Es ist zu bemerken, dass die Voraussetzung, nach der die kompakten Gruppen eine *inverse* Homomorphismenfolge bilden sollen, für die Gültigkeit des letzten Satzes wesentlich ist. Man ersieht daraus, warum in der Pontrjaginschen Verallgemeinerung des Alexanderschen Dualitätssatzes der Koeffizientenbereich für das Kompaktum K als eine *kompakte* und für das Komplement $R_n - K$ als eine *diskrete* Gruppe vorauszusetzen ist.

Note on the projection of irregular linearly measurable plane sets of points.

By

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1. Saks[1]) and Besicovitch[2]) have both constructed irregular linearly measurable[3]) plane sets of positive linear measure whose projections on all directions have Lebesgue measure zero. The existence of these has been the starting point for some investigations[4]) into the structure of irregular sets. One question which suggests itself is whether *parallel projections*, for which the stated property holds, is „privileged", i. e., whether or not it may be possible to prove an analogous result for projection from a finite point in the plane of the set. The justification of the concept „projection from a point" lies in the fact that, if the projection of a plane set from a point of the plane on a straight line not passing through the point has measure zero, then the same is true for any other line which does not contain the point. I shall not prove this statement since the proof is quite trivial, but it is important in that it enables us to say that *the projection from a point is zero or positive*[5]) *without reference to the line on to which the projection is made.*

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[1]) S. Saks, Fundamenta Mathematicae, 9 (1927), pp. 16—24.

[2]) A. S. Besicovitch, *On the fundamental geometrical properties of linearly measurable plane sets of points*; Math. Annalen, 98 (1928) pp. 422—464. See, in particular, pp. 431—434. A description of the set is given in § 3 below.

[3]) i. e. in the sense of Carathéodory, cf. Carathéodory, Gött. Nachrichten, (1914), pp. 404—426.

[4]) cf. *On linearly measurable plane sets of points of upper density* $\frac{1}{2}$, Fund. Math., 22 (1934), pp. 57—69.

[5]) i. e. has zero or positive Lebesgue measure.

2. In this note I consider the set of Besicovitch. I shall, in fact, show that *its projection from any point of the plane outside the set is zero.* The result completes, in a sense, our knowledge of the projections of this set, and it suggests the problem of whether and in what form it is possible to generalize to projection from a finite point the known results on parallel projections. The methods I use here are sufficiently closely related to the methods I use elsewhere [6]) to suggest that one can easily find arguments substantially analogous to those used in the theory of parallel projection. In view of the still incomplete nature of the results of the latter theory I shall make no attempt here to carry out this procedure.

Again, it is immediately obvious from the remarks of § 1 that the property proved here for the special set will hold for all its conical projections on other planes while the upper and lower densities of sets are not, in general, invariant under conical projection. Thus we get a fairly general class of sets which have zero projection from all external points.

3. In this section I define the set in question. To begin with I denote by $c(a, r)$ the circle with centre at the point a and radius r. By *the operation Γ_n on a circle $c(a, r)$*, I mean the construction of n circles as follows:

(a) draw the circle $c[a, (1-1/n)r]$,

(b) divide its circumference into n equal parts,

(c) describe circles of radius $1/nr$ about the points of division.

We now start with a circle C_1 of radius 1. We perform on it the operation Γ_4 and call the resulting set of four circles C_4. On each circle of C_4 we perform the operation Γ_5 and call the resulting set of 20 circles C_5. We proceed in this way constructing C_n by operating with Γ_n on each circle of C_{n-1}. For each n we perform Γ_n in such a way that no circle of C_n touches a circle of C_{n-2}. Finally $A = C_1 \times C_4 \times C_5 \times \dots$

The following properties of A are known [7]):

1. LA (= linear measure of A) = 2.
2. The upper density of A is $\frac{1}{2}$ at every point of A.
3. The lower density of A is equal to $\dfrac{1}{\sqrt{4\pi^2+1}-1}$ at almost all points of A.
4. At almost all points of A every direction is a (weak) density direction.
5. *The (orthogonal) projection of A on every direction has measure zero.*

[6]) See footnote 4) cf. also G. W. Morgan, *The density directions of irregular linearly measurable plane sets*, Proc. London Math. Soc., 38 (1935), pp. 481—484.

[7]) For 1, 2, and 3 see Besicovitch, op. cit. 431—434. Properties 4 and 5 were proved by Morgan, op. cit.

