E_1 et l'ensemble $f(E_1) = \varphi(E_1) \subset \varphi(E) = H$. Comme l'image homéomorphe d'un ensemble toujours de première catégorie, l'ensemble $f(E_1)$ l'est donc aussi. Or, c'est impossible, puisque $f(E_1)$, en tant que sous-ensemble de l'ensemble H jouissant de la propriété $\boldsymbol{L}$, jouit également de cette propriété et par suite, en tant que indénombrable, ne jouit pas de la propriété de Baire [1]).

La fonction $f(x)$ est donc de classe $\leqslant 2$ et elle est discontinue sur tout sous-ensemble indénombrable de l'ensemble E, c. q. f. d.

[1]) Voir p. ex. mon livre cité, p. 41.

On continuous transformations in the plane *).

By

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Introduction.

The purpose of this paper is to present certain remarks which occurred to me while studying the important papers of Schauder and Banach on continuous transformations [1]). While the methods of Banach and Schauder are such that their results are obviously valid in spaces of any finite number of dimensions, it might be of interest to observe that in the two-dimensional case, which is quite important for the applications, it is possible to obtain more complete information. This is essentially due to the existence of transformations which are defined, in terms of complex numbers, by equations of the form $w = z^n$, n an integer. An immediate consequence is the simple lemma in 1.3. Roughly speaking, the lemma states that the image of a plane region, under a continuous transformation, is similar in some respects to the Riemann surfaces employed in the theory of functions of a complex variable [2]). The inference from the lemma is

*) Presented to the American Math. Society at the meeting in Chicago, April 1936.

[1]) S. Banach, *Sur les lignes rectifiables et sur les surfaces dont l'aire est finie*, Fundamenta Mathematicae, vol. 7 (1925), pp. 225—236; J. Schauder, *Über stetige Abbildungen*, Fundamenta Mathematicae, vol. 12 (1928), pp. 47—74.

[2]) The reader will notice that our use of the transformation $w = z^n$, in proving the lemma in 1.3, corresponds to the process of *local uniformization* in the theory of Riemann surfaces. The manner in which the topological index is used in 1.3 is of course familiar to students of the theory of functions of a complex variable, where a similar reasoning is applied to discuss the transformation $w = f(z)$, $f(z)$ an analytic function, in the vicinity of a point where $f'(z) = 0$. — An interesting application of the transformation $w = z^n$ to *the special case of two dimensions* is discussed by S. Saks, *Sur une inégalité de la théorie des fonctions*, Acta Szeged, vol. 44 (1928), pp. 51—55, in particular corollary 2, p. 53.

information concerning the topological index, one of the basic tools in this line of work, which is much more precise than what Schauder used in the n-dimensional case. As a result, we shall obtain, in § 4, more efficient theorems for general continuous transformations. The application of the essentially topological results of § 4 to metrically specialized transformations is rather obvious. As might be expected, the theorems obtained in this manner for the two-dimensional case are considerably more general and final in character than the corresponding results of Schauder for the n-dimensional case. In the way of illustration, some of these theorems are discussed in § 5.

Index of terms and symbols.

§ 1. On the topological index.

1.1 Suppose $w = f(z)$ is a (complex-valued) continuous function in some closed region $\mathfrak{R}$, bounded by a finite number of non-intersecting Jordan curves $C_1, C_2, \ldots, C_n$. Let C be a Jordan curve in $\mathfrak{R}$, and w_0 a complex number such that $f(z) \neq w_0$ on C. If z describes C in the counter-clockwise sense, then the point $w = f(z)$ describes, in the w-plane, a (directed) continuous curve C^*. The (continuously varied) argument of $f(z) - w_0$ changes by a certain amount $2k\pi$, where k is an integer (positive, negative, or zero). The integer k is *the topological index* of the point w_0 with respect to the (directed) continuous curve C^*.

