quential topology τ defined on L will be called „continuous“ if and only if it is continuous in the operations of the system — that is, (1) $\tau\{x_n\}=x$ and $\tau\{y_n\}=y$ imply $\tau\{x_n+y_n\}=x+y$, and (2) if $\{\lambda_n\}$ is any sequence of real numbers converging to λ, and $\tau\{x_n\}=x$, then $\tau\{\lambda_n x_n\}=x$. It is clear that if $x_n\equiv x$ implies $\tau\{x_n\}=x$, then $\tau\{x_n\}=x$ if and only if $\tau\{x_n-x\}=\Theta$. This will be assumed throughout § 7.

Theorem 16: *The join of any number of continuous sequential topologies is itself continuous.*

The proof is that sketched for Theorem 6.

Theorem 17: *The most inclusive continuous sequential topology τ'' included in each of two given continuous sequential topologies τ and τ' exists and is given by the rule $\tau''\{x_n\}=\Theta$ if and only if $\{y_n\}$ and $\{z_n\}$ exist, satisfying $x_n=y_n+z_n$ and $\tau\{y_n\}=\tau'\{z_n\}=\Theta$.*

The only real question is as to whether this rule defines a *continuous* sequential topology; if it does, $\tau''\subset\tau$ and $\tau''\subset\tau'$ are obvious, and is also obvious that any continuous sequential topology including τ and τ' must include τ''. Further, τ'' and homogeneity evidently define a unique sequential topology.

Now suppose $\tau''\{x_n\}=$ and $\tau''\{x'_n\}=x'$. Then we can find $\{y_n\}$, $\{z_n\}$, $\{y'_n\}$ and $\{z'_n\}$ such that $y_n+z_n=x_n-x$, $y'_n+z'_n=x'_n-x'$, $\tau\{y_n\}=\tau'\{z_n\}=\tau\{y'_n\}=\tau'\{z'_n\}=\Theta$. Hence $\tau\{y_n+y'_n\}=\tau'\{z_n+z'_n\}=\Theta$, by definition

$$\tau''\{(x_n+x'_n)-(x+x')\}=\tau''\{(y_n+y'_n)+(z_n+z'_n)\}=\Theta$$

and $\tau''\{x_n+x'_n\}=x+x'$. The proof that $\lambda_n\to\lambda$ and $\tau''\{x_n\}=x$ implies $\tau''\{\lambda_n x_n\}=\lambda x$ follows similar lines.

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On the differentiation of additive functions of rectangles.

By

A. J. Ward (Cambridge).

It has recently been shown that if $f(x, y)$ is a summable function of two variables, it is not in general true that

$$\operatorname*{Lim}_{d(R)\to 0}\frac{1}{\mu R}\int_R\int f\,dx\,dy=f(x_0, y_0)$$

almost everywhere, where R is any rectangle containing the point (x_0, y_0) [1] [2]. Saks raised the question whether it is possible for the upper and lower limits, as $h, k\to 0$, of

$$\frac{1}{4hk}\int_{x_0-h}^{x_0+h}\int_{y_0-k}^{y_0+k} f(x, y)\,dx\,dy$$

to be finite and not identical at the points of a set of positive measure [3]. Besicovitch has solved a slightly different but closely related problem by showing that

$$\overline{\lim_{h,k\to 0}}\frac{1}{hk}\int_{x_0}^{x_0+h}\int_{y_0}^{y_0+k} f(x, y)\,dx\,dy$$

is equal, at almost all points, either to $f(x_0, y_0)$ or to $+\infty$ [4].

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[1]) H. Busemann and W. Feller, Fund. Math. 22 (1934), 226—256; S. Saks, *ibid.*, 257—261.

[2]) Here, as throughout this paper, R denotes a closed rectangle with sides parallel to the co-ordinate axes, $d(R)$ its diameter, and μ Lebesgue plane measure.

[3]) Saks, *loc. cit.*, 260.

[4]) A. S. Besicovitch, Fund. Math. 25 (1935), 209—216.

In this paper I consider the analogous limits obtained when the integral $\iint_R f(x, y)\, dx\, dy$ is replaced by an arbitrary additive function of rectangles $F(R)$. The theorems proved include the result of Besicovitch as a special case.

1. Let $F(x, y)$ be any real function of two real variables, defined in the interior of some square. Given any point (x, y) and two numbers h, k (which may be positive or negative but do not vanish), we write $R(x, y; h, k)$ for the rectangle whose opposite corners are (x, y) and $(x+h, y+k)$. (All rectangles considered have their sides parallel to the coordinate axes.) We write

$$(1) \qquad \Delta F(x, y; h, k) = F(x+h, y+k) - F(x+h, y) - F(x, y+k) + F(x, y);$$

$$(2) \qquad F[R(x, y; h, k)] = \Delta F(x, y; h, k) \operatorname{sgn}(hk).$$

Thus for any rectangle R, $F(R)$ is defined, and has the same value whichever corner we take as (x, y); also $F(R)$ is an additive function of rectangles. We define

$$(3) \qquad \overline{D}(x, y) = \overline{\lim_{d(R)\to 0}} \frac{F(R)}{\mu R}$$

where R is any rectangle containing (x, y) either as an interior point or on the boundary. Since any rectangle containing (x, y) can be divided into at most four rectangles, each of which has (x, y) at one corner, we see from (2) and (3) that we have also

$$(4) \qquad \overline{D}(x, y) = \overline{\lim_{h,k\to 0}} \frac{\Delta F(x, y; h, k)}{hk}.$$

We define further

$$(5) \qquad \overline{D}_m(x, y) = \overline{\lim_{h\to 0}} \frac{\Delta F(x, y; h, h)}{h^2}.$$

There are similar definitions for the lower derivates $\underline{D}(x, y)$, $\underline{D}_m(x, y)$. We have at once

$$(6) \qquad \underline{D}(x, y) \leqslant \underline{D}_m(x, y) \leqslant \overline{D}_m(x, y) \leqslant \overline{D}(x, y).$$

2. We first state an elementary geometrical lemma. By a *principal corner* of any rectangle we mean either the lower left-hand or the upper right-hand corner.

