological index, see for instance Kerékjártó, *Vorlesungen über Topologie* (Berlin, J. Springer, 1923).

[3]) See 6.2.

[3a]) $T[g]$, $T[g_j]$ are supposed to be finite.

other hand, the approximations in general use at present (integral means, Stieltjes polynomials, and the like) are such that the strong assumption $T[g_j] \to T[g]$ is automatically satisfied. Due to these circumstances, our main result leads to a number of applications (in the fields mentioned above), which will be discussed elsewhere. Since for some of these applications the weaker relation $\int\int n_j \to \int\int n$ is quite sufficient, we included, among the corollaries in § 6, a few remarks concerning this relation.

The method used in the present paper depends essentially upon certain results of Banach on functions of bounded variation [4]). These results of Banach were first used by Schauder [5]) to control sequences of index-functions $n_j(x, y)$ with regard to term-wise integration. In the situation which Schauder considered the curves C_j were polygons inscribed in the curve C. For this special situation, Schauder derives from results of Banach a common summable upper bound for the absolute values of the functions $n_j(x, y)$. It is unlikely that such a bound exists in the more general situation considered in our main lemma. At any rate, we proceed in a different fashion. In the terminology of Vitali, we establish the uniformity of the absolute continuity of the integrals, considered as functions of sets, of the functions $n_j(x, y)$. That is, instead of working with *a sufficient condition*, we work with *the necessary and sufficient condition* for the term-wise integration of the sequence $|n_j(x, y)|$.

List of notations and definitions.

[4]) Banach, *Sur les lignes rectifiables et les surfaces dont l'aire est finie*, Fundamenta Mathematicae vol. 7, pp. 225—236.

[5]) Schauder, *Über stetige Abbildungen*, Fundamenta Mathematicae vol. 12, pp. 47—74.

§ 1. The function $N(y)$.

1.1. The purpose of this § 1 is to summarize, for the convenience of the reader, and in a form suitable for the present purposes, certain results of Banach on functions of bounded variation [6]).

1.2. Let $g(t)$ be continuous for $0 \leqq t \leqq 1$. If $g(t)$ is of bounded variation in $0 \leqq t \leqq 1$, then $T[g]$ will denote the total variation of $g(t)$.

1.3. We think of the function $g(t)$ as defining a transformation $y=g(t)$ of the points in $0 \leqq t \leqq 1$ into points on the y-axis. Since $g(t)$ is bounded, the image of $0 \leqq t \leqq 1$ is comprised in some interval $|y| \leqq K$, where K is a finite constant. We shall denote by $\varphi_{t_1}^{t_2}(y)$ the characteristic function of the image, on the y-axis, of the sub-interval $t_1 \leqq t \leqq t_2$. That is, $\varphi_{t_1}^{t_2}(y)=1$ if the equation $y=g(t)$ has at least one root t such that $t_1 \leqq t \leqq t_2$, and $\varphi_{t_1}^{t_2}(y)=0$ if $g(t) \neq y$ for $t_1 \leqq t \leqq t_2$. If we denote by $M_{t_1}^{t_2}$, $m_{t_1}^{t_2}$ the maximum and the minimum respectively of $g(t)$ in $t_1 \leqq t \leqq t_2$, then clearly

$$\varphi_{t_1}^{t_2}(y)=1 \qquad \text{for} \quad m_{t_1}^{t_2} \leqq y \leqq M_{t_1}^{t_2},$$

and

$$\varphi_{t_1}^{t_2}(y)=0 \qquad \text{for} \quad t<m_{t_1}^{t_2} \quad \text{and} \quad t>M_{t_1}^{t_2},$$

and

$$\int_{K}^{K}\varphi_{t_1}^{t_2}(y)\,dy=M_{t_1}^{t_2}-m_{t_1}^{t_2}.$$

1.4. The function $N(y)$ is defined, for $|y| \leqq K$, as the number of distinct roots of the equation $g(t)=y$ in $0 \leqq t \leqq 1$. If this equation has infinitely many distinct roots, then we put $N(y)=\infty$.