1.2. Suppose that w_0 is a complex number such that $f(z) \neq w_0$ on $C_1, C_2, \ldots, C_n$. Let k_i be the topological index of w_0 with respect to the image C_i^* of C_i under the transformation $w = f(z)$ (each of the Jordan curves C_i being described in the counter-clockwise sense). Supposing that C_1 encloses $C_2, \ldots, C_n$, we have the following well-known facts.

a) If $f(z) \neq w_0$ in $\mathfrak{R}$, then $k_1 = k_2 + \ldots + k_n$.

b) Consequently, if $k_1 \neq k_2 + \ldots + k_n$, then the equation $f(z) - w_0 = 0$ has at least one root in the interior of $\mathfrak{R}$.

In particular, for $n=1$, we have the following special statements.

If $f(z)$ is continuous in and on a Jordan curve C, and if $f(z) \neq w_0$ in and on C, then the topological index k of w_0 with respect to the image of C under the transformation $w = f(z)$ is equal to zero. If we only assume that $f(z) \neq w_0$ on C, and if we find that the index k is different from zero, then the equation $f(z) = w_0$ has at least one root in the interior of C.

1.3. One of our main arguments in the sequel will be a more precise form of the last statement. Suppose that C is a Jordan curve in the z-plane which contains $z=0$ in its interior. Let $f(z)$ be a function with the following properties.

a) $f(z)$ is continuous in and on C.

b) $f(0) = 0$, but $f(z) \neq 0$ otherwise in and on C.

c) The topological index k of the point $w=0$ with respect to the image of C under the transformation $w = f(z)$ is different from zero.

Then there exists a $\varrho > 0$ such that for $0 < |w| < \varrho$ the equation $f(z) = w$ has at least $|k|$ distinct roots in the interior of C [3]).

To prove this, let C' denote any Jordan curve interior to C and enclosing $z=0$. Starting from a point z_0 on C', let z describe C' in the counter-clockwise sense. By 1.2, the (continuously varied) argument of $f(z)$ changes then by $2k\pi$ on account of the above assumption c), and hence $f(z)^{1/|k|}$, if varied continuously, returns to its initial value. *That is, we have a singlevalued continuous branch of $f(z)^{1/|k|}$ in and on C.* In other words, we can write

$$f(z) = g(z)^{|k|},$$

[3]) See [2]).

where $g(z)$ is single-valued and continuous in and on C. Clearly, $g(0)=0$, but $g(z)\neq 0$ otherwise in and on C. Furthermore, the topological index of $w=0$ with respect to the image of C under the transformation $w=g(z)$ is $+1$ or -1, and hence different from zero. Consequently, we have a $\delta>0$ such that for $0<|w|<\delta$ the topological index of w with respect to the same image is also different from zero, and $g(z)\neq w$ on C. Hence, by 1.2 for $0<|w|<\delta$ the equation $g(z)=w$ has at least one root interior to C. Suppose now that

$$0<|w|<\varrho=\delta^{|k|}.$$

Let $w_1, w_2, \dots, w_{|k|}$ denote the $|k|$-th roots of w. Then

$$0<|w_j|<\delta, \qquad j=1,2,\dots,|k|.$$

Hence we have, for each w_j, at least one z_j, interior to C, such that $g(z_j)=w_j$, and consequently $f(z_j)=w$. Since $w_1, \dots, w_{|k|}$ are all distinct, and since $g(z)$ is single-valued, these points $z_1, \dots, z_{|k|}$ are all distinct, and the proof is thus finished.

§ 2. The function $N_E(w)$.

2.1. *The continuous transformation* $w=f(z)$. We shall assume in the sequel that $f(z)$ is single-valued and continuous in a closed square S_0 in the z-plane. The boundary of S_0 will be denoted by B_0, its image in the w-plane by B_0^*. Since $|f(z)|$ is bounded on S_0, the image of S_0 is comprised in some finite circle in the w-plane. The set of points in and on this circle will be denoted by K_0^*. If z in S_0 and w in K_0^* are related by the equation $w=f(z)$, then w will be called *the image* of z, and z will be called *a model* of w. A point w might have several, and even infinitely many, models.

