On inner transformations.

By

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1. The chief purpose of this note is to prove the

Theorem. *Let the (single-valued) transformation $\Phi(x)$ be continuous in the topologically complete*[1]) *space X and an inner transformation*[2]) *on a set $A \subset X$; then $\Phi(x)$ is also an inner transformation on $P \supset A$ where P is a G_δ in X.*

This theorem has been proved by Hausdorff[3]) for separable spaces and he has suggested the problem of removing the assumption of separability. The method of proof used here differs from that of Hausdorff only in the formation of the sets Q_1 and P_1 used below.

As in Hausdorff's paper, let $X=\bar{A}$ and take as a basis for X the system Γ_n of all open neighborhoods U which with their transforms $\Phi(U)$ are assumed to have a diameter $<1/n$. Then

$$X = \sum_{\xi_1} U_{\xi_1}, \qquad U_{\xi_1\ldots\xi_n} = \sum_{\xi_{n+1}} U_{\xi_1\ldots\xi_n\xi_{n+1}}$$

where U_{ξ_1} and $U_{\xi_1\ldots\xi_n\xi_{n+1}}$ belong to Γ_n and $\overline{U}_{\xi_1\ldots\xi_n\xi_{n+1}} \subset U_{\xi_1\ldots\xi_n}$. Let $\Phi(A)=B$ and $\Phi(AU_{\xi_1\ldots\xi_n})=BV_{\xi_1\ldots\xi_n}$ where the sets $V_{\xi_1\ldots\xi_n}$ are open in $Y=\Phi(X)$ and have diameters $<1/n$; also the V's can be so chosen[4]) that

$$V_{\xi_1\ldots\xi_n} \supset V_{\xi_1\ldots\xi_n\xi_{n+1}}.$$

[1]) A space is topologically complete if it is homeomorphic with a complete metric space.

[2]) An inner transformation on A is a (single-valued) continuous transformation which transforms every set U open in A into a set V open in $B=\Phi(A)$. See Hausdorff, Fund. Math. 23 (1934), p. 283.

[3]) Loc. cit., p. 283.

[4]) Hausdorff, Loc. cit. p. 285.

Let

(1) $$G_{\xi_1\ldots\xi_n} = U_{\xi_1\ldots\xi_n}\Psi(V_{\xi_1\ldots\xi_n}),$$

where $\Psi(V_{\xi_1\ldots\xi_n})$ is the set of all original points of $V_{\xi_1\ldots\xi_n}$. The set $G_{\xi_1\ldots\xi_n}$ is open in X since $\Psi(V_{\xi_1\ldots\xi_n})$[5]) and $U_{\xi_1\ldots\xi_n}$ are both open in X. Then

$$A = \sum_{\xi_1} AG_{\xi_1}, \qquad AG_{\xi_1\ldots\xi_n} = \sum_{\xi_{n+1}} AG_{\xi_1\ldots\xi_n\xi_{n+1}}.$$

Now let each point of B be contained in a neighborhood V_{ξ_1}; the set $\sum_{\xi_1} V_{\xi_1}$ is open in Y and contains B. Next let each point of B be contained in a neighborhood $V_{\xi_1\xi_2}$; the set $\sum_{\xi_1\xi_2} V_{\xi_1\xi_2}$ is open and contains B. By continuing this process we have at the n-th step a set $\sum_{\xi_1\xi_2\ldots\xi_n} V_{\xi_1\xi_2\ldots\xi_n}$ which contains B and is open in Y. Hence[6])

(2) $$Q_1 = \sum_{\xi_1} V_{\xi_1} \cdot \sum_{\xi_1\xi_2} V_{\xi_1\xi_2} \cdot \sum_{\xi_1\xi_2\xi_3} V_{\xi_1\xi_2\xi_3} \ldots$$

is a G_δ in Y which contains B. Since Φ is a continuous transformation, the set $P_1=\Psi(Q_1)$ is a G_δ in X.

Now let

$$P_2 = \sum_{\xi_1} G_{\xi_1} \cdot \sum_{\xi_1\xi_2} G_{\xi_1\xi_2} \cdot \sum_{\xi_1\xi_2\xi_3} G_{\xi_1\xi_2\xi_3} \ldots$$

The set $P_2 \supset A$ and is a G_δ in X since it is the intersection of a countable number of sets open in X.

