The strong symmetrical cut sets of closed euclidean n-space [1]).

By

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In his paper *Symmetrical cut sets* [2]), W. Dancer shows that the only strong symmetrical cut sets of the topological 2-sphere are the simple closed curves. Dancer's method of proof is entirely set-theoretic, depending principally on the prime end theory of Carathéodory. In the present paper this result is made a special case of a theorem concerning the topological n-sphere, or closed euclidean n-space, H_n, the proof being based mainly on the duality theorem for, and related properties of closed sets of points.

We recall that if M is a point set and K a subset of M, then K is a strong symmetrical cut set of M if $M-K$ is the sum of two mutually separated sets A and B such that there exists a homeomorphism $\Delta(A+K)=B+K$, which on K is the identity.

Theorem. *If M is a strong symmetrical cut set of $H_n (n \geqq 2)$, then M is a generalized closed $(n-1)$-manifold whose Betti numbers $p^i(M)$, $0 \leqq i \leqq n-2$, are all zero. In particular, if $n=3$, M is the topological 2-sphere* [3]).

[1]) Presented to the Amer. Math. Soc., Sept. 12, 1935.

[2]) This volume, pp. 124—136.

[3]) For the definition of generalized closed n-manifold (= g. c. n-m.), see my paper *Generalized closed manifolds in n-space*, Annals of Math., 35 (1934), pp. 876—903 (this paper is hereafter referred to as G. C. M.); or see Fund. Math. 25 (1935), p. 200, footnote [4]). I am informed by Mrs. Lucille Whyburn that, working independently of Dancer and myself, she has found that if, in H_2, M is a fixed set of points under a homeomorphism f of H_2 into itself such that 1) M separates H_2 and 2) at least one component of H_2-M is transformed by f into another component of H_2-M, then it follows that H_2-M has only two components; and that by using methods similar to those employed in part 1) of the proof of my theorem, she is able to extend this to the general case of an H_n. Obviously this would allow a weakening of the hypotheses of Dancer's theorem and the theorem of this paper.

Proof. That M is a closed set of points follows from the fact that no sequence of points can have more than one sequential limit point [4]).

According to the above definition, H_n-M is the sum of mutually separated sets, D_1 and D_2, and there exists a homeomorphism Δ which on M is the identity and such that $\Delta(D_1+M)=D_2+M$. Consequently there exists a homeomorphism f of H_n into itself which on D_1+M agrees with Δ, and on D_2 with Δ^{-1}.

1) *The Betti numbers* $p^i(M)$, $0 \leqq i \leqq n-2$, *are all zero.* Suppose, for some i, that $p^i(M)>0$. Then, by the duality theorem for closed sets, there is a cycle γ_1^{n-i-1} of H_n-M which links M, and which may be assumed, without loss of generality, to lie in D_1. The homeomorph [5]) of γ_1^{n-i-1} in D_2 is a cycle γ_2^{n-i-1} which must also link M (since a chain bounded by γ_2^{n-i-1} in D_2 would necessitate the existence of a homeomorphic chain bounded by γ_1^{n-i-1} in D_1). Furthermore, these two cycles are linearly independent with respect to homologies in H_n-M. Consequently there exist, in M, V-cycles [6]) Γ_1^i and Γ_2^i with which the above cycles are uniquely linked; in particular, Γ_1^i and γ_1^{n-i-1} are linked, but Γ_1^i and γ_2^{n-i-1} are not linked. However, $f(\Gamma_1^i)=\Gamma_1^i$, and $f(\gamma_2^{n-i-1})=\gamma_1^{n-i-1}$, and since f is a homeomorphism of H_n into itself, it follows that the linking properties of Γ_1^i and γ_2^{n-i-1} are the same as for Γ_1^i and γ_1^{n-i-1}, contradicting the fact the latter two are linked, and the former not linked.

2) *The continuum M satisfies condition* 3) *of the definition of g. c. $(n-1)-m.$* [3]). Let P be a point of M and ϵ a positive number. There exists a positive number α such that, if we denote the $(n-1)$-sphere $F(P, \alpha)$ by C, the set $f(C)$ is a subset of $S(P, \epsilon)$. Also, there exists a positive number θ such that $f(C)$ is a subset of $H_n-S(P, \theta)$. Let us select any i such that $0 \leqq i \leqq n-3$, and consider any V-cycle $\gamma^i=\{i_k\}$ on $M \cdot C$. Denoting the set $M \cdot [S(P, \epsilon)-S(P, \theta)]$ by H, the cycle $\gamma^i \sim 0$ on H. For suppose not. Then there is a cycle Γ^{n-i-1} of H_n-H which is linked with γ^i.

[4]) See Dancer, loc. cit., Lemma 1.

[5]) Since, strictly speaking, a chain or cycle γ is the association of a certain algebraic expression $E(\sigma)$ with a certain geometric complex K consisting of cells σ and their boundaries, we mean by the homeomorph of γ under a homeomorphism Δ the analogous association of $E(\sigma)$ with the cells $\Delta(\sigma)$ of the complex $\Delta(K)$.

[6]) By V-cycle is meant Vietoris cycle; see L. Vietoris, *Über den höheren Zusammenhang...*, Math. Ann., 97 (1927), pp. 454—472.