2.2. If E is a point-set in S_0, then the function $N_E(w)$ is defined, in K_0^*, as the number of distinct models, comprised in E, of w [4]). If w has infinitely many models in E, then $N_E(w)=+\infty$. Since we only assume that the transformation $w=f(z)$ is continuous, we cannot assert that $N_E(w)$ is measurable whenever the set E is measurable. However, Banach [5]) proved that $N_E(w)$ *is measurable whenever E is a closed square*. Using his method, the reader will easily verify that $N_E(w)$ is *measurable whenever E is an open set*.

[4]) This function was introduced by Banach, loc. cit. [1]).
[5]) Loc. cit. [1]).

2.3. In particular the function $N_{S_0}(w)$ represents the total number of models of w. This function $N_{S_0}(w)$ is measurable (see 2.2). Thus the set of points where $N_{S_0}(w)=+\infty$ is also measurable. This set will be denoted by Σ_∞^*. Clearly

$$N_E(w)<+\infty \qquad \text{for} \quad w\in K_0^*-\Sigma_\infty^*$$

for every set $E\subset S_0$.

2.4. Obviously, if

$$E=\sum_1^\infty E_n, \quad E_1\subset E_2\subset\dots\subset E_n\subset\dots\subset S_0,$$

then

$$N_{E_n}(w)\to N_E(w) \qquad \text{in} \quad K_0^*.$$

2.5. Obviously, if

$$E=\prod_1^\infty E_n, \quad S_0\supset E_1\supset E_2\supset\dots\supset E_n\supset\dots,$$

then

$$N_{E_n}(w)\to N_E(w) \qquad \text{in} \quad K_0^*-\Sigma_\infty^*.$$

2.6. Obviously, if

$$E=\sum_1^\infty E_n, \quad E_iE_j=0 \qquad \text{for} \quad i\neq j,$$

then

$$N_E(w)=\sum_1^\infty N_{E_n}(w) \qquad \text{in} \quad K_0^*.$$

§ 3. The functions $i(z)$, $N_{I_k}(w)$, $n(w)$.

3.1. A point $z\in S_0$ will be called *a relative* of the point $z_0\in S_0$, if $f(z)=f(z_0)$.

3.2 *The set I.* This set is defined as consisting of all points $z_0\in S_0$ with the following properties.

a) z_0 is *interior* to S_0.

b) There exists a $\varrho=\varrho(z_0)>0$, such that $f(z)\neq f(z_0)$ for $0<|z-z_0|<\varrho$. That is: I is the set of those *interior* points of S_0 which have a neighborhood clear of relatives.

3.3 Clearly

$$N_{S_0}(w)=N_I(w) \qquad \text{for} \quad w\in K_0^*-(\Sigma_\infty^*+B_0^*).$$

3.4. *The function* $i(z)$ will be first defined on the set I as follows [6]). If $z_0 \epsilon I$, then we have a $\varrho = \varrho(z_0) > 0$ such that $f(z) \neq f(z_0)$ for $0 < |z - z_0| < \varrho$. Let C be a Jordan curve, enclosing z_0, in the circular ring $0 < |z - z_0| < \varrho$. The image of C is then a continuous curve C^* which does not pass through the point $w_0 = f(z_0)$. By definition, $i(z_0)$ is then equal to the topological index of $w_0 = f(z_0)$ with respect to C^* (it being supposed that C itself is described in the counterclockwise sense). By 1.2, $i(z_0)$ is then independent of the particular choice of the Jordan curve C. For $z \epsilon S_0 - I$, we define $i(z) = 0$.

3.5. We define the set I_k as the *subset* of I on which $i(z)$ is equal to a given integer k. Thus $z \epsilon I_0$ means that $i(z) = 0$ *and* $z \epsilon I$.