1.5. On the y-axis, let us mark all the points $m_{t_1}^{t_2}$, $M_{t_1}^{t_2}$, where t_1 and $t_2>t_1$ take on all *rational* values in $0 \leqq t \leqq 1$, and let us also mark all the points which correspond, under the transformation $y=g(t)$, to *rational* values of t. The set of points thus marked will be denoted by S^*. The set S^* is denumerable and hence of measure zero.

1.6. If h is a positive integer, we define

$$\psi_h(y)=\sum_{k=1}^{h}\varphi_{(k-1)/h}^{k/h}(y).$$

[6]) See [4]).

1.7. Obviously $\psi_h(y) \leqq 2N(y)$ always and $\psi_h(y) \leqq N(y)$ for y not in S^*.

1.8. Obviously, if H is any number less than $N(y)$, then $\psi_h(y) \geqq H$ for h larger than some $h_0=h_0(y, H)$.

1.9. Obviously (see 1.2).

$$\int_{-K}^{K}\psi_h(y)\,dy=\sum_{k=1}^{h}(M_{(k-1)/h}^{k/h}-m_{(k-1)/h}^{k/h}).$$

1.10. Obviously, it follows from 1.7, 1.8, 1.9 that $\psi_h(y) \to N(y)$ almost everywhere in $|y| \leqq K$. Hence $N(y)$ is measurable. If $g(t)$ is of bounded variation, then we have, from 1.9,

$$\int_{-K}^{K}\psi_h(y)\,dy \to T[g] \qquad \text{for} \quad h \to \infty.$$

The integrals of the functions $\psi_h(y)$ being thus uniformly bounded, it follows that $N(y)$ is summable. We infer finally from 1.7 that

$$\int_{-K}^{K}\psi_h(y)\,dy \to \int_{-K}^{K}N(y)\,dy.$$

Hence, if $g(t)$ is of bounded variation, then $N(y)$ is summable and

$$\int_{-K}^{K}N(y)\,dy=T[g].$$

Conversely, if $N(y)$ is summable, then it follows from 1.7 and 1.9 that

$$\sum_{k=1}^{h}(M_{(k-1)/h}^{k/h}-m_{(k-1)/h}^{k/h}) \leqq 2\int_{-K}^{K}N(y)\,dy.$$

for all values of h. Consequently, if $N(y)$ is summable, then $g(t)$ is of bounded variation.

§ 2. A remark on sequences of positive functions.

2.1. Let $\mathfrak{K}$ denote a set of functions $F(y)$, each of which is defined and summable in the same finite interval $|y| \leqq K$. We shall say that $\mathfrak{K}$ possesses the property $\mathfrak{P}$ if the following condition is satisfied. To every $\varepsilon>0$ there corresponds an $\eta=\eta(\varepsilon)>0$ such that

$$\int_S |F(y)|\,dy<\varepsilon$$

for every function $F(y)$ of $\mathfrak{K}$ and for every measurable set S in $|y| \leqq K$, with a measure less than $\eta(\varepsilon)$ [7]).

2.2. In a finite interval $|y| \leqq K$ let there be given a sequence of functions $F_j(y)$ and a function $F(y)$. Suppose that the following conditions are satisfied.

a) $F(y) \geqq 0$, $F_j(y) \geqq 0$ in $|y| \leqq K$.

b) $\underline{\lim}\, F_j(y) \geqq F(y)$ almost everywhere in $|y| \leqq K$.

c) $F(y)$, $F_j(y)$ are summable in $|y| \leqq K$.

d) $\int_{-K}^{K} F_j(y)\,dy \to \int_{-K}^{K} F(y)\,dy$.

Then the set of functions $F(y), F_j(y)$ possesses the property $\mathfrak{P}$.

2.3. If we replace $\underline{\lim}\, F_j(y) \geqq F(y)$ by $\lim F_j(y) = F(y)$ in condition b), then the above statement reduces to a well-known theorem and can be proved by almost exactly the same argument. The reader is therefore requested to assume the easy task of adapting the proof given loc. cit. [7]) to the situation described in 2.2.

