On certain properties of Fréchet L-spaces.

By

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§ 1. In a recent note[1]) B. Dushnik considered the problem of enumerating all Fréchet limit spaces possessing one or more of the following properties[2]):

A) The closure of every set is closed.

B) Every non-denumerable set contains an element of condensation.

C) Every well-ordered series of decreasing closed sets is denumerable.

D) Every well-ordered series of increasing closed sets is denumerable.

E) Every set Q contains a denumerable set D such that Q is contained in the closure of D.

In this paper Dushnik raises the following questions:

1) *Is it true that any Fréchet L-space possessing properties B and D also possesses property E?*

2) *Is it true that any Fréchet L-space possessing properties C and E also possesses property B?*

The object of the present note is to construct two L-spaces which show that these two questions are to be answered in the negative. We shall make use of the following lemmas.

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[1]) *Concerning Fréchet limit-spaces*, Fund. Math. XXIII, pp. 162—165. I wish to thank Dr. Dushnik for valuable suggestions regarding the present note.

[2]) The implications which exist among these and allied properties of sets in L-space were first considered by W. Sierpiński, *Sur l'équivalence de trois propriétés des ensembles abstraits*, Fund. Math. II, pp. 179—188. See also C. Kuratowski, *Une remarque sur les classes (L) de M. Fréchet*, Fund. Math. III, pp. 41—43.

Lemma 1. *On the linear continuum E a well-ordered series of increasing sets is denumerable if for each set K_α of the series $\overline{K}_\alpha - K_\alpha$ is at most denumerably infinite*[3]).

Suppose the contrary. We may assume that our series $K_1, K_2, \dots K_\alpha, \dots$ is of type Ω. Since $\overline{K}_\alpha - K_\alpha$ is denumerable and the sets K_α are increasing there will exist for any $\alpha < \Omega$ an ordinal β such that $\alpha < \beta < \Omega$ and such that K_β contains a point not in $\overline{K}_\alpha$. Therefore $\overline{K}_\beta - \overline{K}_\alpha \neq 0$. Clearly then we can extract from the series $\overline{K}_1, \overline{K}_2, \dots \overline{K}_\alpha, \dots$ a series of type Ω of increasing sets. But as these sets are all closed this is impossible on E.

Lemma 2. *On the linear continuum E a well-ordered series of decreasing sets is denumerable if each set K_α of the series contains all of its points of condensation.*

Suppose the contrary. We may again assume that our series $K_1, K_2, \dots K_\alpha, \dots$ is of type Ω. Let C_α denote the set of all points of condensation of K_α. Obviously $C_\alpha \supset C_{\alpha+1}$. For any α, since $K_\alpha - C_\alpha$ is denumerable, there will exist an ordinal β such that $\alpha < \beta < \Omega$ and such that C_α contains a point not in K_β. Thus, $C_\alpha - C_\beta \neq 0$. Clearly then we can extract from the series $C_1, C_2, \dots C_\alpha, \dots$ a series of type Ω of decreasing sets. But this is impossible on E since the sets C_α are all closed.

Lemma 3. *If N is any totally imperfect*[4]) *subset of E and K any non-denumerable subset of N then $\overline{\overline{K}\cdot(E-N)}$ contains all points of condensation of K.*

Let p be any point of condensation of K, and I any interval containing p. There must be a point q of $I\cdot(E-N)$ which is a limit point of $K\cdot I$, for otherwise $\overline{K\cdot I}$ would be a closed non-denumerable subset of N, which is impossible since N is totally imperfect. We have then $q \epsilon \overline{K}\cdot(E-N)$ so that $p \epsilon \overline{\overline{K}\cdot(E-N)}$.

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[3]) This proposition appears implicitly in an argument of Kuratowski's. See the reference in footnote 2).

[4]) A non-denumerable set which contains no perfect subset is called totally imperfect. That such sets exist is well known. One may refer, for example, to Sierpiński's book, *Hypothèse du Continu*, p. 30, proposition L_1.

§ 2. *Construction of an L-space which possesses properties B and D and which does not possess property E.*

We shall define this L-space by altering the definition of limit of a sequence in E. (The ordinary limit of a sequence $\{p_n\}$ of points of E will be designated by $\operatorname{Lim} p_n$ and the new limit by $\lim p_n$).

First let N be a totally imperfect subset of E of power $\aleph_1$ and let us write N as a well-ordered series of type Ω

$$x_1,\ x_2, \dots\ x_\alpha, \dots \qquad (\alpha < \Omega)$$

Let $\{p_n\}$ be any sequence of points of E. If $\operatorname{Lim} p_n = p$ we shall put $\lim p_n = p$ unless the following two conditions hold: (1) $p \in N$ and (2) $\{p_n\}$ contains a subsequence all of whose points belong to N and precede p in our well-ordered series for N.

If these conditions hold no limit will be assigned to $\{p_n\}$. Furthermore if $\operatorname{Lim} p_n$ does not exist no limit will be assigned to $\{p_n\}$. In what follows it is to be understood unless otherwise specified that „limit point" means limit point in the *ordinary* sense.

It is clear that if S is any non-denumerable subset of $E - N$ then any point of condensation of S is a point condensation of S in the new sense. Again if S is any non-denumerable subset of N a simple argument yields the same conclusion. Thus our space has property B.

Our space however does not have property E since any denumerable subset of N cannot have more than a denumerable infinity of points of N as limit points (in the new sense).

We shall now show that our space has property D. Let F be any set which is closed in the new sense. In virtue of Lemma 1 it will be sufficient to prove that $\overline{F} - F$ is at most denumerably infinite. Suppose $\overline{F} - F \neq 0$ and consider any point p of $\overline{F} - F$. It is clear that $p \in N$ and that there is a sequence of points $\{p_n\}$ in $F \cdot N$ such that $\operatorname{Lim} p_n = p$. Thus $F \cdot N \neq 0$. Let us denote $F \cdot N$ by A and let us denote $\overline{A} \cdot N$ by A^*. We have $\overline{F} - F = A^* - A$. If A^* is non-denumerable then by Lemma 3 $\overline{A^* \cdot (E - N)} \supset$ all points of condensation of A^*. Since A is dense in A^* we have $\overline{A \cdot (E - N)} = \overline{A^* \cdot (E - N)}$. But in connection with $\overline{A \cdot (E - N)}$ the two meanings of „limit point" are the same. Hence $F \supset \overline{A^* \cdot (E - N)} \supset$ all points of condensation of A^*. But all the points of A^* except at most a denumerable infinity are points of condensation of A^*.

Therefore $A^* - A$ is at most denumerably infinite. Therefore the same is true of $\overline{F} - F$ and it is proved that our space has property D.

§ 3. *Construction of an L-space which possesses properties C and E and which does not possess property B.*

In the definition of our space in § 2 let us replace (2) by (2)' $\{p_n\}$ contains a subsequence all of whose points belong to N and *follow* p in our well-ordered series for N [5]).

That the space so defined possesses property E and does not possess property B is easily proved. As for property C we need merely notice that any set F which is closed in this new sense contains all of its points of condensation so that Lemma 2 applies at once.

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[5]) Parts (2) and (2)' of our definitions were suggested by certain examples of Sierpiński's. See the reference in footnote [2]).