3.6. We proceed to show that the function $i(z)$ and the sets $N_{I_k}(w)$ are measurable [7]). Let k be any integer, and n, m any positive integers. We define the set $E_{k,n,m}$ as consisting of all points $z_0 \epsilon S_0$ with the following properties.

a) z_0 is *interior* to S_0.

b) The closed ring $1/(n+m) \leqslant |z - z_0| \leqslant 1/n$ is interior to S_0 and contains no relatives of z_0.

c) The topological index of the point $w_0 = f(z_0)$ with respect to the image of the circle $|z - z_0| = 1/n$ is equal to k.

3.7. Obviously, the set $E_{k,n,m}$ is open.

3.8. Obviously $E_{k,n,m+1} \subset E_{k,n,m}$.

3.9. Define

$$E_{k,n} = \prod_{m=1}^{\infty} E_{k,n,m}.$$

Obviously, $E_{k,n} \subset E_{k,n+1}$.

3.10. Obviously

$$I_k = \sum_{n=1}^{\infty} E_{k,n}.$$

3.11. Since $E_{k,n,m}$ is open, it follows from 3.10, 3.9 that I_k and therefore $I = \sum_1^{\infty} I_k$ are measurable sets. Consequently, the function $i(z)$ is measurable.

[6]) The function $i(z)$ was introduced by Banach, loc. cit. [1]).
[7]) The measurability of $i(z)$ was already proved by Schauder, loc. cit. [1]), in a different manner.

3.12. Since $E_{k,n,m}$ is open, the function

$$N_{E_{k,n,m}}(w)$$

is measurable (see 2.2). It follows then from 3.10, 3.9, 3.8, 2.4, 2.5, that the functions $N_{I_k}(w)$ are all measurable on $K_0^* - \Sigma_\infty^*$.

3.13. *The function* $n(w)$ [8]). For $w \epsilon K_0^* - B_0^*$, the function $n(w)$ is defined as being equal to the topological index of the point w with respect to the image B_0^* of the boundary B_0 of the fundamental square S_0. For $w \epsilon B_0^*$, we define $n(w) = 0$. The function $n(w)$ is clearly measurable.

§ 4. Relations between $N_{S_0}(w)$, $N_{I_k}(w)$, $n(w)$.

4.1. Given a positive integer n, we define a set $\mathcal{E}_n^*$ in the w-plane as follows: The point $w_0 \epsilon K_0$ belongs to $\mathcal{E}_n^*$ if *first* $N_{S_0}(w_0) < n$, and *second* there exists a $\varrho = \varrho(w_0) > 0$, such that $N_{S_0}(w) \geqq n$ for $0 < |w - w_0| < \varrho$ [9]). Clearly, $\mathcal{E}_n^*$ is an isolated and therefore denumerable set. The set

$$\mathcal{E}^* = \sum_1^{\infty} \mathcal{E}_n^*$$

is thus of measure zero.

4.2. Suppose $w_0 \epsilon K_0^* - (B_0^* + \Sigma_\infty^* + \mathcal{E}^*)$. Then w_0 has a finite number of models

$$z_1, z_2, \ldots, z_m, \qquad m = N_{S_0}(w),$$

all of which are in the set I. It follows then from 1.3 that we have an $\eta = \eta(w_0) > 0$, such that for $0 < |w - w_0| < \eta$ the point w has at least $|i(z_j)|$ models in the vicinity of z_j, for $j = 1, 2, \ldots, m$. Hence

$$N_{S_0}(w) \geqq \sum_1^m |i(z_j)| \qquad \text{for} \quad 0 < |w - w_0| < \eta.$$

Since all the points z_j are in I, we have

$$\sum_1^m |i(z_j)| = \sum_k |k| N_{I_k}(w_0).$$

[8]) Cf. Schauder, loc. cit. [1]).
[9]) The corresponding notion, in the theory of functions of a complex variable, would be the notion of a *branch-point*.

