On the combination of topologies.

By

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1. **Introduction.** The following paper is exclusively concerned with the comparison of different topologies defined over a fixed „space" S of „points" $x, y, z, \ldots$

The basis of comparison is, I believe, already familiar to most topologists. It is that the assertions that a given topology separates points by large distances, that it admits many sets as open and closed, and that it admits few convergent sequences, are all essentially equivalent, and can be united in the single assertion that the topology in question is „inclusive" or „strong".

This directly suggests the following problems,

A) Do there exist a least inclusive topology including, and a most inclusive topology included in, every topology of a given class?

B) If such topologies exist, how can they be constructed?

And it is in the obtaining, under extremely general conditions, of consistent solutions to the above problems, that the main interest of the paper lies.

2. **The combination of distance functions.** First let us be as specific as possible; let us assume that the points of S are actually related by a numerical „distance function" $\rho : \rho(x, y)$, satisfying

$$M1:\quad \rho(x, x) = 0 \quad \text{and} \quad \rho(x, y) \geqq 0$$
$$M2:\quad \rho(x, y) + \rho(y, z) \geqq \rho(z, x).$$

Conditions $M1$—$M2$ are equivalent to the usual metric axioms without the assumption that $\rho(x, y) = 0$ implies $x = y$.

Definition 1: The distance function ρ is said to „include" a second distance function ρ_1 also defined over S [in symbols, $\rho \supset \rho_1$], if and only if $\rho(x, y) \geqq \rho_1(x, y)$ irrespective of x and y.

Clearly if $\rho \supset \rho_1$ and $\rho_1 \supset \rho_2$, then $\rho \supset \rho_2$. Moreover if $\rho \supset \rho_1$ and $\rho_1 \supset \rho$, then ρ and ρ_1 define identical distance functions. Therefore it is legitimate to use the logic and symbolism of an ordering relation.

Theorem 1: There exists a least inclusive distance function including any two given distance functions ρ_1 and ρ_2.

Set $\rho_3(x, y) = \max[\rho_1(x, y), \rho_2(x, y)]$. Clearly ρ_3 is included in any distance function including both ρ_1 and ρ_2. Therefore it is sufficient to show that it satisfies $M1$—$M2$. And this can be done by the reader.

Definition 2: The function ρ_3 just defined is called the „join" of ρ_1 and ρ_2 — in symbols, $\rho_3 = \rho_1 \cup \rho_2$.

Theorem 2: There exists a most inclusive distance function included in any two given distance functions ρ_1 and ρ_2.

Here the construction is less direct. However, set $\rho^*(x, y) = \min[\rho_1(x, y), \rho_2(x, y)]$, and

$$\rho_4(x, y) = \inf_{x = x_0, \ldots, x_r = y} \sum_{k=1}^{r} \rho^*(x_{k-1}, x_k).$$

It is then easy to show that $M1$—$M2$ are satisfied.

Moreover retracing the definition, it is clear that if any distance function ρ_5 satisfies $\rho_1 \supset \rho_5$ and $\rho_2 \supset \rho_5$, then $\rho_5(x, y) \leqq \rho_4(x, y)$ identically. This completes the proof.

Definition 3: The function ρ_4 just defined is called the „meet" of ρ_1 and ρ_2 — in symbols, $\rho_4 = \rho_1 \cap \rho_2$.

Theorem 3: $\rho \supset \rho_1$ if and only if $\rho \cap \rho_1 = \rho_1$. And we have the operational identities

$$L1:\quad \rho_1 \cap \rho_2 = \rho_2 \cap \rho_1 \quad \text{and} \quad \rho_1 \cup \rho_2 = \rho_2 \cup \rho_1.$$
$$L2:\quad \rho_1 \cap (\rho_2 \cap \rho_3) = (\rho_1 \cap \rho_2) \cap \rho_3 \quad \text{and} \quad \rho_1 \cup (\rho_2 \cup \rho_3) = (\rho_1 \cup \rho_2) \cup \rho_3.$$
$$L3:\quad \rho_1 \cap (\rho_1 \cup \rho_2) = \rho_1 \quad \text{and} \quad \rho_1 \cup (\rho_2 \cap \rho_1) = \rho_1.$$

These facts are immediate consequences of the facts stated just after Definition 1, and the definitions of meet and join; they can also be verified independently.

As a corollary of Theorem 3, we have by definition [1]).

