Completely alternating transformations.

By

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1. In an earlier paper [1]) a continuous transformations $T(A)=B$ between two sets in a metric space has been called non-alternating provided that for no two distinct points x and y of B does $T^{-1}(y)$ separate $T^{-1}(x)$ in A. In contrast to this, we shall call a continuous transformation $T(A)=B$ *completely alternating* provided T is not $(1-1)$ and if x and y are any two points of B and x_1 and x_2 any two points of $T^{-1}(x)$, then $T^{-1}(y)$ separates x_1 and x_2 in A, i. e., there exists a separation $A-T^{-1}(y)=A_1+A_2$ where $A_1\supset x_1$, $A_2\supset x_2$ and $\bar{A}_1\cdot A_2=A_1\cdot\bar{A}_2=0$. For example, the transformation $w=z^2$ of the the circle $|z|=1$ into the circle $|w|=1$ is completely alternating. A radial projection of the continuum A consisting of the circle $\varrho=1$ together with the spiral $(\varrho-1)\theta=1$, $\theta\geqslant\pi$, onto the circle $\varrho=1$ is a second example. In this second example, A is not locally connected. If $T(A)=B$ is completely alternating, obviously each of the sets $T^{-1}(x)$, $x\epsilon B$, must be totally disconnected.

A transformation $T(A)=B$ will be said to be *topologically equivalent* or simply *equivalent* to a transformation $W(A')=B'$ provided we can write $T(A)=H_2\,W\,H_1(A)=B$ where $H_1(A)=A'$ and $H_2(B')=B$ are homeomorphisms. For example, any transformation $T(A)=B$ mapping a simple arc A into a simple closed curve by merely drawing the endpoints of A together is equivalent to the transformation $x=\cos t$, $y=\sin t$ on the interval $0\leqslant t\leqslant 1$. Clearly also any two homeomorphisms between two sets A and B are equivalent. It is easily verified that equivalence as here defined is an equable relation.

[1]) See my paper *Non-alternating transformations*, Amer. Jour. Math., 56 (1934), pp. 294—302.

2. Theorem. *If A is a compact continuum and $T(A)=B$ is completely alternating, then A is atriodic* [2]) *and B is a simple closed curve.*

Proof. We shall prove first that B is a simple closed curve. Since T is not $(1-1)$, there exists a point $y\epsilon B$ such that $T^{-1}(y)$ is non-degenerate. We shall prove if $x\epsilon B-y$, then $B-(x+y)$ is disconnected. Let $Y=T^{-1}(y)$. Clearly there exists a separation $Y=Y_1+Y_2$ where Y_1 and Y_2 are closed and disjoint. Let N be a continuum in A irreducible between Y_1 and Y_2 and let $N\cdot Y_1=H$, $N\cdot Y_2=K$. Then $N-(H+K)=Q$ is connected and since T is completely alternating it follows that $T^{-1}(x)\cdot Q=x'$, a single point. Clearly we have a separation $N-x'=N_h+N_k$, $N_h\supset H$, $N_k\supset K$. [For if $A-T^{-1}(x)=A_{h_1}+A_{k_1}$ is a separation between $h_1\epsilon H$ and $k_1\epsilon K$, we have only to take $N_h=N\cdot A_{h_1}$, $N_k=N\cdot A_{k_1}$]. Let $Q_1=Q\cdot N_h=N_h-H$, $Q_2=Q\cdot N_k=N_k-K$. Finally let $B_1=T(Q_1)$, $B_2=T(Q_2)$. Then since each set $T^{-1}(b)$, $b\epsilon B-y$, must intersect Q in exactly one point, it follows that $B_1+B_2=B-(x+y)$ and $B_1\cdot B_2=0$. Furthermore $\bar{B}_1\cdot B_2=0$. For suppose $b_2\epsilon\bar{B}_1\cdot B_2$. Let $x_i\epsilon B_1$, and $x_i\to b_2$. Then by continuity we have

$$L=\limsup\,[T^{-1}(x_i)]\subset T^{-1}(b_2).$$

But this is impossible since $T^{-1}(b_2)$. N is a single point $q_2\epsilon Q_2$, whereas each of the sets $T^{-1}(x_i)$ intersects the set $\bar{Q}_1$ and $\bar{Q}_1\cdot Q_2=0$. Similarly $B_1\cdot\bar{B}_2=0$. Thus $B-(x+y)$ is disconnected for every $x\epsilon B-y$.