Consequently

(1) $$N_{S_0}(w) \geqq \sum_k |k| N_{I_k}(w_0) \quad \text{for} \quad 0 < |w - w_0| < \eta.$$

But w_0 is not in $\mathcal{E}^*$. Hence (1) implies that

(2) $$N_{\dot{S}_0}(w_0) \geqq \sum_k |k| N_{I_k}(w_0),$$

provided that the right-hand member is positive. If the right-hand member is zero, then (2) is obvious. Thus we have the

Theorem. *If* $w \in K_0^* - (B_0^* + \Sigma_\infty^* + \mathcal{E}^*)$, *then*

(3) $$N_{S_0}(w) \geqq \sum_k |k| N_{I_k}(w).$$

4.3. Supposing again that $w \in K_0^* - (B_0^* + \Sigma_\infty^* + \mathcal{E}_0^*)$, all the models

$$z_1, z_2, \ldots, z_m, \qquad m = N_{S_0}(w),$$

are in I, and hence (see 1.2)

(4) $$n(w) = \sum_1^m i(z_j) = \sum_k k\, N_{I_k}(w).$$

From (3) and (4) we obtain, since $n(w) = 0$ on B_0^* and $N_{S_0}(w) = \infty$ on Σ_∞^*, the

Theorem. *If* $w \in K_0^* - \mathcal{E}^*$, *then* $|n(w)| \leqq N_{S_0}(w)$ [10].

§ 5. Applications.

5.1. *Continuous transformations of bounded variation* [11]. We associate with the continuous transformation $w = f(z)$ a function of squares $F(s)$, defined for all closed squares s in S_0 as follows. $F(s)$ is the measure of the image s^* of s (since s^* is a closed set, it is certainly measurable). According to Banach, the continuous transformation $w = f(z)$ is said to be of *bounded variation* if there exists a finite constant $M > 0$ such that $\Sigma F(s_n) < M$, for all sequences s_n of closed squares without common interior points.

[10]) This limitation for $|n(w)|$ seems to be essentially new. Notice that no assumption is made concerning the transformation beyond mere continuity.

[11]) These transformations were studied by Banach, loc. cit [1]).

5.2. According to Banach [12]), the continuous transformation $w = f(z)$ is of bounded variation if and only if the function $N_{S_0}(w)$ is summable. If this condition is satisfied, then the set Σ_∞^* is of measure zero. Since the set $\mathcal{E}^*$ is of measure zero anyhow, and since $n(w) = 0$ on B_0^*, we infer from 4.3 the

Theorem. *If the continuous transformation $w=f(z)$ is of bounded variation, then the index-function $n(w)$ is summable. More exactly, $|n(w)|$ is dominated, almost everywhere in K_0^*, by the summable function $N_{S_0}(w)$* [13]).

5.3. Supposing again that the continuous transformation $w = f(z)$ is of bounded variation, it follows from (3), (4) in 4.2, 4.3 that the absolute values of the partial sums of the series

$$n(w) = \sum_k k\, N_{I_k}(w)$$

are dominated, almost everywhere on the set $K_0^* - B_0^*$, by the summable function $N_{S_0}(w)$. Hence, by a well-known theorem of Lebesgue, we can integrate term by term, which gives the formula

$$\iint_{K_0^* - B_0^*} n(w) = \sum_k k \iint_{K_0^* - B_0^*} N_{I_k}(w).$$

5.4. Let z be a point of S_0, and s a closed square containing z and contained in S_0. If, for $|s| \to 0$, the quotient $F(s)/|s|$ approaches a definite finite limit, then this limit will be called the *area-derivative* $D(z)$ at the point z. According to Banach [14]) for continuous transformations of bounded variation $D(z)$ exists and is finite almost everywhere in S_0, and is summable in S_0.

5.5. *Absolutely continuous transformations* $w = f(z)$ [15]). A continuous transformation $w = f(z)$ is according to Banach absolutely continuous, if it is of bounded variation and if it carries sets of measure zero into sets of measure zero. According to Banach, loc. cit. [1]), we have for such transformations the formula

$$\iint_{S_0} D(z) = \iint_{K_0^*} N_{S_0}(w).$$

[12]) Loc. cit. [1]).