§ 3. A remark on sequences $N_j(y)$.

3.1. In the interval $0 \leqq t \leqq 1$, let there be given a continuous function $g(t)$ and a sequence of continuous functions $g_j(t)$. The symbols

$$_j\varphi_{t_1}^{t_2}(y), \quad _jM_{t_1}^{t_2}, \quad N_j(y), \quad S_j^*, \ldots$$

are defined with respect to $g_j(t)$ in the same way as the symbols

$$\varphi_{t_1}^{t_2}(y), \quad M_{t_1}^{t_2}, \quad N(y), \quad S^*, \ldots$$

were defined with respect to $g(t)$.

3.2. ***Lemma.*** *Suppose that the function $g(t)$ and every function of the sequence $g_j(t)$ is continuous and of bounded variation in $0 \leqq t \leqq 1$. Suppose also that $T[g_j] \to T[g]$ and $g_j(t) \to g(t)$ uniformly in $0 \leqq t \leqq 1$. Then the set of functions $N(y), N_j(y)$ possesses the property $\mathfrak{P}$.*

[7]) The importance of this property in problems on term-wise integration was first stressed by Vitali. See for instance de la Vallée Poussin, *Sur l'intégrale de Lebesgue*, Transactions Amer. Math. Soc., vol. 16, pp. 435—501, in particular p. 446.

Let us observe that under the above assumptions we have a finite interval $|y| \leqq K$ such that all the functions $N(y)$, $N_j(y)$ are equal to zero for $|y| > K$. We can restrict therefore our attention to this interval $|y| \leqq K$.

3.3. Using the notations of § 1, we show first that

$$N(y) \leqq \underline{\lim}\, N_j(y) \quad \text{almost everywhere in } |y| \leqq K.$$

Suppose that y is not in the set $S^* + \sum S_j^*$. Consider, for fixed rational t_1, t_2, the functions $\varphi_{t_1}^{t_2}(y)$, $_j\varphi_{t_1}^{t_2}(y)$. We assert that

$$\varphi_{t_1}^{t_2}(y) \leqq {_j\varphi_{t_1}^{t_2}}(y) \qquad \text{for } j > j_0 = j_0(t_1, t_2, y).$$

If $\varphi_{t_1}^{t_2}(y) = 0$, this is obvious. If $\varphi_{t_1}^{t_2}(y) = 1$, then (cf. 1.2) we have

$$m_{t_1}^{t_2} \leqq y \leqq M_{t_1}^{t_2}.$$

But y is not in S^*, hence we cannot have the sign of equality. Thus

$$m_{t_1}^{t_2} < y < M_{t_1}^{t_2}.$$

Since $g_j(t) \to g(t)$ uniformly, we shall have therefore

$$_jm_{t_1}^{t_2} < y < {_jM_{t_1}^{t_2}}$$

for $j > j_0 = j_0(t_1, t_2, y)$. Hence (cf. 1.2)

$$\varphi_{t_1}^{t_2}(y) = 1 = {_j\varphi_{t_1}^{t_2}}(y) \qquad \text{for } j > j_0(t_1, t_2, y).$$

Consider now the functions (cf. 1.6)

$$\psi_h(y) = \sum_{k=1}^{h} \varphi_{(k-1)/h}^{k/h}(y),$$

$$_j\psi_h(y) = \sum_{k=1}^{h} {_j\varphi_{(k-1)/h}^{k/h}}(y).$$

On account of the preceding remark, we shall have

$$\psi_h(y) \leqq {_j\psi_h}(y) \qquad \text{for } j > j_0^* = j_0^*(h, y).$$

Since y is not in S_j^*, we have (see 1.7)

$$_j\psi_h(y) \leqq N_j(y).$$

Hence

$$\psi_h(y) \leqq N_j(y) \qquad \text{for } j > j_0^* = j_0^*(h, y),$$

and consequently

$$\psi_h(y) \leqq \underline{\lim}\, N_j(y).$$

Since y is not in S^*, we obtain (see 1.10) for $h\to\infty$ the desired inequality

$$N(y)\leqq \underline{\lim}\, N_j(y)$$

for y not in $S^*+\sum S_j^*$, that is almost everywhere in $|y|\leqq K$.