Now let $z\epsilon B-(x+y)$. Then either Q_1 or Q_2, say Q_1, contains exactly one point z' of $T^{-1}(z)$. Furthermore we have the separation $N-z'=N'_h+N'_k$ and if we let $Q_{11}=N'_k\cdot Q_1$, $Q_{12}=N'_h+N_k$ and $B_{11}=T(Q_{11})$, $B_{12}=T(Q_{12})$ we have $B_{11}+B_{12}=B-(x+z)$; and just as above it follows that B_{11} and B_{12} are separated.

Therefore we have shown that any pair of points whatever in B disconnects B so that B is a simple closed curve.

Now to show that A is atriodic, we first prove

(i) If $t=oa+ob+oc$ is any triod in A, then for some $x\epsilon B$ we must have that $t\;T^{-1}(x)$ contains more than one point.

[2]) A set N is called a *triod* provided N is the sum of three continua oa, ob, oc each pair of which have just the point o in common. A set M is said to be *atriodic* provided M contains no triod. See R. L. Moore, Proc. Nt. Acad. Sc. 14 (1928), p. 85.

For otherwise t maps into $T(t)$ topologically, which impossible since by the above proof B is a simple closed curve and hence is atriodic.

Now suppose, contrary to what we wish to show, that A contains a triod $t = oa + ob + oc$. Let $x \epsilon T(t) - T(o)$. Them $T^{-1}(x)$ does not contain o. Let R be the component of $t - t \cdot T^{-1}(x)$ containing o. Now clearly R contains a sub-triod $t = oa' + ob' + oc'$ of t. By (i) there exists a $y \epsilon B$ such that $t' \cdot T^{-1}(y)$ contains at least two points y_1 and y_2. But since $y_1 + y_2 \subset R$ and R is connected and $R \cdot T^{-1}(x) = 0$, $T^{-1}(x)$ cannot separate y_1 and y_2 in A contrary to hypothesis. Thus A is atriodic.

(2.1). ***Corollary.*** *If A is a compact locally connected continuum and $T(A) = B$ is completely alternating, then A is either a simple arc or a simple closed curve and B is a simple closed curve.*

This results at once from (2) since the simple arc and the simple closed curve are the only atriodic locally connected compact continua.

3. ***Theorem.*** *If A is a simple closed curve and $T(A) = B$ is completely alternating, T is equivalent to the transformation $x' = \cos kt$, $y' = \sin kt$ (k an integer) on the circle $x = \cos t$, $y = \sin t$, $0 \leqslant t \leqslant 2\pi$ (or, in other words to the transformation $w = z^k$ on the circle $|z| = 1$)* [3]).

Proof. Since A is a simple closed curve it follows that for each $b \epsilon B$, $T^{-1}(b)$ contains a finite number $k > 1$ of points and k is the same for every $b \epsilon B$. Now let $o \epsilon B$ and let $T^{-1}(o) = o_1 + o_2 + \ldots + o_k$. Then $A - T^{-1}(o) = \sum_1^k \alpha_i$, where α_i is an open arc with end points o_i and o_{i+1}, where $o_{k+1} = o_1$. Furthermore if $b \epsilon B - o$, then since T is completely alternating it follows that $T^{-1}(p) = p_1 + \ldots + p_k$, where $p_i \epsilon \alpha_i$. Now let A' denote the circle

$$x = \cos t, \qquad y = \sin t, \qquad 0 \leqslant t \leqslant 2\pi.$$

Let α_1' denote the arc of A' given by $0 < t < 2\pi/k$. Let h be a homeomorphism mapping $\bar{\alpha}_1$ into $\bar{\alpha}_1'$ so that $h(o_1)$ is the point $(t = 0)$.

We now define $H_1(A) = A'$ as follows. Let $p \epsilon A$. If $p \epsilon \bar{\alpha}_1$, let $H_1(p) = h(p)$. If $p = o_i$ for some i, let $H_1(o_i)$ be the point on A' given by $t = (i-1)2\pi/k$. Finally if $p \epsilon \alpha_i$, let $p_1 = T^{-1}[T(p)] \cdot \alpha_1$ and let t_1 be the value of t corresponding to $h(p_1)$; we then define $H_1(p)$ to be the point on A' given by

$$t = 2\pi(i-1)/k + t_1.$$

Clearly H_1 is a homeomorphism.