[13]) I was unable to find in the literature any theorem of comparable generality.

[14]) Loc. cit. [1]).

[15]) Studied by Banach and subsequently by Schauder, loc. cit. [1]).

According to Schauder[16]), we have more generally

$$\iint_E D(z) = \iint_{K_0^*} N_E(w)$$

for all measurable sets E in S_0.

5.6. For an absolutely continuous transformation, the set B_0^* is of measure zero. Hence, by 5.3 and 5.5, we can write

$$\iint_{K_0^*} n(w) = \sum_k k \iint_{I_k} D(z) = \sum_k \iint_{I_k} i(z)\, D(z) = \iint_I i(z)\, D(z), \quad ^{17})$$

since, by definition, $i(z) = k$ on I_k. Since by definition $i(z) = 0$ on $S_0 - I$, we obtain finally the

Theorem. *If the transformation $w=f(z)$ is absolutely continuous, then*

$$\iint_{K_0^*} n(w) = \iint_{S_0} i(z)\, D(z).$$

Schauder[18]) proved this formula under the additional assumption that $i(z)$ is bounded. Our argument shows that this restriction is superfluous.

5.7. As a last application we shall give a (very incomplete) discussion of those points z for which the index-function $i(z)$ is different from ± 1. Let I_0^* be the image of I_0, and consider a point

$$w \in K_0^* - (B_0^* + \Sigma_\infty^* + \mathcal{S}^* + I_0^*).$$

We have then

$$N_{S_0}(w) = \sum_{k \neq 0} N_{I_k}(w).$$

since all the models of w are in I and none of them is in I_0. On the other hand, we have (see 4.2)

$$N_{S_0}(w) \geqq \sum_k |k|\, N_{I_k}(w).$$

[16]) Loc. cit. [1]).

[17]) The last step is justified by the remark that the partial sums of the series

$$\sum_k \iint_{I_k} |i(z)|\, D(z) = \sum_k |k| \iint_{I_k} D(z) = \sum_k |k| \iint_{K_0^*} N_{I_k}(w)$$

are dominated, on account of the theorem in 4.2, by the finite constant $\iint_{K_0^*} N_{S_0}(w)$.

[18]) Loc. cit. [1]).

It follows that

$$\sum_{k \neq 0} (1 - |k|)\, N_{I_k}(w) \geqq 0.$$

Hence we must have

$$N_{I_k}(w) = 0 \quad \text{for } |k| > 1, \qquad w \in K_0^* - (B_0^* + \Sigma_\infty^* + \mathcal{S}^* + I_0^*).$$

Hence the following

Theorem. *If $w=f(z)$ is a continuous transformation, then for $|k| > 1$ the image I_k^* of the set I_k is a subset of the set $B_0^* + \Sigma_\infty^* + \mathcal{S}^* + I_0^*$.*

5.8. In particular, if the transformation is absolutely continuous, the sets B_0^*, Σ_∞^*, $\mathcal{S}_0^*$ are of measure zero. Hence we have, as an immediate corollary of 5.7, the

Theorem. *If an absolutely continuous transformation is such that the set of points where $i(z) = 0$ is carried into a set of measure zero, then the set of points where $i(z) \neq \pm 1$ is also carried into a set of measure zero.*

5.9. If the transformation $w = f(z)$ is absolutely continuous, then a measurable set E in S_0 is carried into a set of measure zero if and only if $D(z)=0$ almost everywhere on the set[19]). Combined with the theorem in 5.8, this leads to the

Theorem. *Suppose that the absolutely continuous transformation $w=f(z)$ is such that the area-derivative $D(z)$ is positive almost everywhere in the square S_0. Suppose also that the index-function $i(z)$ is different from zero almost everywhere in S_0. Then $i(z)=\pm 1$ almost everywhere in S_0.*

[19]) Schauder, loc. cit. [1]).

The Ohio State University,
February 1936.