3.4. On account of 1.10, the assumption $T[g_j]\to T[g]$ implies that

$$\int_{-K}^{K} N_j(y)\,dy \to \int_{-K}^{K} N(y)\,dy.$$

3.5. The lemma in 3.2 appears now as a direct consequence of the remark made in § 2. Indeed, the functions $N(y)$, $N_j(y)$ satisfy condition a) in 2.2 by definition, condition b) by 3.3, condition c) by 1.10 and condition d) by 3.4.

§ 4. Preliminary remarks on the topological index.

4.1. Let the closed continuous curve C be defined by the equations $x=f(t)$, $y=g(t)$, $0\leqq t\leqq 1$, where $f(t)$, $g(t)$ are continuous in $0\leqq t\leqq 1$ and $f(0)=f(1)$, $g(0)=g(1)$. Let (x,y) be a point not on C, and let (ξ,η) be a point which describes C in the sense of increasing t-values. Then the continuously varied argument of the complex number $\xi+i\eta-(x+iy)$ changes by a certain amount $2k\pi$, where k is an integer (positive, negative, or zero). This integer k is the topological index of the point (x,y) with respect to the (directed) closed continuous curve C.

4.2. The topological index of the point (x,y) with respect to C is a function of x,y which we shall denote by $n(x,y)$. For (x,y) on C, we put $n(x,y)=0$. Clearly, $n(x,y)$ is a measurable function.

4.3. The closed continuous curve C being given as above, suppose we have a sequence of closed continuous curves

$$C_j: x=f_j(t),\quad y=g_j(t),\quad 0\leqq t\leqq 1,\quad f_j(0)=f_j(1),\quad g_j(0)=g_j(1),$$

where $f_j(t)$, $g_j(t)$ are continuous in $0\leqq t\leqq 1$. Suppose $f_j(t)\to f(t)$, $g_j(t)\to g(t)$ uniformly for $0\leqq t\leqq 1$. Let $n_j(x,y)$ have the same meaning with respect to C_j as that of $n(x,y)$ with respect to C. Then $n_j(x,y)\to n(x,y)$ for (x,y) not on C, as it is well known.

It is also well known that the following more precise statement is true: if k is a closed circular disc which has no point in common with C, then $n_j(x,y)=n(x,y)$ on k for j greater than some $j_0=j_0(k)$. It follows then, on account of the Heine-Borel theorem: if S is a closed set in the (x,y)-plane which has no point in common with C, then $n_j(x,y)=n(x,y)$ on S for j greater than some $j_0=j_0(S)$.

4.4. Suppose we have a subdivision of $0\leqq t\leqq 1$ by points $0=t_0<\ldots<t_{k-1}<t_k<\ldots<t_h=1$, such that the functions $f(t)$, $g(t)$ are both linear in each of the closed subintervals $t_{k-1}\leqq t\leqq t_k$. Then the curve C is a polygon and will be denoted by Π. Let (x,y) be a point not on Π, and draw from (x,y) a ray r which does not pass through any of the vertices of Π. Let $P_0,\ldots,P_{k-1},P_k,\ldots,P_h,P_0$ be the vertices of Π, numbered in the sense of increasing t-values. If r intersects the side $P_{k-1}P_k$ of Π in a point A_k, and if (x,y), P_{k-1}, P_k is the counter-clockwise sense around the triangle (x,y), P_{k-1}, P_k, then A_k is called a point of *positive crossing*. If (x,y), P_{k-1}, P_k is the clock-wise sense around the triangle (x,y), P_{k-1}, P_k, then A_k is called a point of *negative crossing* (remember that the ray r does not pass through any vertex.) Clearly, the topological index of (x,y) with respect to Π is equal to the difference between the number of points of positive crossing and the number of points of negative crossing. Hence the absolute value of this topological index is certainly not greater than the total number of the sides of Π which are intersected by a line l through (x,y) which does not contain any vertex of Π.