Now let W denote the transformation sending $(x, y) \epsilon A'$ into $(x', y') \epsilon B'$ where

$$x' = \cos kt, \qquad y' = \sin kt, \qquad 0 \leqslant t \leqslant 2\pi.$$

For convenience we shall regard B' as a different circle from A'.

Since for each $b' = (x', y') \epsilon B'$, there correspond k values of t expressible in the form $t_1, t_1 + 2\pi/k, \ldots, t_1 + 2(k-1)\pi/k$, it follows from the definition of H_1 and of W that $H_1^{-1} W^{-1}(b') = T^{-1}(b)$ for some $b \epsilon B$. Thus if for each $b' \epsilon B'$ we define

$$H_2(b') = T H_1^{-1} W^{-1}(b'),$$

we have $H_2(B') = B$ and it is readily seen that H_2 is a homeomorphism.

Now $H_2 = T H_1^{-1} W^{-1}$ gives $T = H_2 W H_1$, by applying W and H_1 successively on the right; and thus T is equivalent to W.

4. ***Theorem.*** *If A is a simple arc ab, then is T equivalent to the transformation $x = \cos kt$, $y = \sin kt$, (k an integer), or to the transformation $x = \cos(k+1/2)t$, $y = \sin(k+1/2)t$ on the circle $x = \sin t$, $y = \sin t$, $0 \leqslant t \leqslant 2\pi$, according as $T(a) = T(b)$ or $T(a) \neq T(b)$.*

Proof. (i) Suppose $T(a) = T(b) = o$. Let $k+1$ be the number of points in $T^{-1}(o)$ and let $T^{-1}(o) = o_0 + o_1 + \ldots + o_k$ where $o_0 = a$, $o_k = b$ and o_i precedens o_{i+1} on the arc ab in the order a, b. Now $A - T^{-1}(o) = \sum_1^k \alpha_i$, where α_i is the open arc $o_{i-1} o_i$ on ab. Furthermore if $p \epsilon B - o$, then $T^{-1}(p) = \sum_1^k p_i$, where $p_i \epsilon \alpha_i$. Now let A' be the interval $(0, 2\pi)$ and let h be a homeomorphism mapping $\bar{\alpha}_1$ on to the interval $(0, 2\pi/k)$ so that $h(a) = 0$, $h(o_1) = 2\pi/k$. Now define $H_1(A) = A'$ as follows: if $p \epsilon \alpha_1$, let $H_1(p) = h(p)$; if $p = o_i$ for some i, let $H_1(p) = 2\pi i/k$; if $p \epsilon \alpha_i$, let $p_1 = T^{-1}[T(p)] \cdot \alpha_1$ and define

$$H_1(p) = 2\pi(i-1)/k + H_1(p_1).$$

[3]) Although in this case the transformation is more simply described by means of the complex variable, we shall use the language of the real parameter t since is definitely more advantageous in the case of an arc A as treated below in § 4.

Clearly H_1 is a homeomorphism. Now if we let W be the transformation $x=\cos kt$, $y=\sin kt$, $0\leqslant t\leqslant 2\pi$, sending the interval $A'=(0,2\pi)$ into the circle B', and define $H_2(b')=TH_1^{-1}W^{-1}(b')$, $b'\in B'$, it follows just as in the preceeding proof that H_2 is a homeomorphism and that $T=H_2WH_1$ so that T is equivalent to W.

(ii) Suppose $T(a)=a'$, $T(b)=b'$, $a'\neq b'$. Since there exist at least one $p\in B$ such that $T^{-1}(p)$ is non degenerate and since T is completely alternating it follows that both $T^{-1}(a')$ and $T^{-1}(b')$ are non degenerate. Let $k+1$ be the number (clearly finite) of points in $T^{-1}(a')$ and write $T^{-1}(a')=a_0+a_1+\ldots+a_k$, where $a_0=a$ and a_i precedes a_{i+1} on ab in the order a, b. Then $T^{-1}(b')$ contains just $k+1$ points and we can write $T^{-1}(b')=b_0+b_1+\ldots+b_k$ where $b_k=b$ and b_i $(i<k)$ lies on the open arc a_ia_{i+1}. Let A' be the interval $(0,2\pi)$ and let h be a homeomorphism mapping the arc a_0a_1 into the interval $[0, 2\pi/(k+1/2)]$ so that $h(a_0)=0$ and $h(b_0)=\pi/(k+1/2)$.

