On inner transformations.

By


1. The chief purpose of this note is to prove the

Theorem. Let the (single-valued) transformation \( \Phi(x) \) be continuous in the topologically complete \(^1\) space \( X \) and an inner transformation \(^2\) on a set \( A \subset X \); then \( \Phi(x) \) is also an inner transformation on \( P \subset A \) where \( P \) is a \( G_\delta \) in \( X \).

This theorem has been proved by Hausdorff \(^3\) for separable spaces and he has suggested the problem of removing the assumption of separability. The method of proof used here differs from that of Hausdorff only in the formation of the sets \( Q_1 \) and \( P_1 \) used below.

As in Hausdorff's paper, let 

\[ X = A \]

and take as a basis for 

\( X \) the system \( I_n \), of all open neighborhoods \( U \) with their transformations \( \Phi(U) \) assumed to have a diameter \( < 1/n \). Then 

\[ X = \bigcup_{i \in I_n} U_i, \quad U_{i_1 \ldots i_n} = \bigcup_{i_{n+1} \in I_{n+1}} U_{i_1 \ldots i_n i_{n+1}} \]

where \( U_i \) and \( U_{i_1 \ldots i_{n+1}} \) belong to \( I_n \) and \( \bigcup_{i_{n+1} \in I_{n+1}} U_{i_1 \ldots i_n i_{n+1}} \subset U_{i_1 \ldots i_n} \).

Let \( \Phi(A) = B \) and \( \Phi(A U_{i_1 \ldots i_n}) = B V_{i_1 \ldots i_n} \) where the sets \( V_{i_1 \ldots i_n} \) are open in \( Y = \Phi(X) \) and have diameters \( < 1/n \); also the \( V_i \)s can be so chosen \(^4\) that 

\[ V_{i_1 \ldots i_n} \supset V_{i_1 \ldots i_n i_{n+1}}. \]

(1) 

\[ G_{i_1 \ldots i_n} = U_{i_1 \ldots i_n} V_{i_1 \ldots i_n}, \]

where \( V_{i_1 \ldots i_n} \) is the set of all original points of \( V_{i_1 \ldots i_n} \). The set \( G_{i_1 \ldots i_n} \) is open in \( X \) since \( V_{i_1 \ldots i_n} \) and \( U_{i_1 \ldots i_n} \) are both open in \( X \). Then 

\[ A = \bigcup_{i_n} AG_{i_1 \ldots i_n}, \quad AG_{i_1 \ldots i_n} = \bigcup_{i_{n+1} \in I_{n+1}} AG_{i_1 \ldots i_n i_{n+1}} \]

Now let each point of \( B \) be contained in a neighborhood \( V_{i_1} \); the set \( \bigcup_{i_1} V_{i_1} \) is open in \( Y \) and contains \( B \). Next let each point of \( B \) be contained in a neighborhood \( V_{i_1 \ldots i_n} \); the set \( \bigcup_{i_1 \ldots i_n} V_{i_1 \ldots i_n} \) is open and contains \( B \). By continuing this process we have at the \( n \)-th step a set \( \bigcup_{i_1 \ldots i_n} V_{i_1 \ldots i_n} \) which contains \( B \) and is open in \( Y \). Hence \(^5\)

\[ Q_1 = \bigcup_{i_1} V_{i_1} \cdot \bigcup_{i_1 \ldots i_n} V_{i_1 \ldots i_n} \cdot \bigcup_{i_1 \ldots i_{n+1}} V_{i_1 \ldots i_{n+1}} \ldots \]

is a \( G_\delta \) in \( Y \) which contains \( B \). Since \( \Phi \) is a continuous transformation, the set \( P_1 = \Phi(Q_1) \) is a \( G_\delta \) in \( X \).

Now let 

\[ P_2 = \bigcup_{i_1} G_{i_1} \cdot \bigcup_{i_1 \ldots i_n} G_{i_1 \ldots i_n} \cdot \bigcup_{i_1 \ldots i_{n+1}} G_{i_1 \ldots i_{n+1}} \ldots \]

The set \( P_2 \subset A \) and is a \( G_\delta \) in \( X \) since it is the intersection of a countable number of sets open in \( X \).