Lemma. *If in any rectangle R there is given a finite number of rectangles R_ν, not touching or over-lapping each other or the sides of R; then the remaining area $R - \sum R_\nu$ can be divided into a finite number of rectangles R'_ν, each of which has as one of its corners a principal corner of one of the rectangles R_ν.*

The rectangles R'_ν may be constructed thus: Produce each horizontal side of every rectangle R_ν as far as possible to the left, until it meets either the left-hand side of R or the right-hand side of some other R_ν. Then produce each right-hand vertical side of R_ν as far as possible, both ways, without crossing any of the horizontal lines already in existence. It may be shown [5]) that the lines thus drawn divide $R - \sum R_\nu$ into a set of rectangles R'', each of which contains a principal corner of some R_ν, either at a corner on on a side. If on a side, we have merely to divide R'' into two parts by a line drawn from that point. Thus we obtain the required subdivision into rectangles R'_ν.

3. ***Theorem 1.*** *If at each point (x, y) of a set E, we have $-\infty < \underline{D}(x, y) \leqslant \overline{D}(x, y) < \infty$; then, almost everywhere in E, $\underline{D}(x, y) = \overline{D}(x, y)$.*

We shall show that, almost everywhere in E,

$$(7) \qquad \underline{D}(x, y) \geqslant \overline{D}_m(x, y).$$

Then by a similar argument we have almost everywhere in E

$$\overline{D}(x, y) \leqslant \underline{D}_m(x, y),$$

and combining these inequalities with (6) we obtain the desired result.

Suppose on the contrary that in a subset of E of positive outer measure, we have

$$-\infty < \underline{D}(x, y) < \overline{D}_m(x, y) \leqslant \overline{D}(x, y) < \infty.$$

Then we can find M so large and $\eta > 0$ so small that in a set E_1, of positive outer measure we have

$$(8) \qquad -M < \underline{D}(x, y) < \overline{D}_m(x, y) - 3\eta < \overline{D}(x, y) < M.$$

We can then find an integer A such that in a set E_2 of positive outer measure, (8) holds and also

$$(9) \qquad \underline{D}(x, y) < (A-2)\eta,$$

$$(10) \qquad \overline{D}_m(x, y) > A\eta.$$

[5]) The proof is easy and will not be given.

From (8) (9) and (10) we see that

(11) $$|A\eta|<M.$$

For any point (x, y) of E_2, there exists $\delta(x, y)>0$ such that

(12) $$-M\leqslant\frac{F[R(x, y;\ h, k)]}{|hk|}\leqslant M$$

whenever $0<|h|<\delta(x, y)$; $0<|k|<\delta(x, y)$; from (8). Hence, we can find a fixed $\delta>0$ and a set E_3, contained in E_2, of positive outer measure, such that (12) is true whenever $0<|h|<\delta$, $0<|k|<\delta$, and (x, y) lies in E_3. Let E_4 be the set of points of E_3 which are points of outer density of E_3 in the strong sense [6]). Then $\mu_e E_4=\mu_e E_3>0$.

4. Let (x, y) be any point of E_4. We shall show that there exists a sequence of rectangles $R(x, y;\ h_n, k_n)$, $n=1, 2,\dots$ such that

(13) $$h_n k_n>0, \qquad \tfrac{1}{2}|k_n|<|h_n|<2|k_n|,$$

(14) $$h_n, k_n\to 0 \quad \text{as} \quad n\to\infty,$$

(15) $$(x+h_n, y+k_n) \text{ belongs to } E_3,$$

(16) $$F[R(x, y;\ h_n, k_n)]>(A-1)\eta h_n k_n.$$

For from (10) and the fact that (x, y) is a point of outer density of E_3, we can find a sequence of squares $S_n=R(x, y;\ l_n, l_n)$ such that

(17) $$|l_n|<\delta \quad \text{and} \quad l_n\to 0 \quad \text{as} \quad n\to\infty,$$

(18) $$F(S_n)>A\,\eta\, l_n^2,$$

(19) $$\mu_e(E_3 S_n)>(1-\varepsilon^2)\, l_n^2,$$

where

(20) $$\varepsilon=\eta/16M.$$

From (8) clearly $0<\varepsilon<\frac{1}{2}$. Consider one such square S_n, and suppose for example that $l_n>0$. By (19) there exist h_n, k_n such that

(21) $$(1-\varepsilon)\, l_n<h_n<l_n,$$

(22) $$(1-\varepsilon)\, l_n<k_n<l_n,$$

[6]) The point (x, y) is a point of outer density of set E, in the strong sense, if $\lim\limits_{d(R)\to 0}\frac{\mu_e(ER)}{\mu(R)}=1$, where R is any rectangle, containing (x, y), with sides parallel to the axes. S. Saks, *Théorie de l'Intégrale*, Warsaw 1933, pp. 223, 231.