Let us now define $H_1(A)=A'$ as follows. Let $p\in A$. if $p=a_i$ for some i, let

$$H_1(p)=2\pi i/(k+1/2).$$

If not, let a_i be the first point of $T^{-1}(a')$ on the arc pa in the order p, a and let $p_1=T^{-1}[T(p)]\cdot a_0a_1$; then define

$$H_1(p)=2\pi i/(k+1/2)+h(p_1).$$

Clearly H_1 is a homeomorphism.

Now let W denote transformation

$$x=\cos(k+1/2)t,\qquad y=\sin(k+1/2)t,\qquad 0\leqslant t\leqslant 2\pi,$$

sending the interval $A'=(0,2\pi)$ into the unit circle B'. Then if we define $H_2(z)=TH_1^{-1}W^{-1}(z)$, $z\in B'$, it follows just as in the proceeding proofs that H_2 is a homeomorphism and that $T=H_2WH_1$, so that T is equivalent to W.

Note. In case (ii) clearly the fraction $1/2$ could be replaced by any fraction θ, $0<\theta<1$.

5. Componentwise alternating transformations. A continuous transformation $T(A)=B$ which is not monotone [4]) will be said to be *completely componentwise alternating* provided that if $x, y\in B$ and X_1 and X_2 are components of $T^{-1}(x)$, then $T^{-1}(y)$ separates X_1 and X_2 in A.

[4]) T is said to be monotone provided that for each $b\in B$, $T^{-1}(b)$ is connected. See C. B. Morrey, Amer. Jour. Math., 57 (1935), pp. 17—50.

Now it is known [5]) that if A is compact, then any continuous transformation $T(A)=B$ can be factored into the form $T_2T_1(A)$, where T_1 is monotone and T_2 has the property that for each $x\in B$, $T_2^{-1}(x)$ is of dimension 0. We proceed to prove the following:

Theorem. *If A is a compact continuum and $T(A)=B$ is completely componentwise alternating and if T be factored into the form T_2T_1 as above, then T_2 is completely alternating (and consequently B is a simple closed curve and $T_1(A)$ is atriodic). Conversely, if $T_1(A)=A'$ is monotone and $T_2T_1(A)=T_2(A')=B$ is completely alternating, then $T(A)=T_2T_1(A)=B$ is completely componentwise alternating.*

To prove the first statement, let $x, y\in B$, $x_1, x_2\in T_2^{-1}(x)$, $X_i=T_1^{-1}(x_i)$, $(i=1,2)$. Then since X_1 and X_2 are components of $T^{-1}(x)$, there exists a separation

$$\text{(i)}\quad A-T^{-1}(y)=A_1+A_2,\quad \text{where}\quad A_i\supset X_i\quad (i=1,2).$$

Applying T_1 to this and letting $T_1(A)=A'$ we get

$$\text{(ii)}\quad A'-T_2^{-1}(y)=T_1(A_1)+T_1(A_2).$$

Now since T_1 is monotone, it follows that

$$\text{(iii)}\quad T_1^{-1}T_1(A_i)=A_i\qquad (i=1,2).$$

For if not, there exist a $p\in T_1(A_1)$, say, so that $T_1^{-1}(p)\cdot T^{-1}(y)\neq 0$, since $T_1^{-1}(p)$ is connected. Then if $q\in T_1^{-1}(p)\cdot T^{-1}(y)$ and $r\in A_1\cdot T_1^{-1}(p)$, we have $T_1(q)=T_1(r)=p$. Whence $T_2T_1(q)=T_2T_1(r)=T(q)=T(r)=y$ since $q\in T^{-1}(y)$; and this is impossible since $r\in A_1\subset A-T^{-1}(y)$.

Now from (iii) it follows that $T_1(A_1)$ and $T_1(A_2)$ are disjoint and separated since [See (1.1) of my paper cited in ref. 1] if $p\in T_1(A_1)\cdot\overline{T_1(A_2)}$ we would have a point $q\in T_1^{-1}(p)\subset A_1$ belonging to $\overline{A}_2$ which is impossible. Thus since $T_1(A_i)\supset T_1(X_i)=x_i$ $(i=1,2)$, (ii) gives the required separation of $A'-T_2^{-1}(y)$ between x_1 and x_2. Accordingly T_2 is completely alternating.