Theorem 4: *The different distance functions defined over any abstract „space", are a lattice.*

Of course, one can obtain from any distance function ϱ, a metric geometry by the simple expedient of stating

$M3$: $\quad x = y$ if and only if $\varrho(x, y) = 0$

as a definition of equality. Reflexiveness, symmetry, and transitivity then appear as consequences of $M1$—$M2$.

But this does not detach all interest from the subclass of distance functions satisfying $M3$ ab initio. This subclass is obviously closed under the operation of join. That it is not closed under the dual operation of meet, is shown by

Example 1: There exist two metrics giving the line its proper topology, whose meet identifies all points — that is, is the degenerate distance function ϱ: $\varrho(x, y) \equiv 0$.

We shall merely sketch the construction. Let ϱ_1 be defined dyadically by induction and continuity from

(1) $\quad \varrho_1(0, 1) = 1.$

(2) $\quad$ If $x \leqq y \leqq z$, then $\varrho_1(x, z) = \varrho_1(x, y) + \varrho_1(y, z)$.

(3) If $a = 2k/2^n$, $b = (2k+1)/2^n$, $c = (2k+2)/2^n$, then $\varrho_1(a, b) = 2^{-4n}\varrho_1(a, c)$ and so [by (2)] $\varrho_1(b, c) = (1 - 2^{-4n})\varrho_1(a, c)$.

And let ϱ_2 be defined by $\varrho_2(x, y) = \varrho_1(1 - x, 1 - y)$.

Then it is easy to show that if $\varrho = \varrho_1 \cap_2$, then $\varrho(0, 1) = 0$, whence $\varrho(x, y) \equiv 0$.

3. The combination of sequential topologies. In a space S with a distance function ϱ, a given sequence $\{x_n\}$ is said to „converge" to a given point x if and only if $\lim_{n\to\infty} \varrho(x_n, x) = 0$. This suggests as a generalization

[1]) Cf. the author's *On the lattice theory of ideals*, Bull. Am. Math. Soc. 40 (1934), 613—19. O. Ore uses the term „structure" in exactly the same sense, in an article *On the foundations of abstract algebra*, Annals of Math. 36 (1935), 406—37.

Definition 4: Let S be any „space". By a „sequential topology" of S, is meant a rule τ stating when a given sequence $\{x_n\}$ of points of S „converges" to a given point x of S — in symbols, when $\tau\{x_n\} = x$.

Moreover it is obvious that if $\varrho \supset \varrho'$, and $\lim_{n\to\infty} \varrho(x_n, x) = 0$, then $\lim_{n\to\infty} \varrho'(x_n, x) = 0$. This suggests similarly.

Definition 5: A sequential topology τ on S will be said to „include" a second topology τ' [in symbols, $\tau \supset \tau'$], if and only if $\tau\{x_n\} = x$ implies $\tau'\{x_n\} = x$.

The legitimacy of Definition 5 is shown by the fact that if we regard τ simply as a relation between the class $\{S\}$ of sequences of points of S and the class S, then „inclusion" as defined is simply the inverse of the usual notion of inclusion for the corresponding relations.

Theorem 5: *There exists a least inclusive sequential topology including any given class of sequential topologies τ_k.*

This is of course the sequential topology τ defined by the rule $\tau\{x_n\} = x$ if and only if $\tau_k\{x_n\} = x$ for every k. We shall call it the „join" of the τ_k.

Corollary: *Any two sequential topologies τ_1 and τ_2 have a join $\tau = \tau_1 \cup \tau_2$.*

Theorem 6: *Each of the following properties is true of the join τ of a given set of sequential topologies τ_k, provided it is true of every τ_k:*

1^0 *If $x_n = x$ for every n, then $\tau\{x_n\} = x$.*
2^0 $\tau\{x_n\} = x$ and $n(i) \uparrow \infty$ *imply* $\tau\{x_{n(i)}\} = x$.
3^0 *If to every $n(i) \uparrow \infty$ there exists $i(j) \uparrow \infty$ such that $\tau\{x_{n(i(j))}\} = x$, then $\tau\{x_n\} = x$.*
4^0 *If $\tau\{x_n\} = x$ and $\tau\{x_n\} = y$, then $x = y$.*

The following property is true of $\tau_1 \cup \tau_2$, provided it is true of τ_1 and of τ_2.