and $(x+h_n, y+k_n)$ belongs to E_3. We can express the rectangle $R(x, y;\ h_n, k_n)$ as

$$R(x, y;\ h_n, k_n)=S_n-R_{n1}-R_{n2}-R_{n3}$$

where R_{n1}, R_{n2}, R_{n3} have each one corner at $(x+h_n, y+k_n)$ and their opposite corners at $(x, y+l_n)$; $(x+l_n, y+l_n)$; $(x+l_n, y)$ respectively. The sides of these rectangles are less than l_n and so less than δ; hence as $(x+h_n, y+k_n)$ belongs to E_3 we have from (12)

$$F(R_{n1})+F(R_{n2})+F(R_{n3})\leqslant M\sum_{i=1}^{3}\mu(R_{ni})$$

and so from (18) (21) (22) (20) and (11), since $F(R)$ is additive,

$$\begin{aligned} F[R(x, y;\ h_n, k_n)] &> A\,\eta\, l_n^2-M(l_n^2-h_n k_n)\\ &> A\,\eta\, h_n k_n-(M-A\,\eta)\,(l_n^2-h_n k_n)\\ &> A\,\eta\, h_n k_n-2M\, l_n^2[1-(1-\varepsilon)^2]\\ &> A\,\eta\, h_n k_n-4M\,\varepsilon\, l_n^2\\ &> A\,\eta\, h_n k_n-\eta\, l_n^2/4\\ &> (A-1)\,\eta\, h_n k_n \end{aligned}$$

since from (21) and (22) we have certainly $h_n k_n>l_n^2/4$, for $\varepsilon<\frac{1}{2}$. We have now shown that h_n, k_n satisfy conditions (15) and (16). From (21) and (22), since $\varepsilon<\frac{1}{2}$, (13) is also satisfied. If $l_n<0$ we can proceed in a similar way; in this case h_n and k_n will be negative. Thus h_n, k_n are defined for all n, and since $l_n\to 0$ we have clearly $h_n, k_n\to 0$ as $n\to\infty$.

5. Consider now a fixed point (x_0, y_0) of E_4; since E_3-E_4 is of measure zero, (x_0, y_0) is also a point of outer density of E_4 in the strong sense. From (9) we see that there exist h_0, k_0 such that, writing $R_0=R(x_0, y_0;\ h_0, k_0)$,

(23) $$F(R_0)<(A-2)\,\eta\,\mu(R_0),$$

(24) $$0<|h_0|<\delta, \qquad 0<|k_0|<\delta,$$

(25) $$\mu_e(E_4 R_0)>(1-\varepsilon)\,\mu(R_0).$$

With each point (x, y) of $E_4 R_0$ is associated a sequence of rectangles satisfying the conditions of the last paragraph; and in particular condition (13). Hence we can apply Vitali's theorem and obtain, by (25) and (16), a finite set of such rectangles $R_1,\dots R_n$, say, not touching each other or the sides of R_0, such that

(26) $$\sum_{\nu=1}^{n} \mu(R_\nu) > (1-\varepsilon)\,\mu(R_0),$$

(27) $$\sum_{\nu=1}^{n} F(R_\nu) > (A-1)\,\eta \sum_{\nu=1}^{n} \mu(R_\nu).$$

From (13), (15) we see that the principal corners of each R_ν are points of E_3. Apply the lemma; we can write

(28) $$R_0 = \sum_{\nu=1}^{n} R_\nu + \sum_{\nu=1}^{n'} R'_\nu,$$

where each R'_ν has one corner belonging to E_3. We have therefore, by (12) and (24)

(29) $$F(R'_\nu) \geqslant -M\,\mu(R'_\nu)$$

and so from (20), (26), (27) and (28), since $F(R)$ is additive,

$$\begin{aligned} F(R_0) &> (A-1)\,\eta \sum_{\nu=1}^{n} \mu(R_\nu) - M \sum_{\nu=1}^{n'} \mu(R'_\nu) \\ &> (A-1)\,\eta\,\mu(R_0) - \{M + (A-1)\,\eta\} \sum_{\nu=1}^{n'} \mu(R'_\nu) \\ &> (A-1)\,\eta\,\mu(R_0) - 2M\,\varepsilon\,\mu(R_0) \\ &> (A-2)\,\eta\,\mu(R_0), \end{aligned}$$

(since $(A-1)\eta < M$). This contradicts (23) and so (7) is established.

6. A system of rectangles $R_n(x,y)$ associated with a point (x,y) will be said to be *completely regular* if

(i) Each $R_n(x,y)$ contains (x,y);

(ii) $d(R_n) \to 0$ as $n \to \infty$;

(iii) There exists $\alpha > 0$ such that, for all n,

(30) $$\mu(R_n) > \alpha\,[d(R_n)]^2,$$

(31) $$d(R_{n+1}) > \alpha\, d(R_n).$$

Theorem 2. If $\underline{D}(x,y) > -\infty$ at each point of a set E, then, except at the points of a set N of measure zero [independent of the systems $R_n(x,y)$], we have at each point of E

$$\lim_{n\to\infty} \frac{F[R_n(x,y)]}{\mu[R_n(x,y)]} = \underline{D}(x,y)$$

for every completely regular system $R_n(x,y)$.