4. *I proceed to prove for A the property stated in* § 2. To begin with I suppose that the point P from which we project is exterior to C_1. Let O be the centre of C_1 and take axes with O for origin and OP as axis Ox. The co-ordinates of P are $(h, 0)$ (say) and we shall consider the projection of A on to some fixed straight line. The following lemma, which does not require proof, will be fundamental:

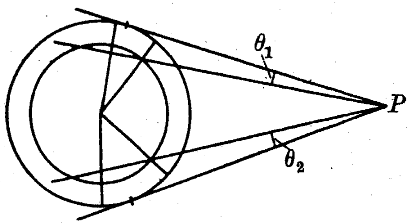
Lemma I. *Under the conditions described above there exist two positive constants K_1, K_2 with the following properties:*

(i) *If l is the length of any linear interval included in the closed circle C_1 then its projection has length less than $K_1 l$.*

(ii) *If λ is the length of any interval included in the projection of C_1 and θ the angle which the interval subtends at P then*

$$K_2 < \lambda/\theta < K_1.$$

Now divide each circle of each C_n into N equal sectors, where N is an arbitrary and fixed number, and take one of the dividing radii of each circle parallel to Ox.



Let s_n be some circle of C_n; the radius of s_n is $6/n!$ The set $s_n \cdot A$ lies in the ring bounded by s_n and the concentric circle of radius $\frac{6}{n!}\left(1-\frac{2}{n+1}\right)$.

Consider the plane areas which this ring has in common with the sectors containing the points of contact of the tangents to s_n from P and let θ_1, θ_2 be the angles which they subtend at P [See Fig.].

Then it is easily verified that, for sufficiently large n,

$$\theta_i \leqslant \frac{24\pi^2}{h-1}\cdot\frac{1}{n!}\cdot\frac{1}{N^2}. \qquad (i=1,2)$$

Hence, by (ii) of lemma, the projections of these areas are each less than

$$K_1\frac{4\pi^2}{h-1}\cdot\frac{1}{n!}\cdot\frac{1}{N^2}=\frac{K_3}{n!\cdot N^2} \qquad \text{(say)}.$$

Since the number of these areas obtained from all the circles of C_n is $\frac{n!}{3}$, we see that the total projection of all these areas is less than $\frac{K_4}{N^2}$, where K_4 is a constant. Now, for large n, the linear measure of the subset of A contained in these areas is (asymptotically) $\frac{2}{N}\cdot LA$. *Hence, given N, we can find n_0 so that, if $n \geqslant n_0$ and we perform the above operatiou on s_n, a subset of A of linear measure greater than $\frac{1}{N}\cdot LA$ projects into a set of Lebesgue measure less than $\frac{K_4}{N^2}$.*

Now consider the circles of C_{n+1} which were not projected in the given operation; their proportion to the whole of the circles of C_{n+1} is (asymptotically) $1-\frac{2}{N}$. We perform the same operation on each of them, then again on those circles of C_{n+2} which remain, etc. Suppose that we have performed this operation on $C_n, C_{n+1}, \ldots, C_{n+m}$.

The measure of the projection so far obtained is for large m, n less than

$$\frac{K_4}{N^2}\left[1+\left(1-\frac{1}{N}\right)+\left(1-\frac{1}{N}\right)^2+\ldots+\left(1-\frac{1}{N}\right)^m\right]$$
$$\leqslant\frac{K_4}{N^2}\cdot\frac{1}{1-\left(1-\frac{1}{N}\right)}=\frac{K_4}{N}.$$

The linear measure of the part of the set not yet projected is less than $\left(1-\frac{1}{N}\right)^m$.

Since N was arbitrary and m can be as large as we please, we have

Lemma II. Given any positive number ε, we can write $A = A_1 + A_2$ where (i) $LA_1 < \varepsilon$, and (ii) the projection of A has measure less than ε.

Now, by the definition of Carathéodory measure and (i) of Lemma II, we can enclose A_1 in a set of areas the sum of whose diameters is less than 2ε. It follows, by (i) of Lemma I, that the projection of A_1 has measure less than $2K_1\varepsilon$. Combining this with Lemma II (ii), we see that the projection of A has measure less than $(2K_1+1)\varepsilon$.

Since ε is arbitrary we see that *the projection of A from P has measure zero.*

5. In § 4 we saw that A has zero projection from every point external to C_1. Now A is closed and so, if Q is any point exterior to A, we can find ν such that Q is exterior to C_ν. We can then apply to Q and each circle of C_ν precisely the same argument as we applied above to P and C_1. It follows that the subset of A in each circle of C_ν has zero projection from Q and so, therefore, has A. *This completes the result.*