5^0 *Let $\tau\{x_j^i\} = x$ identically in i. Then $\tau\{x_i\} = x$ if and only if $N(i)$ exists, so large that $j(i) > N(i)$ implies $\tau\{x_{j(i)}^i\} = x$.*

In each of cases 1^0—4^0, the proof consists simply of showing that the hypotheses are true of every τ_k [by definition of τ], hence [by assumption] so is the conclusion, and consequently [again by definition of τ] that the conclusion holds for τ. The proof of 5^0 is very similar.

In fact, the list 1^0—5^0 could be considerably amplified to include all of the significant properties of convergence in metric spaces.

Theorem 7: *There exists a most inclusive sequential topology included in any given class of sequential topologies τ_k.*

This is of course the sequential topology τ defined by the rule $\tau\{x_n\}=x$ if and only if $\tau_k\{x_n\}=x$ for some k. We shall call it the „meet" of the τ_k.

Theorem 8: *If either of properties 1^0—2^0 of Theorem 6 is true of every τ_k, then it is true of their meet.*

The proof is left to the reader.

It is interesting that the „L-spaces" originally[1]) defined by Fréchet are simply sequential topologies satisfying conditions 1^0—2^0. This and the fact that the definitions of join and meet are simply those of the ordinary logical product and logical sum of the corresponding relations, enable one to state

Theorem 9: *The sequential topologies on a fixed class of points are a „distributive" lattice [that is, they satisfy $\tau\cup(\tau_1\cap\tau_2)=(\tau\cup\tau_1)\cap(\tau\cap\tau_2)$], of which the „L-spaces" of Fréchet are a sublattice.*

The dual of Theorem 6 is not true of properties 3^0—5^0. However, if there exists one sequential topology satisfying a set σ of conditions for which Theorem 6 holds, and also included in all the topologies of a given class $\sum$ of sequential topologies, then the join of all sequential topologies satisfying σ exists and satisfies σ — which defines a restricted operation of „meet" relative to σ, and shows that the sequential topologies satisfying σ and including a fixed sequential topology satisfying σ, constitute a lattice.

4. The combination of closure topologies. In a space S with a distance function ϱ, a set X is called „closed" if and only if

$C0$: $\operatorname{Inf}_{x\epsilon X}\varrho(x,y)=0$ implies $y\epsilon X$.

[1]) M. Fréchet, *Les dimensions d'un ensemble abstrait*, Math. Ann. **68** (1910), 145—68.

And X is called „open" if and only if its complement X' is closed. Again, it is true that

$C1$: The sum of any two, and the product of any number of closed sets is closed.

This suggests as a convenient generalization

Definition 6: By a „closure topology" defined over a space S, is meant a system Γ of closed sets satisfying $C1$. A set X of X is called „open" under Γ if and only if its complement X' is in Γ.

Moreover it is obvious that if ϱ and ϱ' are any two distance functions defined over the space S, then $\varrho\supset\varrho'$ implies that the class of sets closed under ϱ contains as a subclass all the subsets closed under ϱ'. This suggests

Definition 7: A closure topology Γ will be said to „include" a second closure topology Γ', if and only if Γ contains all the sets of Γ'.

This terminology is obviously legitimate. Furthermore

Theorem 10: *There exists a most inclusive closure topology included in any class of closure topologies Γ_k.*

This is namely the set-theoretical intersection of the Γ_k, which can easily be shown to satisfy $C1$. We shall call it the „meet" of the Γ_k.

Theorem 11: *There exists a least inclusive closure topology including any two given closure topologies Γ_1 and Γ_2.*

This topology, the „join" $\Gamma_1\cup\Gamma_2$ of Γ_1 and Γ_2, could be defined descriptively (using Theorem 10) as the meet of all closure topologies including Γ_1 and Γ_2. But it is more satisfactory to define it constructively — the proof will be omitted[1]) — as the aggregate Γ of all subsets of S which can be expressed as the intersection $\prod_i(X_i+Y_i)$ of an unrestricted set of sums (X_i+Y_i) of a subset $X_i\epsilon\Gamma_1$ and a subset $Y_i\epsilon\Gamma_2$.

Corollary: *The closure topologies on any given set are a lattice.*

[1]) $\Gamma_1\cup\Gamma_2$ is evidently the class of all subsets which can be obtained from the subsets of Γ_1 and Γ_2 by finite sums, unrestricted products, and iteration. The equivalence of this with the aggregate Γ' is the substance of § 16 of the author's *On the combination of subalgebras*, Proc. Camb. Phil. Soc. 29 (1933), 441—64.