Suppose the theorem false; then there exists a subset E_1 of E, of positive outer measure, with each point of which is associated a completely regular sequence of rectangles $R_n(x,y)$ such that

$$\varlimsup_{n\to\infty} \frac{F[R_n(x,y)]}{\mu[R_n(x,y)]} > \underline{D}(x,y).$$

The number α occurring in (30) and (31) will depend on (x,y), but we can find a fixed α so small that (30) and (31) are satisfied, for this value of α, at all points of a subset E_2 of E_1 with positive outer measure. Let B be an integer such that

(32) $$B\alpha^3 > 576.$$

Arguing as in § 3 we show that there exist $\eta > 0$, $\delta > 0$, an integer A, and a set E_3, contained in E_2, of positive outer measure such that, for (x,y) in E_3, $0<|h|<\delta$, $0<|k|<\delta$, we have

(33) $$A\eta < \underline{D}(x,y) \leqslant (A+1)\,\eta;$$

(34) $$\varlimsup_{n\to\infty} \frac{F[R_n(x,y)]}{\mu[R_n(x,y)]} > (A+B)\,\eta;$$

(35) $$F[R(x,y;h,k)] > A\eta\,|hk|;$$

and also

(36) $$d[R_1(x,y)] \geqslant \delta;$$

(37) $$F[R_n(x,y)] > (A+B)\,\eta\,\mu[R_n(x,y)]$$

whenever $d(R_n) < \delta$.

Consider a point (x_0, y_0) which belongs to E_3 and is a point of outer density of E_3 in the strong sense. Then from (33) it follows that we can find h, k such that the rectangle $R_0 = R(x_0, y_0; h, k)$ satisfies

(38) $$0<|h|<\tfrac{1}{2}\delta, \qquad 0<|k|<\tfrac{1}{2}\delta,$$

(39) $$F(R_0) < (A+2)\,\eta\,\mu(R_0),$$

(40) $$\mu_e(E_3 R_0) > \mu(R_0)\left(1 - \frac{\alpha^3}{1000}\right).$$

It is convenient to suppose $0 < h \leqslant k$ (it will readily appear how to apply the argument in the other possible cases); making this supposition we define the integer m such that

(41) $$m \geqslant 2, \qquad mh \leqslant 2k < (m+1)\,h.$$

We consider the $2m$ squares S_{pq} defined by

(42) $$x_0+(p-1)\,h/2 \leqslant x \leqslant x_0+ph/2, \qquad (p=1,2)$$

(43) $$y_0+(q-1)\,h/2 \leqslant y \leqslant y_0+qh/2, \qquad (q=1,2\ldots m)$$

and the two rectangles R_p,

(44) $$x_0+(p-1)\,h/2 \leqslant x \leqslant x_0+ph/2, \qquad (p=1,2)$$

(45) $$y_0+mh/2 \leqslant y \leqslant y_0+k,$$

except that if $2k=mh$ exactly, these rectangles are naturally omitted. A square S_{pq} will be said to be of class A if

(46) $$\mu_e(E_3\,S_{pq}) > \mu(S_{pq}) - \alpha^3 h^2/144;$$

otherwise it will be said to be of class B. Let m' be the number of squares of class B; then clearly we have

$$\mu_e(E_3\,R_0) \leqslant \mu(R_0) - m'\,\alpha^3 h^2/144$$

and so from (40) and (41) we obtain

$$m'\,\alpha^3 h^2/144 < \alpha^3 hk/1000 < \alpha^3 h^2(m+1)/2000$$

and so

(47) $$m' < 144\,(m+1)/2000 < (m+1)/12 < m/6.$$

From this it follows that each of the two columns of m squares given by $p=1$ and $p=2$ contains some square of class A. Then the squares of each column, together with the corresponding rectangle R_p (if it exists), can be grouped together into rectangles, each of which contains exactly one square of class A. We suppose this process carried out for each column. Of the resulting rectangles, some will consist each of a single square of class A; these we enumerate as $S_1, S_2, \ldots S_M$. The remaining rectangles, each of which contains one or more squares of class B or possibly a rectangle R_1 or R_2, we enumerate as $R'_1, R'_2, \ldots R'_{M'}$. Clearly $M' \leqslant m'+2$, and so, since by the construction $M=(2m-m')-M'$, we have from (41) and (47)

(48) $$M > 2m - m/3 - 2 > m/2.$$

Now by the additive property of $F(R)$

(49) $$F(R_0) = \sum_{\nu=1}^{M} F(S_\nu) + \sum_{\nu=1}^{M'} F(R'_\nu).$$

We shall obtain a lower bound for each of these terms.

7. Consider a square S_ν (one of the squares S_{pq}); let (x_ν, y_ν) be its middle point, so that its principal corners are $(x_\nu-h/4, y_\nu-h/4)$ and $(x_\nu+h/4, y_\nu+h/4)$. S_ν is of class A; hence by (46) there is certainly[7] a point (x'_ν, y'_ν) of E_3 such that

(50) $$x_\nu - h/12 < x'_\nu < x_\nu + h/12,$$

(51) $$y_\nu - h/12 < y'_\nu < y_\nu + h/12.$$

Let $R_N(x'_\nu, y'_\nu)$ be the first of the sequence of rectangles $R_n(x'_\nu, y'_\nu)$ which satisfies

$$d[R_n(x'_\nu, y'_\nu)] < h/12;$$

then by (31), (36) and (38) we have

(52) $$\alpha h/12 < d[R_N(x'_\nu, y'_\nu)] < h/12,$$

and so by (30)

(53) $$\mu[R_N(x'_\nu, y'_\nu)] > \alpha^3 h^2/144.$$

Let $(\xi_{1\nu}, \eta_{1\nu})$ and $(\xi_{2\nu}, \eta_{2\nu})$ be the principal corners of $R_N(x'_\nu, y'_\nu)$, so that, by (50), (51) and (52) we have