4.5. Let the closed continuous curve C be given by equations $x=f(t)$, $y=g(t)$ as in 4.1. Denote by Π_h the inscribed polygon corresponding to the subdivision of $0\leqq t\leqq 1$ into h equal parts. Let us use the notations of § 1, the function $g(t)$ of § 1 being identified with the y-coordinate of C. Take a point (x,y) which is not on C and whose y-coordinate is not in the set S^* (see 1.4). Let l be the line through (x,y) parallel to the x-axis. This line does not pass then through any vertex of any of the polygons Π_h. Therefore we have, for the topological index $n_h(x,y)$ of (x,y) with respect to Π_h, the inequality $|n_h(x,y)|\leqq\psi_h(y)$ where $\psi_h(y)$ is the function defined in 1.5. We have therefore, on account of 1.7, $|n_h(x,y)|\leqq N(y)$. For

$h\to\infty$ it follows that $|n(x,y)|\leqq N(y)$, where $n(x,y)$ is defined as in 4.2. Since the set S^* is of measure zero, we see that $|n(x,y)|\leqq N(y)$ almost everywhere [8]).

4.6. On account of 1.10, it follows that if the y-coordinate of the closed continuous curve C is of bounded variation, then the topological index $n(x,y)$ is a summable function.

§ 5. The lemma.

5.1. Let there be given a closed continuous curve

$$C: x=f(t),\quad y=g(t),\quad 0\leqq t\leqq 1,\quad f(0)=f(1),\quad g(0)=g(1),$$

and a sequence of closed continuous curves

$$C_j: x=f_j(t),\quad y=g_j(t),\quad 0\leqq t\leqq 1,\quad f_j(0)=f_j(1),\quad g_j(0)=g_j(1).$$

All the functions $f(t)$, $g(t)$, $f_j(t)$, $g_j(t)$ are supposed to be continuous in $0\leqq t\leqq 1$. In dealing with the functions $g(t)$, $g_j(t)$, we shall use the notations of § 3. The symbols $n(x,y)$, $n_j(x,y)$ are defined as in 4.3.

5.2. ***Lemma.*** *The closed continuous curves C, C_j being defined as in* 5.1, *suppose that*

a) $f_j(t)\to f(t)$, $g_j(t)\to g(t)$ *uniformly in* $0\leqq t\leqq 1$;
b) *the functions $g(t)$, $g_j(t)$ are of bounded variation in* $0\leqq t\leqq 1$;
c) $T[g_j]\to T[g]$.

Under these conditions, all the curves C, C_j are comprised in some finite square, $|x|\leqq K$, $|y|\leqq K$, and we have the relation

$$\int_{-K}^{K}\int_{-K}^{K}|n(x,y)-n_j(x,y)|\,dx\,dy\to 0 \qquad \text{for } j\to\infty.$$

5.3. To prove this, let there be given any $\sigma>0$. On account of 5.2, condition b), the set of points (x,y) on C has its (two-dimensional) measure equal to zero. Hence we have, in the square $|x|\leqq K$, $|y|\leqq K$, an open set Q_σ with $|Q_\sigma|<\sigma$ which covers C. We denote by $q_\sigma(x,y)$ the characteristic function of Q_σ. That is, $q_\sigma(x,y)=1$ for (x,y) in Q_σ, and $q_\sigma(x,y))=0$ otherwise.

[8]) As far as the present author is aware, this type of limitation for $|n(x,y)|$ was first used by Schauder, loc. cit. [5]).

5.4. We denote by Q_σ^* the complement of Q_σ with respect to the closed square $|x|\leqq K$, $|y|\leqq K$. Then Q_σ^* is a closed set. Hence, by 4.3, we have a j_0 such that $C_j\subset Q_\sigma$ and $n_j(x,y)=n(x,y)$ on Q_σ^* for $j>j_0$.