Theorem 12: If $\Gamma \supset \Gamma'$, and one of the following conditions is true of Γ', then it is true of Γ:

$C2$: *S is closed, and any set consisting of a single point is closed.*

$C3$: *Given $x \in S$ and $y \in S$, there exist disjoint open sets containing x and y respectively.*

The proof is trivial.

A closure topology satisfying $C2$ will be called a „Riesz" topology, and one satisfying $C3$, a „Hausdorff" topology.

Corollary: The join of any two Riesz topologies is a Riesz topology; the join of any two Hausdorff topologies is a Hausdorff topology.

In a Riesz topology, the „closure" $\bar{X}$ of any set X in S is defined as the product of all closed sets which contain X, and from this it is easy to deduce the usual [1]) definition of an abstract closure topology.

It should be remarked that from $C3$ it can be proved that the intersection of all closed sets containing a given point x contains only x, and also that an empty set is open — whence S is closed. Therefore any Hausdorff topology is a fortiori a Riesz topology.

It is also known [cf. Hausdorff, *loc. cit.*] that to be a „Hausdorff topology" is equivalent to being a Hausdorff space in the usual sense.

Finally, we have the trivial

Theorem 13: The meet of any class of Riesz topologies is a Riesz topology.

Corollary: The Riesz topologies over any space are a sublattice of the lattice of closure topologies.

Definition 7: By a „basis" of a Hausdorff topology Γ, is meant a special class of open sets, of which any set open under Γ is the sum of a subclass. A Hausdorff topology is called „separable" if and only if it has a countable basis.

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[1]) As stated, for example, in Kuratowski's *Topologie I*, p. 15, and due originally to F. Riesz. F. Hausdorff, *Mengenlehre*, Berlin 1927, pp. 227—8, gives the definition by $C1$—$C2$ and the proof of the equivalence with the usual definition.

The significance of the notion of basis derives in large part from the fact that the sets of any basis of a Hausdorff topology again satisfy Hausdorff's [but not our] axioms.

Theorem 14: The join of any two separable Hausdorff topologies is again separable.

The proof is a direct consequence of the fact that if Γ has a basis B, and Γ_1 a basis B_1, then the class of all products $X_i \cdot Y_j$ $[X_i \in B,\ Y_j \in B_1]$ is a basis for the Hausdorff topology $\Gamma \cup \Gamma_1$.

A similar relation can be proved for local separability; the questions of the preservation of regularity and normality are however more difficult.

5. Sequential vs. neighborhood topologies. I am much indebted to a letter from R. Baer, for the definitions and results of the present section.

In a metric space, it is well-known that

$D1$: A sequence $\{x_n\}$ converges to a point x if and only if every open set containing x contains almost every point of $\{x_n\}$ [1]).

$D2$: A set X is open if and only if it contains almost all the terms of any sequence converging to one of its points.

Theorem 15: Condition $D1$ and any „primitive" closure topology Γ determine a „derivative" sequential topology $\tau(\Gamma)$ satisfying 1^0—2^0. Similarly condition $D2$ and any „primitive" sequential topology τ determine a „derivative" closure topology $\Gamma(\tau)$.

The only conceivable question is as to the validity of 1^0—2^0 and of condition $C1$. And these can be verified almost by inspection [2]).

Theorem 16: If $\Gamma \supset \Gamma_1$, then $\tau(\Gamma) \supset \tau(\Gamma_1)$, and if $\tau \supset \tau_1$, then $\Gamma(\tau) \supset \Gamma(\tau_1)$.

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[1]) It is evident that $D1$ determines the same sequential topology from any basis for the open sets of a closure topology, that it does from the class of all open sets.

[2]) It is interesting that if τ satisfies 2^0, then $D2$ is equivalent to saying that

$D2'$: X is closed if and only if $\{x_n\} \subset X$ and $\tau\{x_n\} = x$ imply $x \in X$.

For if X is closed under $D2$, then it is clearly closed under $D2'$. While if X is not closed under $D2$, then $n(i) \uparrow \infty$ exists such that $\{x_{n(i)}\} \subset X$, by 2^0 $\tau\{x_{n(i)}\} = x$, and yet by assumption x *non* $\in X$, whence X is not closed under $D2'$.

The proof involves only substitution in the definitions, and use of a fortiori reasoning.