(54) $$x_\nu - h/6 < \xi_{1\nu} < \xi_{2\nu} < x_\nu + h/6;$$

(55) $$y_\nu - h/6 < \eta_{1\nu} < \eta_{2\nu} < y_\nu + h/6;$$

also from (52) and (53), since $\mu(R_N) < d(R_N)\,(\xi_{2\nu}-\xi_{1\nu})$ we have

(56) $$\xi_{2\nu} - \xi_{1\nu} > \alpha^3 h/12.$$

Consider now the four rectangles

(57) $$x_\nu - h/4 < x < \xi_{1\nu}, \qquad y_\nu - h/4 < y < y_\nu + h/4;$$

(58) $$\xi_{2\nu} < x < x_\nu + h/4, \qquad y_\nu - h/4 < y < y_\nu + h/4;$$

(59) $$\xi_{1\nu} < x < \xi_{2\nu}, \qquad y_\nu - h/4 < y < \eta_{1\nu};$$

(60) $$\xi_{1\nu} < x < \xi_{2\nu}, \qquad \eta_{2\nu} < y < y_\nu + h/4;$$

which together with $R_N(x'_\nu, y'_\nu)$ make up the square S_ν. By (54),

[7]) α is obviously less than 1.

(55) and (56) we see that each of these has area at least $(\alpha^3 h/12)(h/12)=\alpha^3 h^2/144$; and so by (46) each rectangle contains a point of E_3; say P_1, P_2, P_3, P_4 respectively. It follows that the square S_ν can be divided into the rectangle $R_N(x'_\nu, y'_\nu)$ and sixteen other rectangles, say $R_{\nu 1}, R_{\nu 2}, \dots R_{\nu,16}$, each of which has at one corner a point

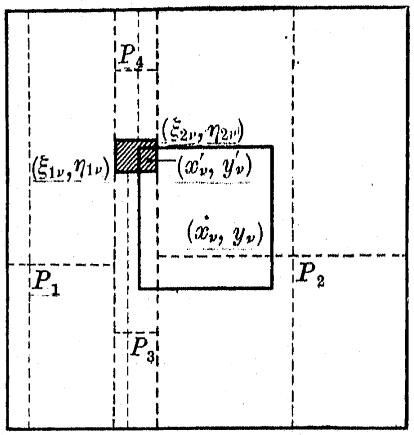

Fig. 1.

of E_3 [8]). The diameters of all these rectangles are less than h and so, by (38), less than δ; hence by (32), (35), (37) and (53) we have

$$(61)\qquad \begin{aligned} F(S_\nu) &= F[R_N(x'_\nu, y'_\nu)] + \sum_{\lambda=1}^{16} F(R_{\nu\lambda}) \\ &> (A+B)\,\eta\,\mu[R_N(x'_\nu, y'_\nu)] + A\,\eta \sum_{\lambda=1}^{16} \mu(R_{\nu\lambda}) \\ &> A\,\eta\,\mu(S_\nu) + B\,\eta\,\mu[R_N(x'_\nu, y'_\nu)] \\ &> A\,\eta\,\mu(S_\nu) + B\,\eta\,\alpha^3 h^2/144 \\ &> A\,\eta\,\mu(S_\nu) + 4\,\eta\,h^2. \end{aligned}$$

Consider finally the rectangles R'_ν; every such rectangle contains a square of class A and so certainly some point of E_3. Hence it can be divided into at most four rectangles, each of which has one corner at such a point of E_3. Their diameters are less than $h+k$ and so less than δ. From (35) we obtain by addition over these sub-rectangles

$$(62)\qquad F(R'_\nu) > A\,\eta\,\mu(R'_\nu).$$

[8]) Each of the four rectangles (57)—(60) is divided into four by lines through the points P_1, P_2, P_3, P_4 respectively.

From (49), (61) and (62) we obtain

$$F(R_0) > A\,\eta\,[\sum_{\nu=1}^{M} \mu(S_\nu) + \sum_{\nu=1}^{M'} \mu(R'_\nu)] + 4\,M\,\eta\,h^2$$

and therefore by (39) and (48)

$$(A+2)\,\eta\,\mu(R_0) > F(R_0) > A\,\eta\,\mu(R_0) + 2m\,\eta\,h^2$$

and so

$$2\eta\,h\,k > 2m\,\eta\,h^2 \geqslant (m+1)\,\eta\,h^2$$

which contradicts (41). We obtain a similar contradiction if R_0 has any other possible shape, instead of that given by $0<h\leqslant k$. Hence the theorem is proved.

Corollary 1. *If at each point of a set E we have $\underline{D}(x, y) > -\infty$; then, almost everywhere in E, $\underline{D}(x, y) = \underline{D}_m(x, y)$.*

It is only necessary to consider the completely regular system $R_n(x, y) = R(x, y; 2^{-n}, 2^{-n})$.

Corollary 2. *If the function $F(R)$ is of bounded variation, in particular if $F(R) = \iint_R f(x, y)\,dx\,dy$; then almost everywhere either*

$$\underline{D}(x, y) = -\infty$$

or

$$\underline{D}(x, y) = \underline{D}_m(x, y).$$

For in this case it is known that $D_m(x, y)$ exists almost everywhere [9]).

8. It will be observed that measurability has played no part in the preceding arguments. It is easy to show, however, that $\overline{D}(x, y)$ and $\overline{F}(x, y)$ are measurable functions [10]). This will enable us to prove another theorem similar to Theorem 1.