5.5. Consider any one of the functions $n_j(x,y)$. By 4.6, $n_j(x,y)$ is summable, and by 4.5 we have $|n_j(x,y)|\leqq N_j(y)$ almost everywhere in $|x|\leqq K$, $|y|\leqq K$. Hence

$$\iint_{Q_\sigma}|n_j(x,y)|\,dx\,dy=$$

$$=\int_{-K}^{K}\int_{-K}^{K}|n_j(x,y)|\,q_\sigma(x,y)\,dx\,dy\leqq\int_{-K}^{K}\left(\int_{-K}^{K}q_\sigma(x,y)\,dx\right)N_j(y)\,dy.$$

Denote by E_σ the set of those points y in $|y|\leqq K$ for which

$$\int_{-K}^{K}q_\sigma(x,y)\,dx \quad\text{exists and is}\quad >|Q_\sigma|^{1/2},$$

and let E_σ^* be the complement of E_σ which respect to $|y|\leqq K$. Then

$$|Q_\sigma|=\int_{-K}^{K}\int_{-K}^{K}q_\sigma(x,y)\,dx\,dy\geqq\int_{E_\sigma}\left(\int_{-K}^{K}q_\sigma(x,y)\,dx\right)dy>|Q_\sigma|^{1/2}|E_\sigma|,$$

and consequently

$$|E_\sigma|<|Q_\sigma|^{1/2}<\sigma^{1/2}.$$

5.6. It follows that

$$\iint_{Q_\sigma}|n_j(x,y)|\,dx\,dy\leqq\int_{E_\sigma}\left(\int_{-K}^{K}q_\sigma(x,y)\,dx\right)N_j(y)\,dy+$$

$$+\int_{E_\sigma^*}\left(\int_{-K}^{K}q_\sigma(x,y)\,dx\right)N_j(y)\,dy\leqq 2K\int_{E_\sigma}N_j(y)\,dy+|Q_\sigma|^{1/2}\int_{E_\sigma^*}N_j(y)\,dy.$$

But, by 1.10,

$$\int_{E_\sigma^*}N_j(y)\,dy\leqq\int_{-K}^{K}N_j(y)\,dy=T[g_j].$$

Thus we obtain, since $|Q_\sigma|<\sigma$, the inequality

$$\iint_{Q_\sigma}|n_j(x,y)|\,dx\,dy\leqq 2K\int_{E_\sigma}N_j(y)\,dy+\sigma^{1/2}\,T[g_j].$$

The same reasoning, applied to $n(x, y)$, leads to

$$\iint_{Q_\sigma} |n(x, y)|\, dx\, dy \leqq 2K \int_{E_\sigma} N(y)\, dy + \sigma^{1/2}\, T[g].$$

Let us recall that $|E_\sigma| < \sigma^{1/2}$.

5.7. By 3.2, the set of functions $N(y)$, $N_j(y)$ possesses the property $\mathfrak{P}$. That is, given any $\varepsilon > 0$, we have an $\eta = \eta(\varepsilon) > 0$ such that

$$\int_S N(y)\, dy < \varepsilon, \qquad \int_S N_j(y)\, dy < \varepsilon,$$

for all values of j and for all measurable sets S in $|y| \leqq K$ with $|S| < \eta(\varepsilon)$. Suppose $\varepsilon > 0$ is prescribed at pleasure. Choose the σ of 5.3 so that $\sigma^{1/2} < \eta(\varepsilon)$ and $\sigma^{1/2} < \varepsilon$. Then the inequalities of 5.6 yield [9])

$$\iint_{Q_\sigma} |n_j(x, y)|\, dx\, dy < 2K\varepsilon + T[g_j]\,\varepsilon,$$

$$\iint_{Q_\sigma} |n(x, y)|\, dx\, dy < 2K\varepsilon + T[g]\,\varepsilon.$$

By 5.4 we have then, for $j > j_0$,

$$\int_{-K}^{K}\int_{-K}^{K} |n_j(x, y) - n(x, y)|\, dx\, dy = \iint_{Q_\sigma} |n_j(x, y) - n(x, y)|\, dx\, dy <$$
$$< (4K + T[g] + T[g_j])\,\varepsilon.$$