Theorem 17: *Let Γ_1 and τ_1 be any primitive topologies. Then $\Gamma(\tau(\Gamma_1)) \supset \Gamma_1$ and $\tau(\Gamma(\tau_1)) \subset \tau_1$.*

For if X is open under Γ_1, then by $D1$ X contains almost all the points of any sequence $\{x_n\}$ converging under $\tau(\Gamma_1)$ to any of its points, and hence by D2 X is open under $\Gamma(\tau(\Gamma_1))$. The proof that $\tau(\Gamma(\tau_1)) \subset \tau_1$ is similar.

Definition 8: A sequential topology τ_1 and a closure topology Γ_1 are called „equivalent" [in symbols, $\tau_1 \sim \Gamma_1$] if and only if $\tau(\Gamma_1) = \tau_1$ and $\Gamma(\tau_1) = \Gamma_1$.

Theorem 18: *Any derivative sequential topology Γ_1 is equivalent with the closure topology which it determines.*

Let $\Gamma_1 = \Gamma(\tau_1)$. Then $\tau(\Gamma_1) \subset \tau_1$ by Theorem 17, whence $\Gamma(\tau(\Gamma_1)) \subset \Gamma_1$ by Theorem 16. But by Theorem 17, $\Gamma(\tau(\Gamma_1)) \supset \Gamma_1$; hence $\Gamma(\tau(\Gamma_1)) = \Gamma_1$, proving the theorem.

It would be very interesting to know the exact class of pairs of equivalent sequential and closure topologies. Theorem 18 shows that it contains the class of derivative topologies. One can also prove

Theorem 19: *Any Hausdorff topology Γ satisfying the first countability axiom is equivalent with its derivative sequential topology.*

By Theorem 17, it is sufficient to show that $X \epsilon \Gamma(\tau(\Gamma))$ implies that $X \epsilon \Gamma$. But suppose $X non \epsilon \Gamma$ — that is, that the complement X' of X is not the sum of those sets open under Γ which it contains. Then X' will contain some point x having no neighborhood totally in X. And by the first countability axiom one can find $\{x_n\}$, having almost every point in every neighborhood of x, and lying entirely in X, completing the proof.

6. The consistency of the definitions of join and meet. Theorem 16 shows that the two extensions of the notion of inclusiveness stated in Definitions 5 and 7 are consistent for „equivalent" topologies. The present section will be devoted to studying the extent of the consistency of the extensions of the operations of join and meet.

An important fact is

Theorem 20: *If two sequential or closure topologies correspond to distance functions ϱ_1 and ϱ_2, then their join corresponds to $\varrho_1 \cup \varrho_2$. It is a corollary that the join of any two metrizible topologies is itself metrizible.*

The proof is trivial.

Example 1 of § 2 shows that the corresponding theorem for the operation of meet is false. Nevertheless

Theorem 21: *$\Gamma(\tau \cap \tau_1) = \Gamma(\tau) \cap \Gamma(\tau_1)$, irrespective of the primitive sequential topologies τ and τ_1.*

For $X \epsilon \Gamma(\tau \cap \tau_1)$ if and only if every sequence converging under τ or τ_1 to a point in the complement X' of X, has almost all its points in X'. But that is to say, if and only if X is in $\Gamma(\tau)$ and in $\Gamma(\tau_1)$ — and hence, by definition, in $\Gamma(\tau) \cap \Gamma(\tau_1)$.

Theorem 22: *$\tau(\Gamma \cup \Gamma_1) = \tau(\Gamma) \cup \tau(\Gamma_1)$, irrespective of the primitive closure topologies Γ and Γ_1.*

For $\{x_n\}$ converges to x under $\tau(\Gamma \cup \Gamma_1)$ if and only if every intersection $X \cdot X_1$ of a set X open under Γ and a set X_1 open under Γ_1 contains almost every term of $\{x_n\}$. But this is true if and only if every X and X_1 contains almost every term of $\{x_n\}$ — which is to say, if and only if $\{x_n\}$ converges to x under $\tau(\Gamma)$ and under $\tau(\Gamma_1)$.

It is a corollary that if $\tau \sim \Gamma$ and $\tau_1 \sim \Gamma_1$, then $\tau(\Gamma \cup \Gamma_1) = \tau \cup \tau_1$ and $\Gamma(\tau \cap \tau_1) = \Gamma \cap \Gamma_1$. However it should be cautioned that unless the closure topology obeys Hausdorff's first countability axiom — it is usually true that more sets are open under $\Gamma(\tau \cup \tau_1)$ than under $\Gamma \cup \Gamma_1$. And it is almost invariably true that more sequences converge under $\tau(\Gamma \cap \Gamma_1)$ than under $\tau \cap \tau_1$.

