On the Differentiability of Functions and Summability of Trigonometrical Series.

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Introduction.

§ 1. The present paper consists of five chapters, in each of which a different problem is treated. Fundamental for the whole paper is Theorem 1, which will be enunciated in a moment.

Let $f(x)$ be a function [1]) defined in the interval (a, b), and let

$$(1)\quad \Delta_k(x, h; f) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x+jh-\tfrac{1}{2}kh) \quad (h>0,\ k=1, 2, \ldots).$$

The finite limit $\lim_{h\to 0} \Delta_k(x, h; f)/h^k$, if it exists, will be called the *k-th Riemann derivative* of f, at the point x, and denoted by $D_k f(x)$. In the cases $k=1, 2$ we obtain the first symmetric derivative and the Schwarz derivative respectively.

If, for a given x, we have an equation

$$(2)\qquad f(x+t) = a_0 + a_1 t + a_2 t^2/2! + \ldots + a_k t^k/k! + o(t^k),$$

where the numbers $a_j = a_j(x)$ are independent of t, then a_k will be called the k-th de la Vallée-Poussin derivative of f at the point x, and will be denoted by $f_{(k)}(x)$. If $f_{(k)}(x)$ exists, so does $f_{(k-1)}(x)$. The existence of $f_{(k)}(x)$ implies that of $D_k f(x)$, which is then equal to $f_{(k)}(x)$.

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[1]) In what follows we shall only consider measurable functions. All the operations which will in this paper be applied to $f(x)$, lead to measurable functions. The proofs offer no real difficulties, and may be left to the reader. For f non-measurable, and $k>1$, Theorem 1 is false.

Theorem 1. *If* $\overline{\lim_{h\to 0}}|\Delta_k(x, h; f)/h^k| < \infty$ *for* $x \in E$, $|E| > 0$ [1]), *then* $f_{(k)}(x)$ *exists almost everywhere in E. Moreover, for almost every* $x \in E$,

$$f_{(j+1)}(x) = \left(\frac{d}{dx}\right)_a f_j(x), \qquad j=0, 1, \dots, k-1, \tag{3}$$

where the symbol $(d/dx)_a$ *denotes the approximate derivative* [2]).

This theorem will be established in Chapter 1. Its proof is long and is based on a series of lemmas, some of which are interesting in themselves. The most important and difficult is the first part of the theorem (the second is easy, cf. § 19). The argument uses some rather deep results from the theory of analytic functions. A special case, viz. when $k=2$ and $D_2 f(x)$ exists for $x \in E$, can be obtained by an elementary argument (cf. § 17).

§ 2. Chapter 2 generalises and completes the following theorem recently established by Kuttner[3]).

Theorem A. *If a trigonometrical series*

$$\tfrac{1}{2}a_0 + \sum_{\nu=1}^{\infty}(a_\nu \cos \nu x + b_\nu \sin \nu x) = \sum_{\nu=0}^{\infty} A_\nu(x) \tag{4}$$

converges in a set E, $|E|>0$, *and the conjugate series*

$$\sum_{\nu=1}^{\infty}(a_\nu \sin \nu x - b_\nu \cos \nu x) = \sum_{\nu=1}^{\infty}\bar{A}_\nu(x) \tag{5}$$

is summable $(C, 1)$ *in E, then the series* (5) *converges almost everywhere in E.*

Since the idea of Kuttner's proof will be repeatedly used in Chapter 2, we reproduce here Kuttner's argument, in a simplified form. It is based on the following

Lemma A. *If* (4) *converges in E,* $|E| > 0$, *then, for almost every* $\xi \in E$, *there is a sequence* $\{\mu_n\} = \{\mu_n(\xi)\}$, $1 < \mu_n < 2$, *such that*

$$\sum_{\nu=1}^{n}\bar{A}_\nu(\xi)\sin\frac{\nu\mu_n}{n} \to 0 \qquad \text{as } n\to\infty. \tag{6}$$

[1]) By $|E|$ we denote the Lebesgue measure of a set E.

[2]) For $k=1$ the theorem was established (even in a more general form) by Khintchine [1], p. 217 sqq. The case $k=2$ was enunciated by Denjoy [1], p. 1220, without proof. Cf. also Denjoy [2].

[3]) Kuttner [1].

