The strong symmetrical cut sets of closed euclidean \( n \)-space 1).

By


In his paper Symmetrical cut sets 2), W. Dancer shows that the only strong symmetrical cut sets of the topological 2-sphere are the simple closed curves. Dancer's method of proof is entirely set-theoretic, depending principally on the prime end theory of Carathéodory. In the present paper this result is made a special case of a theorem concerning the topological \( n \)-sphere, or closed euclidean \( n \)-space, \( H_n \), the proof being based mainly on the duality theorem for, and related properties of closed sets of points.

We recall that if \( M \) is a point set and \( K \) a subset of \( M \), then \( K \) is a strong symmetrical cut set of \( M \) if \( M-K \) is the sum of two mutually separated sets \( A \) and \( B \) such that there exists a homeomorphism \( \Delta (A-K)=B-K \), on which \( K \) is the identity.

**Theorem.** If \( M \) is a strong symmetrical cut set of \( H_n \), then \( M \) is a generalized closed \( (n-1) \)-manifold whose Betti numbers \( \gamma'(M) \), \( 0 \leq i \leq n-2 \), are all zero. In particular, if \( n=3 \), \( M \) is the topological 2-sphere 3).

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2) This volume, pp. 124—136.
3) For the definition of generalized closed \( n \)-manifold (= g. c. n-m.), see my paper Generalized closed manifolds in \( n \)-space, Annals of Math., 35 (1934), pp. 872—903 (this paper is hereafter referred to as G. C. M.); or see Fund. Math. 25 (1935), p. 200, footnote 4. I am informed by Mrs. Lucille Whyburn that, working independently of Dancer and myself, she has found that if, in \( H_n \), \( M \) is a fixed set of points under a homeomorphism \( \varphi \) of \( H_n \), then \( M \) separates \( H_n \) and 2) at least one component of \( H_n-M \) is transformed by \( \varphi \) into another component of \( H_n-M \), then it follows that \( H_n-M \) has only two components; and that by using methods similar to those employed in part 1) of the proof of my theorem, she is able to extend this to the general case of an \( H_n \). Obviously this would allow a weakening of the hypotheses of Dancer's theorem and the theorem of this paper.

**Proof.** That \( M \) is a closed set of points follows from the fact that no sequence of points can have more than one sequential limit point 4).

According to the above definition, \( H_n-M \) is the sum of mutually separated sets, \( D_1 \) and \( D_2 \), and there exists a homeomorphism \( \Delta \) which on \( M \) is the identity and such that \( \Delta (D_1+M)=D_2+M \). Consequently there exists a homeomorphism \( \varphi \) of \( H_n \) into itself which on \( D_1+M \) agrees with \( \Delta \), and on \( D_2 \) with \( \Delta^{-1} \).

1) The Betti numbers \( \gamma'(M) \), \( 0 \leq i \leq n-2 \), are all zero. Suppose, for some \( i \), that \( \gamma'(M) \neq 0 \). Then, by the duality theorem for closed sets, there is a cycle \( \gamma_i^{n-1} \) of \( H_n-M \) which links \( M \), and which may be assumed, without loss of generality, to lie in \( D_1 \). The homeomorphism \( \varphi \) of \( \gamma_i^{n-1} \) in \( D_1 \) is a cycle \( \gamma_i^{n-1} \) which must also link \( M \) (since a chain bounded by \( \gamma_i^{n-1} \) in \( D_1 \) would necessitate the existence of a homeomorphic chain bounded by \( \gamma_i^{n-1} \) in \( D_2 \)). Furthermore, these two cycles are linearly independent with respect to homologies in \( H_n-M \). Consequently there exist, in \( M \), \( V \)-cycles \( \varphi_i^{n-1} \) and \( \gamma_i^{n-1} \) with which the above cycles are uniquely linked; in particular, \( \varphi_i^{n-1} \) and \( \gamma_i^{n-1} \) are linked, but \( \varphi_i^{n-1} \) and \( \gamma_i^{n-1} \) are not linked. However, \( \varphi_i^{n-1} \), \( i \neq j \), and \( \gamma_i^{n-1} \), \( i \neq j \), and since \( f \) is a homeomorphism of \( H_n \) into itself, it follows that the linking properties of \( \varphi_i^{n-1} \) and \( \gamma_i^{n-1} \) are the same as for \( \varphi_i^{n-1} \) and \( \gamma_i^{n-1} \), contradicting the fact the latter two are linked, and the former not linked.

2) The continuum \( M \) satisfies condition 3) of the definition of g. c. \( (n-1)-m. \). Let \( P \) be a point of \( M \) and \( r \) a positive number. There exists a positive number \( \theta \) such that, if we denote the \( (n-1) \)-sphere \( F(P, r) \) by \( G \), the set \( f(G) \) is a subset of \( S(P, \theta) \). Also, there exists a positive number \( \theta \) such that \( f(G) \) is a subset of \( H_n-S(P, \theta) \). Let us select any \( i \) such that \( 0 \leq i \leq n-3 \), and consider any \( V \)-cycle \( \gamma_i^{n-1} \) on \( M \). Denoting the set \( M \cdot [S(P, e)-S(P, \theta)] \) by \( H \), the cycle \( \gamma_i^{n-1} \) on \( H \). For suppose not. Then there is a cycle \( \gamma_i^{n-1} \) of \( H_n-M \) which is linked with \( \gamma_i^{n-1} \).