On C let $K^{i+1}=\{T_k\}\to\gamma^i$ be a V-chain [7]). Hereafter, we let $\bar{i}_k$ and $\bar{T}_k$ denote the elements i_k and T_k of γ^i and K^{i+1}, respectively, with the basic cells geometrically realized on C. We define transformations Δ'_k and Δ''_k of $\bar{T}_k$ as follows: At points of M or D_1, Δ'_k is the identity; at points of D_2, $\Delta'_k\equiv\Delta^{-1}$. At points of M or D_2, Δ''_k is the identity, and at points of D_1, $\Delta''_k\equiv\Delta$. These transformations are easily seen to be continuous.

Let J be the smallest carrier of γ^i on $M\cdot C$. Let β be a positive number less than $\varrho(J,|\Gamma^{n-i-1}|)$ [8]), and let ξ be a number such that $0<\xi<\beta$, and

$$f[S(J,\xi)]\subset S(J,\beta).$$

We can then so choose k that 1) $|\bar{i}_k|\subset S(J,\xi)$; 2) Γ^{n-i-1} is linked with $\bar{i}_k$; 3) $\Delta'_k(\bar{i}_k)$ and $\Delta''_k(\bar{i}_k)$ together bound an L^{i+1} that fails to meet Γ^{n-i-1}. Let

$$K_1^{i+1}=\Delta'_k(\bar{T}_k)\to\Delta'_k(\bar{i}_k),$$
$$K_2^{i+1}=\Delta''_k(\bar{T}_k)\to\Delta''_k(\bar{i}_k),$$

The chains K_1^{i+1}, K_2^{i+1} satisfy the following relations:

$$|K_1^{i+1}|\subset H+D_1\cdot[S(P,\epsilon)-\overline{S(P,\theta)}],$$
$$|K_2^{i+1}|\subset H+D_2\cdot[S(P,\epsilon)-\overline{S(P,\theta)}].$$

From the chains K_1^{i+1}, K_2^{i+1} and L^{i+1} we may form a cycle N^{i+1} which, since $i+1\leqq n-2$, bounds a chain S^{i+2} in $S(P,\epsilon)-\overline{S(P,\theta)}$. By Lemma 1 of G. C. M., the intersection of $|S^{i+2}|$ and $|\Gamma^{n-i-1}|$ contains a connected set N which joins $|K_1^{i+1}|$ and $|K_2^{i+1}|$. But i) $N\subset|\Gamma^{n-i-1}|\cdot[S(P,\epsilon)-\overline{S(P,\theta)}]\subset(H_n-H)\cdot[S(P,\epsilon)-\overline{S(P,\theta)}]$, and therefore N meets $|K_1^{i+1}|$ and $|K_2^{i+1}|$ only in points of D_1 and D_2, respectively. But then N of necessity contains a point of H, contradicting i).

For the case $i=n-2$, we proceed as in the above argument except that $H=M\cdot[H_n-S(P,\theta)]$, and in general $H_n-\overline{S(P,\theta)}$ takes the place of $S(P,\epsilon)-\overline{S(P,\theta)}$.

[7]) By a V-chain we mean a chain bounded by a V-cycle; see G. C. M., footnote [7]).

[8]) If L denotes either a chain or a complex, we denote the set of points on L by $|L|$.

3) *The point sets D_1 and D_2 are connected.* If D_1 is not connected, there is a cycle γ_1^0 of D_1 (based on a pair of points in distinct components of D_1) which links M. Then the homeomorph, γ_2^0, of γ_1^0 in D_2 links M and these cycles are uniquely linked with independent members of the $(n-1)$-basis of M. The proof of 3) from here on is similar to that of 1).

4) *The boundary, B, of the domain D_1 is a g. c. $(n-1)$-m.* By 1), M satisfies condition 2) of the definition of g. c. $(n-1)$-m., and we have shown in 2) that it satisfies condition 3) of that definition. Since, by Principal Theorem D of G. C. M., the boundary of any domain complementary to a continuum satisfying these conditions is a g. c. $(n-1)$-m., 4) follows at once.

5) *The set M is a g. c. $(n-1)$-m.* As $\Delta(B)=B$, it is clear that B is the common boundary of D_1 and D_2. On the other hand, a g. c. $(n-1)$-m. separates H_n into just two domains of which it is the common boundary, and therefore

ii) $$H_n-B=D_1+D_2.$$

But by hypothesis,

iii) $$H_n-M=D_1+D_2.$$

From relations ii) and iii) it follows that B and M have the same complements and hence are identical.

For $n=2$, M is a simple closed curve, and for $n=3$, M is a topological 2-sphere. This follows from 1) and 5) above, and the results of G. C. M.

Remark. Although the property of being a strong symmetrical cut set of H_2 characterizes the simple closed curve in H_2, in the sense that it is a *necessary* as well as a sufficient condition for a point set to be a simple closed curve [4]), an analogous statement about H_3 and the topological 2-sphere would apparently not hold, in view of an example of Alexander [9]). In the latter case, however, it would be interesting to know whether the property of being a strong symmetrical cut set of H_3 characterises those 2-spheres whose complementary domains are 3-cells.

[9]) J. W. Alexander, *An example of a simply connected surface bounding a region which is not simply connected*, Proc. Nat. Acad. Sc., vol. 10 (1924), pp. 8—10.