7. The combination of topologies in linear spaces. Because of their importance in applications, linear spaces deserve special treatment. In fact, it is in connection with the different definitions of convergence over a fixed class of functions that the most interesting instances of the above theory arise.

Accordingly, let L be any system of elements $x, y, z, \ldots$ in which the operations of vector addition and scalar multiplication are defined, with the usual [1]) symbolism and restrictions. A se-

[1]) For these, cf. S. Banach, *Théorie des opérations linéaires*, Warsaw 1932, p. 26.

quential topology τ defined on L will be called „continuous" if and only if it is continuous in the operations of the system — that is, (1) $\tau\{x_n\}=x$ and $\tau\{y_n\}=y$ imply $\tau\{x_n+y_n\}=x+y$, and (2) if $\{\lambda_n\}$ is any sequence of real numbers converging to λ, and $\tau\{x_n\}=x$, then $\tau\{\lambda_n x_n\}=x$. It is clear that if $x_n\equiv x$ implies $\tau\{x_n\}=x$, then $\tau\{x_n\}=x$ if and only if $\tau\{x_n-x\}=\Theta$. This will be assumed throughout § 7.

Theorem 16: *The join of any number of continuous sequential topologies is itself continuous.*

The proof is that sketched for Theorem 6.

Theorem 17: *The most inclusive continuous sequential topology τ'' included in each of two given continuous sequential topologies τ and τ' exists and is given by the rule $\tau''\{x_n\}=\Theta$ if and only if $\{y_n\}$ and $\{z_n\}$ exist, satisfying $x_n=y_n+z_n$ and $\tau\{y_n\}=\tau'\{z_n\}=\Theta$.*

The only real question is as to whether this rule defines a *continuous* sequential topology; if it does, $\tau''\subset\tau$ and $\tau''\subset\tau'$ are obvious, and is also obvious that any continuous sequential topology including τ and τ' must include τ''. Further, τ'' and homogeneity evidently define a unique sequential topology.

Now suppose $\tau''\{x_n\}=$ and $\tau''\{x'_n\}=x'$. Then we can find $\{y_n\}$, $\{z_n\}$, $\{y'_n\}$ and $\{z'_n\}$ such that $y_n+z_n=x_n-x$, $y'_n+z'_n=x'_n-x'$, $\tau\{y_n\}=\tau'\{z_n\}=\tau\{y'_n\}=\tau'\{z'_n\}=\Theta$. Hence $\tau\{y_n+y'_n\}=\tau'\{z_n+z'_n\}=\Theta$, by definition

$$\tau''\{(x_n+x'_n)-(x+x')\}=\tau''\{(y_n+y'_n)+(z_n+z'_n)\}=\Theta$$

and $\tau''\{x_n+x'_n\}=x+x'$. The proof that $\lambda_n\to\lambda$ and $\tau''\{x_n\}=x$ implies $\tau''\{\lambda_n x_n\}=\lambda x$ follows similar lines.

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On the differentiation of additive functions of rectangles.

By

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It has recently been shown that if $f(x, y)$ is a summable function of two variables, it is not in general true that

$$\operatorname*{Lim}_{d(R)\to 0}\frac{1}{\mu R}\int_R\int f\,dx\,dy=f(x_0, y_0)$$

almost everywhere, where R is any rectangle containing the point (x_0, y_0) [1] [2]. Saks raised the question whether it is possible for the upper and lower limits, as $h, k\to 0$, of

$$\frac{1}{4hk}\int_{x_0-h}^{x_0+h}\int_{y_0-k}^{y_0+k}f(x, y)\,dx\,dy$$

to be finite and not identical at the points of a set of positive measure [3]. Besicovitch has solved a slightly different but closely related problem by showing that

$$\overline{\lim_{h,k\to 0}}\frac{1}{hk}\int_{x_0}^{x_0+h}\int_{y_0}^{y_0+k}f(x, y)\,dx\,dy$$

is equal, at almost all points, either to $f(x_0, y_0)$ or to $+\infty$ [4].

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[1]) H. Busemann and W. Feller, Fund. Math. 22 (1934), 226—256; S. Saks, *ibid.*, 257—261.

[2]) Here, as throughout this paper, R denotes a closed rectangle with sides parallel to the co-ordinate axes, $d(R)$ its diameter, and μ Lebesgue plane measure.

[3]) Saks, *loc. cit.*, 260.

[4]) A. S. Besicovitch, Fund. Math. 25 (1935), 209—216.