Let $\{R\}$ be any collection of closed rectangles, contained in some square; the cardinal number of the collection is quite arbitrary. Then we say that *the set $E = \sum R$ is measurable.* For with each point of E can be associated a sequence of squares, each containing the point and with diameters tending to zero, which lie in one of the rectangles R. By Vitali's theorem, for any positive ε there is a

[9]) S. Saks, *Théorie de l'Intégrale*, p. 49.

[10]) $\overline{F}(x, y)$ is defined as $\lim\limits_{d(S)\to 0} \dfrac{F(S)}{\mu(S)}$ for all squares S containing (x, y); we have clearly $\overline{D}_m(x, y) \leqslant \overline{F}(x, y) \leqslant \overline{D}(x, y)$. Saks, loc. cit. p. 46.

finite non-overlapping set of such squares, say $S_1, \dots S_n$, such that

$$\mu(\sum_{\nu=1}^{n} S_\nu) = \sum_{\nu=1}^{n} \mu(S_\nu) > \mu_e(E) - \varepsilon.$$

But each square S_ν is included in E; and so

$$\mu_i(E) \geqslant \mu(\sum_{\nu=1}^{n} S_\nu) > \mu_e(E) - \varepsilon.$$

Since ε is arbitrary, this shows that E is measurable. Let $E(K\,\delta)$ be the sum of all rectangles R such that $F(R) > K\mu(R)$ and $\delta(R) < \delta$. Then by the above remark $E(K, \delta)$ is measurable. The set of points at which $\overline{D}(x, y) \geqslant K$ is identical with the set $\prod_{n=1}^{\infty} E(K - 1/n, 1/n)$, and so $\overline{D}(x, y)$ is measurable [11]). Again, the set of points (x, y) such that $F(R) \leqslant K\mu(R)$ for any rectangle, containing (x, y), of diameter less than δ, is the complement of $E(K, \delta)$, and therefore is measurable.

9. ***Theorem 3.*** [12]). *If at each point of a set E we have $-\infty < \underline{D}(x, y) \leqslant \overline{F}(x, y) < \infty$; then almost everywhere in E, $\underline{D}(x, y) = \underline{D}_m(x, y) = \overline{D}_m(x, y)$.*

Suppose the theorem false; then arguing as in the proof of Theorem 1 we find $M > 0$, $\eta > 0$, an integer A, $\delta > 0$, and a set E_3 of positive outer measure such that, for (x, y) in E_3,

(63) $$-M < \underline{D}(x, y) < (A - 2)\,\eta < A\,\eta < \overline{D}_m(x, y) \leqslant \overline{F}(x, y) < M;$$

(64) $$-M\,\mu(R) \leqslant F(R)$$

whenever $d(R) < \delta$ and R contains (x, y); and finally

(65) $$F(S) \leqslant M\,\mu(S)$$

whenever $d(S) < \delta$ and the square S contains (x, y). Let H be the set of points which satisfy the last two conditions, (64) and (65), but not necessarily (63); then by the preceding arguments H is measurable. It follows that almost all points (x_0, y_0) of H, and *a fortiori* almost all points of E_3, are such that the sub-set of H lying on the line $y - y_0 = x - x_0$ is linearly measurable and has linear density 1

[11]) Substituting 'squares' for 'rectangles' we see that $\overline{F}(x, y)$ is measurable. In this case the proof is virtually identical with that given by Saks; l. cit. p. 47.

[12]) This theorem was suggested to me by a conversation with Mr. Besicovitch.

at the point (x_0, y_0) [13]). Let E_4 be the set of those points of E_3 which have this property; then $\mu_e E_4 = \mu_e E_3 > 0$. We shall show that with each point of E_4 can be associated a sequence of squares $S_n(x, y) = R(x, y; h_n, h_n)$ such that

(66) $$h_n \to 0 \quad \text{as} \quad n \to \infty;$$

(67) $$(x + h_n, y + h_n) \text{ belongs to } H;$$

(68) $$F[S_n(x, y)] > (A - 1)\,\eta\, h_n^2.$$

Thereafter the proof proceeds exactly as in § 5, by considering a fixed point (x_0, y_0) of E_4 which is also a point of outer density of E_4 in the strong sense. (H takes the place of E_3). It remains only to show how to form the sequences of squares.

10. Let (x_1, y_1) be a point of E_4. From (63) and the fact that (x_1, y_1) is a point of linear density of H, we can find a sequence of squares $S'_n = R(x_1, y_1; l_n, l_n)$ such that

(69) $$l_n < \tfrac{1}{4}\delta \quad \text{and} \quad l_n \to 0 \quad \text{as} \quad n \to \infty;$$

(70) $$F(S'_n) > A\,\eta\, l_n^2;$$

(71) $$m[H; 0 \leqslant x - x_1 = y - y_1 \leqslant 2l_n] > (2 - \varepsilon)\sqrt{2}\, l_n;$$

where m denotes *linear* Lebesgue measure [14]) and

(72) $$\varepsilon \leqslant \eta/8M < 1.$$

Consider one such square S'_n and suppose for example that $l_n > 0$. By (71) there exists h_n such that

(73) $$l_n \leqslant h_n < (1 + \varepsilon)\, l_n;$$

(74) $$(x_1 + h_n, y_1 + h_n) \text{ belongs to } H.$$

Then $S_n(x_1, y_1)$ is the square $R(x_1, y_1; h_n, h_n)$. (If $l_n < 0$ we proceed similarly; the inequalities in (73) are reversed). To show that this sequence of squares satisfies the required conditions it remains only to prove (68).