Since $T[g_j] \to T[g]$, it follows that

$$\overline{\lim_{j\to\infty}} \int_{-K}^{K}\int_{-K}^{K} |n_j(x, y) - n(x, y)|\, dx\, dy \leqq (4K + 2T[g])\,\varepsilon.$$

Since $\varepsilon > 0$ was arbitrarily given, this implies

$$\lim_{j\to\infty} \int_{-K}^{K}\int_{-K}^{K} |n_j(x, y) - n(x, y)|\, dx\, dy = 0,$$

and the lemma is proved.

[9]) Essentially, these inequalities express the fact that the set of functions $n(x, y)$, $n_j(x, y)$ possesses the property $\mathfrak{P}$, stated for functions of two variables. Cf. the introduction.

§ 6. Corollaries.

6.1. With regard to applications to be discussed elsewhere, we state first an immediate consequence of the lemma in 5.2. The closed continuous curves C, C_j being given as in 5.2 and satisfying the conditions stated there, we have some finite square $|x| \leqq K$, $|y| \leqq K$ which contains all these curves. Let there be given, in this square, a uniformly bounded sequence of measurable functions $F_j(x, y)$ which converge almost everywhere to a (bounded and measurable) function $F(x, y)$. *Then we have the relation*

$$\int_{-K}^{K}\int_{-K}^{K} F_j(x, y)\, n_j(x, y)\, dx\, dy \to \int_{-K}^{K}\int_{-K}^{K} F(x, y)\, n(x, y)\, dx\, dy.$$

Indeed, we have

$$\left|\iint Fn - \iint F_j n_j\right| \leqq \iint |F - F_j|\, |n| + \iint |F_j|\, |n - n_j|,$$

the integrals being extended over the square $|x| \leqq K$, $|y| \leqq K$. By assumption we have $|F_j| < M$, $|F - F_j| < 2M$ in this square, M being some finite constant independent of j. Since $|n|$ is summable, the relations $(F - F_j)\, n \to 0$ almost everywhere and $|(F - F_j)n| < 2M|n|$ imply that $\iint |F - F_j|\, |n| \to 0$, while $\iint |F_j|\, |n - n_j| < M \iint |n - n_j| \to 0$ follows directly from 5.2.

6.2. Let there be given a closed continuous curve C and a sequence C_j of such curves. The statement that $C_j \to C$ in the sense of Fréchet is then equivalent to the statement that these curves admit of simultaneous representations

$$C: x = f(t),\quad y = g(t),\quad 0 \leqq t \leqq 1,\quad f(0) = f(1),\quad g(0) = g(1)$$
$$C_j: x = f_j(t),\quad y = g_j(t),\quad 0 \leqq t \leqq 1,\quad f_j(0) = f_j(1),\quad g_j(0) = g_j(1),$$

where f, g, f_j, g_j are continuous in $0 \leqq t \leqq 1$ and $f_j \to f$, $g_j \to g$ uniformly in $0 \leqq t \leqq 1$. Suppose now that all these curves are rectifiable and that the length $l(C_j)$ of C_j converges to the length $l(C)$ of C. According to Adams and Lewy [10]), these assumptions imply that

[10]) C. R. Adams and Hans Lewy, *On convergence in length*, Duke Math. Journal, vol. I, pp. 19—26. See their theorem 1 on page 20. For the sake of accuracy, it should be observed that these authors only consider sequences of arcs given in the non-parametric form $y = f(x)$. However, their method applies, with trivial modifications, to the more general situation considered in our text.

$T[g_j] \to T[g]$, $T[f_j] \to T[f]$. We have therefore, on account of 5.2, the following theorem.