Now let $P=P_1 \cdot P_2$ and $Q=\Phi(P)$. The set P is a G_δ in X, which contains A and the method of proof used by Hausdorff shows that in this case also Φ is an inner transformation on P. For, as in the proof of Hausdorff,

$$\Phi(PG_{\xi_1\ldots\xi_n}) \subset QV_{\xi_1\ldots\xi_n}$$

since $G_{\xi_1\ldots\xi_n} \subset \Psi(V_{\xi_1\ldots\xi_n})$. To show that

$$QV_{\xi_1\ldots\xi_n} \subset \Phi(PG_{\xi_1\ldots\xi_n})$$

let $y \in QV_{\xi_1\ldots\xi_n} \subset Q_1V_{\xi_1\ldots\xi_n}$. By (2), $y \in \prod_p V_{\xi_1\ldots\xi_p}$ for a suitably chosen sequence $\{\xi_p\}$ and as $\prod_p V_{\xi_1\ldots\xi_p}$ contains but a single point, $y=\prod_p V_{\xi_1\ldots\xi_p}$.

[5]) Hausdorff, *Mengenlehre*, p. 194.

[6]) This set Q_1 differs from the set Q_1 used by Hausdorff and his remark on page 287 does not apply to it.

Since X is complete and as $\overline{U}_{\xi_1\ldots\xi_n}\subset U_{\xi_1\ldots\xi_{n+1}}$ we have $x=\prod_p U_{\xi_1\ldots\xi_p}$. Then as shown by Hausdorff $y=\Phi(x)$ and $x\in\prod_p G_{\xi_1\ldots\xi_p}\subset P_2$; also $x\in P$, since $x\in\Psi(Q_1)$. Therefore

$$Q(PG_{\xi_1\ldots\xi_n})=QV_{\xi_1\ldots\xi_n}.$$

From (1) we see that the sets $P\cdot G_{\xi_1\ldots\xi_n}$ are arbitrarily small for n sufficiently great and form a basis for P since every point of P is contained in arbitrarily small neighborhoods $G_{\xi_1}, G_{\xi_1\xi_2}, G_{\xi_1\xi_2\xi_3}\ldots$ Therefore Φ is an inner transformation on P and the theorem is true for all topologically complete spaces.

2. We will now prove the following theorem:

Theorem. *Every semi-metric*[7] *space E which satisfies the axiom*[8] *given below is an inner transformation of a Baire space*[9] *A.*

Axiom A. For each point a and each positive number k there is a positive number r such that, if b is a point for which $ab\geqq k$ and c is any point, $ac+bc\geqq r$.

Hausdorff[10] has proved this theorem for metric spaces. Hence since any semi-metric space E in which the above axiom is satisfied is homeomorphic[11] with a metric space the extension to semi-metric spaces is immediate. For E is homeomorphic with the metric space B which is a continuous transformation of a Baire space A. The transformation from A to E is an inner transformation since that from A to B is.

The above theorem also holds in topological spaces satisfying the axioms:

A. Every point x has at least one neighborhood U_x; x is contained in U_x.

B. If U_x and V_x are neighborhoods of x there exists a neighborhood $W_x=U_x\cdot V_x$.

C. For each U_x there exists a V_x such that, if y lies in V_x, some $U_y\subset U_x$.

D. If x and y are distinct points, some U_x does not contain y.

E. For each point a and each integer n there is an integer $m=g(a,n)$ such that m increases indefinitely with n and the relation $V^n(a)\cdot V^n(b)\neq 0$ implies that b lies in $V^m(a)$ and a lies in $V^m(b)$.

F. If x is a point and $\{U_x\}$ the set of this neighborhoods, there is a countable subset $\{V_x^i\}$ of these neighborhoods having only x in common and such that each U_x contains some V_x^i.

Wilson has shown[12] that a topological space Z satisfying the above axioms is homeomorphic with a semi-metric space satisfying Axiom A. Hence the theorem under consideration is true in the space Z.

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[7]) A space is semi-metric if for any two points a and b there corresponds a number ab called the distance between the points which satisfies the axioms:

I, $ab=ba$; II, $ab=0$ if and only if $a=b$.

[8]) W. A. Wilson, *On Semi-Metric Spaces*, Amer. Journ. of Math., LIII, Nº 2, p. 361.

[9]) By a Baire space we understand the product $X=(B_1, B_2, B_3,\ldots)$ of a sequence of sets $B_n\neq 0$, that is the set of sequences $x=(b_1, b_2, b_3,\ldots)$, $b_n\in B_n$. If $\xi=(\beta_1,\beta_2,\beta_3,\ldots)\neq x$, then the distance $x\xi=1/n$ when n is the first difference place between x and ξ.

[10]) Loc. cit., p. 288.

[11]) W. A. Wilson, loc. cit., p. 366.

[12]) Loc. cit., p. 373.