Let R_{n1}, R_{n2} be the rectangles, respectively.

(75) $$x_1 + l_n \leqslant x \leqslant x_1 + h_n; \qquad y_1 \leqslant y \leqslant y_1 + h_n;$$

[13]) Saks, loc. cit. p. 226.

[14]) It is clear that if l_n is negative we must read $m[H; 0 \geqslant x - x_1 = y - y_1 \geqslant 2l_n]$ in (71).

(76) $x_1 \leqslant x \leqslant x_1 + h_n, \qquad y_1 + l_n \leqslant y \leqslant y_1 + h_n;$

and let S_n'' be the square

(77) $x_1 + l_n \leqslant x \leqslant x_1 + h_n, \qquad y_1 + l_n \leqslant y \leqslant y_1 + h_n.$

Since $F(R)$ is additive we have

(78) $$F(S_n) = F(S_n') + F(R_{n1}) + F(R_{n2}) - F(S_n'').$$

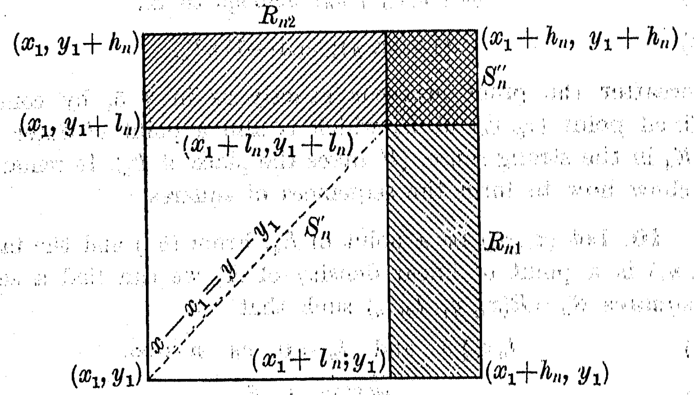

Fig. 2.

Since the point $(x_1 + h_n, y_1 + h_n)$ belongs to H and is a corner of each of the rectangles R_{n1}, R_{n2}, S_n'', we can apply (64) and (65); from (73) and (78) we derive thus

(79) $$F(S_n) \geqslant F(S_n') - M\{\mu R_{n1}) + \mu(R_{n2}) + \mu(S_n'')\}$$
$$\geqslant F(S_n') - M l_n^2 \varepsilon (2 + 3\varepsilon).$$

From (63), (70), (72), (73) and (79) we obtain finally

$$\begin{aligned} F(S_n) &> A\eta l_n^2 - 5 M l_n^2 \varepsilon \\ &> A\eta h_n^2 - 5 M l_n^2 \varepsilon - M(h_n^2 - l_n^2) \\ &> A\eta h_n^2 - 5 M l_n^2 \varepsilon - M l_n^2 (\varepsilon^2 + 2\varepsilon) \\ &> A\eta h_n^2 - 8 M l_n^2 \varepsilon \\ &> A\eta h_n^2 - 8 M h_n^2 \varepsilon \\ &> (A-1)\eta h_n^2; \end{aligned}$$

which is the result (68) required. The argument is practically the same in the case when l_n and h_n are negative.[15]).

[15]) If $F(x, y)$ is continuous we can replace the condition $\overline{F}(x, y) < \infty$, in Theorem 3, by the weaker condition $\overline{D}_m(x, y) < \infty$. The argument is unchanged, except that (65) is replaced by the condition (65 A): $F[R(x, y; h, h)] \leqslant Mh^2$ whenever $|h| < \delta$ and (x, y) is in E_3. The set of points satisfying (65A) is closed, and so H is measurable.

11. For measurable functions of one variable, there is a theorem which states that if any of the four derivates is finite in a set E, then it is the approximate derivative almost everywhere in E. An analogous theorem, *in the weak sense*, can be proved for functions of rectangles, but the corresponding theorem in the strong sense is false. Let (x_1, y_1) be a point such that $\underline{D}(x_1, y_1) > -\infty$. Given $\eta > 0$, we write $G(x_1, y_1, \eta)$ for the set of points (x, y) such that

(80) $$|\Delta F(x_1, y_1; x - x_1, y - y_1) - (x - x_1)(y - y_1)\underline{D}(x_1, y_1)| >$$
$$> \eta |(x - x_1)(y - y_1)|.$$

Theorem 4. *If at each point of a set E $\underline{D}(x, y) > -\infty$, then almost everywhere in E, for all $\eta > 0$,*

$$\lim_{d(S)\to 0} \frac{\mu_e[SEG(x, y, \eta)]}{\mu S} = 0,$$

where S is any square containing (x, y).

Suppose the theorem untrue; then by the usual arguments we can find fixed $\varepsilon > 0$, $\eta > 0$, an integer A, $\delta > 0$, such that, in a subset E_1 of E with positive outer measure we have

(81) $$\overline{\lim_{d(S)\to 0}} \frac{\mu_e[SEG(x, y, 3\eta)]}{\mu S} > 8\varepsilon;$$

(82) $$A\eta < \underline{D}(x, y) \leqslant (A+1)\eta;$$

and

(83) $$F(R) \geqslant A\eta\,\mu(R)$$

if $d(R) < \delta$ and R contains (x, y).