If the closed continuous rectifiable curves C, C_j *satisfy the conditions* $C_j \to C$ *in the sense of* Fréchet *and* $l(C_j) \to l(C)$, *then* $\iint |n - n_j| \to 0$.

6.3 We shall now make a few remarks concerning *theorems of the weak type* $\iint n_j \to \iint n$. In deriving these theorems, we shall replace the assumption $T[g_j] \to T[g]$ by weaker assumptions requiring only uniform boundedness of the total variation. While it may seem that this will increase the scope of applications, it should be observed that all the methods of approximation in general use at present are such that the condition $T[g_j] \to T[g]$ is automatically satisfied. In the following statements of the weak type $\iint n_j \to \iint n$, no effort has been made to reduce the assumptions to a minimum.

6.4. Let Π be a closed polygon given as in 4.4. Let $n(x, y)$ be the index-function relative to Π, and let $P_0, P_1, \ldots, P_{k-1}, P_k, \ldots, P_j = P_0$ be the vertices of Π in the order of increasing t-values. If (x_k, y_k) are the coordinates of P_k, then we have (as an easy consequence of the remarks made in 4.4) the relation

$$\iint n(x, y)\, dx\, dy = \tfrac{1}{2} \sum_{k=1}^{j} x_{k-1}(y_k - y_{k-1}) - \tfrac{1}{2} \sum_{k=1}^{j} y_{k-1}(x_k - x_{k-1}).$$

6.5. Consider now a closed continuous curve C given as in 6.2. Denote by Π_j the inscribed polygon corresponding to the subdivision of $0 \leqq t \leqq 1$ into j equal parts. Let $x = f_j(t)$, $y = g_j(t)$ be the equations of Π_j, the functions f_j, g_j being linear between adjacent points of division. Clearly

$$T[g_j] = \sum_{k=1}^{j} \left| g\left(\frac{j}{k}\right) - g\left(\frac{k-1}{j}\right) \right|.$$

Hence, if $g(t)$ is of bounded variation, we have $T[g_j] \to T[g]$. Thus the lemma of 5.2 applies. Since, by 6.4, obviously

$$\iint n_j \to \tfrac{1}{2} \int_0^1 f\, dg - \tfrac{1}{2} \int_0^1 g\, df = \int_0^1 f\, dg,$$

we obtain the result that *if the y-coordinate $g(t)$ of the closed continuous curve C is of bounded variation, then*

$$\iint n(x, y)\, dx\, dy = \int_C x\, dy,$$

where the double integral is taken over any square $|x| \leqq K$, $|y| \leqq K$ containing C [11]).

6.6. To illustrate the way in which the preceding remark may be utilized, we consider the following situation. *Let the closed continuous curves* C, C_j *be given as in 6.2, and suppose that all the functions* f, g, f_j, g_j *are of bounded variation. Suppose that* $T[f_j] < M$, $T[g_j] < M$, *where M is some finite constant independent of j. Then* $\iint n_j \to \iint n$, the integration being extended over any square containing all the curves C, C_j.

The proof is immediate. By 6.4, the assertion $\iint n_j \to \iint n$ is equivalent to

$$\int_0^1 f_j\, dg_j \to \int_0^1 f\, dg,$$

and this relation is a well-known direct consequence of the identity

$$\int_1^1 f dg - \int_0^1 f_j\, dg_j = \int_0^1 (g_j - g) df + \int_0^1 (f - f_j) dg_j. \quad ^{12})$$

6.7. Using again the theorem of Adams and Lewy (see 6.2) we obtain the following corollary of the preceding statement. *If the closed continuous rectifiable curves* C, C_j *satisfy the conditions* $C \to C_j$ *in the sense of* Fréchet *and* $l(C_j) < M$ *(a finite constant independent of j), then* $\iint n_j \to \iint n$.

11) This result could be obtained also without using the main lemma in 5.2.

12) For the simple properties of the Riemann-Stieltjes integral needed here see for instance Hobson, *The theory of functions of a real variable*, vol. I (second edition), p. 507.

The Ohio State University,
March 1936.