We proceed now to prove the converse statement. It results at once from the definition that any completely alternating transformation $W(X)=Y$ has the property that for each $y\in Y$, $W^{-1}(y)$ is of dimension 0. Thus T_2 has this property and since the factorization

[5]) See my paper, loc. cit.; also see S. Eilenberg, Fund. Math. 22 (1934), pp. 292—296.

$T=T_2T_1$ where T_1 is monotone and T_2 has this same property, is unique [6]), it follows that for any $p \epsilon T_1(A)$, $T_1^{-1}(p)$ is a component of $T^{-1}T_2(p)$.

Now let $x, y \epsilon B$ and let X_1 and X_2 be distinct components of $T^{-1}(x)$. Let $x_1=T_1(X_1)$, $x_2=T_1(X_2)$. By what we have just shown, x_1 and x_2 are distinct points of $T_2^{-1}(x)$. Thus if we let $A'=T_1(A)$ we have a separation

$$A'-T_2^{-1}(y)=A_1'+A_2', \quad \text{where} \quad x_i \subset A_i' \qquad (i=1,2).$$

Applying T_1^{-1} to this we get

$$A-T^{-1}(y)=T_1^{-1}(A_1')+T_1^{-1}(A_2'), \quad \text{since} \quad T_1^{-1}T_2^{-1}=T^{-1};$$

and this must be a separation since T_1 is continuous. Finally, $T_1^{-1}(A_i') \supset X_i$ so that T is completely componentwise alternating.

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[6]) To prove this it suffices to show that if $T=T_2T_1$ is any such factorization, then for any $p \epsilon T_1(A)$, $T^{-1}(p)$ is a component of $T^{-1}T_2(p)$. Now since $T=T_2T_1$, we must have $T_1^{-1}(p) \subset T^{-1}T_2(p)$. Also, since T_1 is monotone, $T_1^{-1}(p)$ is connected. Thus $T_1^{-1}(p)$ is contained in some single component X of $T^{-1}T_2(p)$. It remains to show that $T_1(X)=p$. If not, then $T_1(X)$ is a non-degenerate, continuum; but then since T_2 maps only 0-dimensional sets into single points, it would follow that $T_2T_1(X)=T(X)$ could not reduce to a single point, contrary to the fact that $T(X)=x$.

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Sur les fonctions de deux variables réelles.

Par

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Le but de cette note est de montrer que l'on peut, à l'aide d'un léger changement de la méthode de démonstration, remplacer quelques résultats de M. Blumberg [1]) et de Mlle Schmeiser [2]) par des résultats plus précis. Pour ne pas compliquer inutilement la note actuelle, nous n'envisagerons que le théorème 2^a de Mlle Schmeiser, auquel nous allons donner une forme plus précise que voici:

Théorème. *Soit π le plan euclidien; soit s une droite de ce plan; soit $f(x,y)=f(P)$ une fonction réelle* [3]) *définie dans π. P étant un point quelconque de s et d étant une direction quelconque de P, désignons par $\overrightarrow{Pd}$ la demidroite issue du point P dans la direction d (le point P étant regardé comme n'appartenant pas à $\overrightarrow{Pd}$).*

Enfin, désignons par $E(P,d)$ l'ensemble de tous les nombres ξ jouissant de la propriété suivante: il existe une suite de points $P_1, P_2, \ldots$ telle que

$$P_n \epsilon \overrightarrow{Pd}, \qquad P_n \to P, \qquad f(P_n) \to \xi \qquad \text{pour} \quad n \to \infty \text{ [4])}.$$

Alors il existe un ensemble dénombrable $D \epsilon s$ jouissant de la propriété suivante: d_1, d_2 étant deux directions quelconques, situées d'un même côté de la droite s, on a

$$(1) \qquad E(P, d_1) \cdot E(P, d_2) \neq 0$$

pour chaque $P \epsilon s - D$.

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[1]) Fund. Math. 16 (1930), p. 17—24.

[2]) Fund. Math. 22 (1934), p. 70—76.

[3]) La démonstration s'applique d'ailleurs aussi dans le cas d'une fonction *complexe* de deux variables *réelles*.

[4]) On admet aussi les valeurs $\xi=\pm\infty$.