Let H be the set of points (x, y) satisfying the last condition, (83); then H is measurable and contains E_1. Hence $(E - H)$ has zero density at almost all points of E_1, say at the points of a set E_2. Thus we have, if (x, y) is in E_2,

(84) $$\overline{\lim_{d(S)\to 0}} \frac{\mu_e[SHG(x, y, 3\eta)]}{\mu S} > 8\varepsilon.$$

Now consider any square S containing (x, y), in which the average outer density of the set $HG(x, y, 3\eta)$, (that is, the quotient occurring in (84)), is greater than 8ε. Let S_0 be the smallest square with centre (x, y) and containing S; then $\mu S_0 \leqslant 4\mu S$ and so the average density of the set in S_0 is greater than 2ε. Divide S_0 by lines through (x,y) into four equal squares; in one of these at least the average density is greater than 2ε. It follows that at each point of E_2, either

$$(85)\qquad \overline{\lim_{h\to 0}}\ \frac{\mu_e[HG(x,y,3\eta)\,R(x,y;h,h)]}{h^2} > 2\varepsilon$$

or

$$(86)\qquad \overline{\lim_{h\to 0}}\ \frac{\mu_e[HG(x,y,3\eta)\,R(x,y;h,-h)]}{h^2} > 2\varepsilon.$$

Hence one at least of (85) and (86) must be true in a set E_3 of positive outer measure; and we may suppose it is (85), for in the other case we could consider $\Phi(x,y) = -F(x,-y)$.

Let (x_1, y_1) be a point of E_3. There is a sequence l_n, $|l_n| < \frac{1}{2}\delta$, $l_n \to 0$, such that

$$(87)\qquad \frac{\mu_e[HG(x_1,y_1,3\eta)\,R(x_1,y_1;l_n,l_n)]}{l_n^2} > 2\varepsilon.$$

Hence for each n there is a point (x_1+h_n, y_1+k_n), say, lying in $HG(x_1,y_1,3\eta)\,R(x_1,y_1;\ l_n, l_n)$ and such that

$$(88)\qquad |h_n| > \varepsilon|l_n|, \qquad |k_n| > \varepsilon|l_n|.$$

It follows that

$$(89)\qquad h_n k_n > 0, \qquad \varepsilon|h_n| < |k_n| < \varepsilon^{-1}|h_n|$$

and from (80),

$$F[R(x_1,y_1;h_n,k_n)] - \underline{D}(x_1,y_1)\,h_n k_n| > 3\eta\, h_n k_n$$

Comparing this with (82) and (83) we have

$$F[R(x_1,y_1;h_n,k_n)] > (A+3)\,\eta\, h_n k_n.$$

We note finally that (x_1+h_n, y_1+k_n) belongs to H. From this point the argument is exactly parallel to those used in proving Theorems 1 and 3; the detail is left to the reader.

Note. The theorem obtained by replacing „any square" by „any rectangle" in the enunciation of Theorem 4 is certainly false; this may be seen by considering the case of a positive function of rectangles (for example $\iint_R f(x,y)\,dx\,dy$) which has at a set of positive measure $\overline{D}(x,y) = +\infty$, $\underline{D}(x,y)$ finite.

Sur les théorèmes de séparation dans la Théorie des ensembles.

Par

Casimir Kuratowski (Warszawa).

Soit A une famille de sous-ensembles d'un espace composé d'éléments arbitraires. On dit que cette famille satisfait au *premier théorème de séparation*, lorsqu'à chaque couple d'ensembles disjoints A_1 et A_2 appartenant à A correspond un ensemble B qui, ainsi que son complémentaire, appartient à A et qui satisfait aux formules $A_1 \subset B$ et $BA_2 = 0$. Le *deuxième théorème de séparation* est satisfait, lorsqu'à chaque couple d'ensembles A_1 et A_2 appartenant à A correspond un couple d'ensembles disjoints C_1 et C_2 dont les complémentaires appartiennent à A et qui remplissent les formules $A_1 - A_2 \subset C_1$ et $A_2 - A_1 \subset C_2$.

Des exemples surtout importants de familles satisfaisant aux deux théorèmes de séparation présentent: la famille des ensembles boreliens de classe multiplicative $\beta > 0$ [1]), celle des ensembles analytiques [2]), celle des ensembles projectifs de classe $CPCA$ [3]).

[1]) c. à d. G_δ, $F_{\sigma\delta}$, $G_{\delta\sigma\delta}$ etc. Les théorèmes de séparation pour ces familles d'ensembles ont été démontrés par MM. Lusin et Sierpiński. Voir W. Sierpiński, Fund. Math. 6 (1924), p. 2 et Bulet. Soc. St. de Cluj 6 (1932), p. 461, et N. Lusin, Fund. Math. 16 (1930), pp. 57 et 60.

Il importe de remarquer que la famille des ensembles fermés *satisfait au deuxième théorème de séparation* et cependant — si l'espace est connexe — elle *ne satisfait pas au premier:* à chaque couple E_1, E_2 d'ensembles fermés correspond un couple H_1, H_2 d'ensembles ouverts disjoints tel que $E_1 - E_2 \subset H_1$ et $E_2 - E_1 \subset H_2$ (cf. par exemple ma *Topologie I*. p. 99, 2 et 6), tandis que l'espace ne contient aucun vrai sous-ensemble non vide qui soit simultanément fermé et ouvert.

[2]) Les théorèmes de séparation pour les ensembles analytiques ont été démontrés par M. Lusin sous le nom du „premier et deuxième principes". Ils jouent un rôle fondamental dans la théorie de ces ensembles.

[3]) Les théorèmes de séparation pour les ensembles $CPCA$ ont été démontrés par M. Novikoff, Fund. Math. 25 (1935), p. 459.