Now let \( P = P_1 \cdot P_2 \) and \( Q = \Phi(P) \). The set \( P \) is a \( G_\delta \) in \( X \), which contains \( A \) and the method of proof used by Hausdorff shows that in this case also \( \Phi \) is an inner transformation on \( P \). For, as in the proof of Hausdorff,

\[ \Phi(PG_{i_1 \ldots i_n}) \subset QV_{i_1 \ldots i_n} \]

since \( G_{i_1 \ldots i_n} \subset \Phi(V_{i_1 \ldots i_n}) \). To show that 

\[ QV_{i_1 \ldots i_n} \subset \Phi(PG_{i_1 \ldots i_n}) \]

let \( y \in QV_{i_1 \ldots i_n} \subset Q_1 V_{i_1 \ldots i_n} \). By (2), \( y \in \bigcup_{i_1 \ldots i_n} V_{i_1 \ldots i_n} \) for a suitably chosen sequence \( \{ \xi_i \} \) and as \( \bigcup_{i_1 \ldots i_n} V_{i_1 \ldots i_n} \) contains but a single point, \( y \in \bigcap_{i_1 \ldots i_n} V_{i_1 \ldots i_n} \).

\(^1\) Hausdorff, Mengenlehre, p. 194.

\(^2\) This set \( Q_1 \) differs from the set \( Q_1 \) used by Hausdorff and his remark on page 287 does not apply to it.

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Since $X$ is complete and as $\overline{U_{i+1} \cup U_{i+2}} \subseteq U_{i+3}$ we have $x = \cap_{i} U_{i+3}$.

Then as shown by Hausdorff $y = \varphi(x)$ and $x \in \cap_{i} \{G_{i+3} \cap P_{i+3}\}$, also $x : P$, since $x \in \varphi(Q_1)$. Therefore

$$Q(PG_{i+3}) = QV_{i+3}.\]

From (1) we see that the sets $P \cdot G_{i+k}$ are arbitrarily small for a sufficiently great and form a basis for $P$ since every point of $P$ is contained in arbitrarily small neighborhoods $G_{i+k}, G_{i+k}, G_{i+k}, ..., G_{i+k}$. Therefore $G$ is an inner transformation on $P$ and the theorem is true for all topologically complete spaces.

2. We will now prove the following theorem:

**Theorem.** Every semi-metric space $E$ which satisfies the axiom $A$ given below is an inner transformation of a Baire space.

**Axiom A.** For each point $a$ and each positive number $k$ there is a positive number $r$ such that, if $b$ is a point for which $ab \geq k$ and $b$ is any point, $ab + br \geq r$.

Hausdorff has proved this theorem for metric spaces. Hence since any semi-metric space $E$ in which the above axiom is satisfied is homeomorphic with a metric space the evidence to semi-metric spaces is immediate. For $E$ is homeomorphic with the metric space $B$ which is a continuous transformation of a Baire space $A$. The transformation from $A$ to $B$ is an inner transformation since that from $A$ to $B$ is.

The above theorem also holds in topological spaces satisfying the axioms:

A. Every point $x$ has at least one neighborhood $U_x$; $x$ is contained in $U_x$.

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1) A space is semi-metric if for any two points $a$ and $b$ there corresponds a number $ab$ called the distance between the points which satisfies the axioms:

I. $ab = ba$; II. $ab = 0$ if and only if $a = b$.


3) By a Baire space we understand the product $X = \{B_1, B_2, B_3, ...\}$ of a sequence of sets $B_n + 0$, that is the set of sequences $x = \{b_1, b_2, b_3, ...\}$, $b_n \in B_n$. If $x = \{b_1, b_2, b_3, ...\} + x$, then the distance $x = 1/n$ when $n$ is the first difference place between $x$ and $y$.


5) W. A. Wilson, loc. cit., p. 386.