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4) See Dancer, loc. cit., Lemma 1.
5) Since, strictly speaking, a chain or cycle \( \gamma \) is the association of a certain algebraic expression \( E(\gamma) \) with a certain geometric complex \( K \) consisting of cells \( \alpha \) and their boundaries, we mean by the homeomorphic of \( \gamma \) under a homeomorphism \( \Delta \) the analogous association of \( E(\Delta) \) with the cells \( \Delta(\alpha) \) of the complex \( \Delta(K) \).
6) By \( V \)-cycle is meant Victoria cycle; see L. Vietoris, Über den höheren Zusammenhang..., Math. Ann., 97 (1927), pp. 454—472.
On C let $K^{n+1} = (T_k \rightarrow y')$ be a V-chain 7). Hereafter, we let $	ilde{t}_k$ and $T_k$ denote the elements $t_k$ and $T_k$ of $y'$ and $K^{n+1}$, respectively, with the basic cells geometrically realized on C. We define transformations $\Delta_k'$ and $\Delta_k''$ of $\tilde{T}_k$ as follows: At points of $M$ or $D_1$, $\Delta'_k = \Delta''_k = \Delta_k$ is the identity; at points of $D_2$, $\Delta'_k = \Delta''_k = \Delta_k$. These transformations are easily seen to be continuous.

Let $J$ be the smallest carrier of $y'$ on $M \cdot C$. Let $\beta$ be a positive number less than $\varepsilon(J, |P^{n+1}|)$, and let $\xi$ be a number such that $0 < \xi < \beta$, and

$$f[S(J, \xi)] \subset S(J, \beta).$$

We can then so choose $k$ that 1) $\tilde{t}_k \subset S(J, \xi)$; 2) $P^{n+1}$ is linked with $\tilde{t}_k$; 3) $\Delta'_k(\tilde{t}_k)$ and $\Delta''_k(\tilde{t}_k)$ together bound an $L^{n+1}$ that fails to meet $P^{n+1}$.

Let

$$K_1^{n+1} = \Delta'_k(\tilde{T}_k) \rightarrow \Delta'_k(\tilde{t}_k),$$

$$K_2^{n+1} = \Delta''_k(\tilde{T}_k) \rightarrow \Delta''_k(\tilde{t}_k).$$

The chains $K_1^{n+1}, K_2^{n+1}$ satisfy the following relations:

$$[K_1^{n+1}] \subset H + D_1 \cdot [S(P, e) - \overline{S(P, \theta)}],$$

$$[K_2^{n+1}] \subset H + D_2 \cdot [S(P, e) - \overline{S(P, \theta)}].$$

From the chains $K_1^{n+1}, K_2^{n+1}$ and $L^{n+1}$ we may form a cycle $\lambda^{n+1}$ which, since $i+1 \leq n-2$, bounds a chain $S^{n+2}$ in $S(P, e) - \overline{S(P, \theta)}$. By Lemma 1 of G. C. M., the intersection of $[S^{n+2}]$ and $[P^{n+1}]$ contains a connected set $N$ which joins $[K_1^{n+1}]$ and $[K_2^{n+1}]$. But $N \subset [P^{n+1}] \cdot [S(P, e) - \overline{S(P, \theta)}] \subset (H_n-B) \cdot [S(P, e) - \overline{S(P, \theta)}]$, and therefore $N$ meets $[K_1^{n+1}]$ and $[K_2^{n+1}]$ only in points of $D_1$ and $D_2$, respectively. But then $N$ of necessity contains a point of $H$, contradicting i).

For the case $i=n-2$, we proceed as in the above argument except that $H = M \cdot [H_n - \overline{S(P, \theta)}]$ and in general $H_n - \overline{S(P, \theta)}$ takes the place of $S(P, e) - \overline{S(P, \theta)}$.

3) The point sets $D_1$ and $D_2$ are connected. If $D_1$ is not connected, there is a cycle $\gamma_k'$ of $D_1$ (based on a pair of points in distinct components of $D_1$) which links $M$. Then the homeomorph, $\gamma_k''$ of $\gamma_k'$ in $D_1$ links $M$ and these cycles are uniquely linked with independent members of the $(n-1)$-basis of $M$. The proof of 3) from here on is similar to that of 1).

4) The boundary, $B$, of the domain $D_1$ is a g. c. $(n-1)$-m. By 1), $M$ satisfies condition 2) of the definition of g. c. $(n-1)$-m., and we have shown in 2) that it satisfies condition 3) of that definition. Since, by Principal Theorem D of G. C. M., the boundary of any domain complementary to a continuum satisfying these conditions is a g. c. $(n-1)$-m., 4) follows at once.

5) The set $M$ is a g. c. $(n-1)$-m. As $D(B)=B$, it is clear that $B$ is the common boundary of $D_1$ and $D_2$. On the other hand, a g. c. $(n-1)$-m. separates $M$ into just two domains of which it is the common boundary, and therefore

$$H_n - B = D_1 + D_2.$$ But by hypothesis,

$$H_n - M = D_1 + D_2.$$ From relations ii) and iii) it follows that $B$ and $M$ have the same complements and hence are identical.

For $n=2$, $M$ is a simple closed curve, and for $n=3$, $M$ is a topological 2-sphere. This follows from 1) and 5) above, and the results of G. C. M.

Remark. Although the property of being a strong symmetrical cut set of $H_n$ characterizes the simple closed curve in $H_n$, in the sense that it is a necessary as well as a sufficient condition for a point set to be a simple closed curve 4), an analogous statement about $H_n$ and the topological 2-sphere would apparently not hold, in view of an example of Alexander 5). In the latter case, however, it would be interesting to know whether the property of being a strong symmetrical cut set of $H_n$ characterizes those 2-spheres whose complementary domains are 3-cells.

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7) By a V-chain we mean a chain bounded by a V-cycle; see G. C. M., footnote 9).

9) If $L$ denotes either a chain or a complex, we denote the set of points on $L$ by $|L|$.  

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