Let $s_n(x)$ and $\bar{s}_n(x)$ denote the n-th partial sums of the series (4) and (5) respectively. Let $\mathcal{E} \subset E$, $|\mathcal{E}|>0$, be a set where $s_n(x)$ tends uniformly to its limit $s(x)$; in particular $s(x)$ is continuous over $\mathcal{E}$. Let $\xi \in \mathcal{E}$ be a point of density of $\mathcal{E}$. Since, for n large enough, the relative density of $\mathcal{E}$ in the intervals $(\xi+n^{-1}, \xi+2n^{-1})$ and $(\xi-2n^{-1}, \xi-n^{-1})$ exceeds $1/2$, there is a number $1<\mu_n<2$ such that $\xi \pm \mu_n/n \in \mathcal{E}$. Therefore

$$s_n(\xi+\mu_n/n) - s_n(\xi-\mu_n/n) \to 0,$$

which is equivalent to (6). Since $|E-\mathcal{E}|$ may be arbitrarily small, the lemma is established.

The rest of the proof is the same as in Kuttner's paper. For every sequence $\{u_\nu\}$ we write

$$\Delta_n\{u_\nu\} = \Delta_n^1\{u_\nu = u_n\} - u_{n+1}; \quad \Delta_n^k\{u_\nu\} = \Delta_n^{k-1}\{u_\nu\} - \Delta_{n+1}^{k-1}\{u_\nu\}, \tag{7}$$

for $k=2,3,\dots$ Applying Abel's transformation to the left-hand side of (6) twice, we get

$$\sum_{m=1}^{n-2}\bar{s}_m^{(1)}(\xi)\,\Delta_m^2\left(\sin\frac{\nu\mu_n}{n}\right) + \bar{s}_{n-1}^{(1)}(\xi)\,\Delta_{n-1}\left(\sin\frac{\nu\mu_n}{n}\right) + \bar{s}_n(\xi)\sin n\mu_n \to 0, \tag{8}$$

where $\bar{s}_m^{(1)} = \bar{s}_1 + \bar{s}_2 + \dots + \bar{s}_m$. Assuming, as we may, that the $(C, 1)$ sum of (5) at ξ is 0, we have $s_m^{(1)} = o(m)$. Since $|\Delta_m^k(\sin\nu\mu_n)/n| \leqslant (2/n)^k$, the absolute value of the first two terms on the left of (8) is less than

$$\left(\frac{2}{n}\right)^2\sum_{m=1}^{n-2}\cdot o(m) + \frac{2}{n}\cdot o(n) = o(1). \tag{9}$$

Since $\sin n\mu_n > \sin 1$, from (8) and (9) we obtain $\bar{s}_n(\xi) \to 0$, i. e. Theorem A [1]).

§ 3. The importance of theorem A is due to the fact that the series conjugate to Fourier series are summable $(C, 1)$ almost everywhere. For an analogous reason, the following theorem may also be of some interest.

Theorem B. *If* (4) *is summable* $(C, 1)$ *in E,* $|E|>0$, *to sum* $s(x)$, *and if* $\overline{\lim_{n\to\infty}}\, s_n(x) < \infty$ *for* $x \in E$, *then, for almost every* $x \in E$, *we have*

$$\underline{\lim}\, s_n(x) > -\infty, \qquad s(x) = \tfrac{1}{2}\{\underline{\lim}\, s_n(x) + \overline{\lim}\, s_n(x)\}. \tag{10}$$

[1]) Let $\varrho_1 < \varrho_2 < \dots \to \infty$. The argument of Theorem A shows that, if (5) is summable $(C, 1)$ for $x \in E$, and $s_n = O(\varrho_n)$ for $x \in E$, then $\bar{s}_n = O(\varrho_n)$ for almost every $x \in E$. The O may be replaced by o.

For let $\mathcal{E}\subset E$, $|\mathcal{E}|>0$, be such that $s^*(x)$ is continuous over $\mathcal{E}$, and

(11) $$s_n(x)\leqslant s^*(x)+\varepsilon_n, \qquad (x\in\mathcal{E})$$

where ε_n is a numerical sequence tending to 0; we write $s^*(x)=\overline{\lim}\, s_n(x)$, $s_*(x)=\underline{\lim}\, s_n(x)$. Let $\xi\in\mathcal{E}$ be a point of density of $\mathcal{E}$. Considering the relative density of $\mathcal{E}$ in the intervals $(\xi+\pi(1-\eta)/n, \xi+\pi(1+\eta)/n)$, $(\xi-\pi(1+\eta)/n, \xi-\pi(1-\eta)/n)$, where $\eta>0$ is fixed but arbitrarily small, we see, as in Lemma A, that there is a sequence $\delta_n\to 0$ such that $\xi\pm(\pi+\delta_n)/n\in\mathcal{E}$. Let $(\pi+\delta_n)/n=\lambda_n$, and suppose that $s(\xi)=0$. Applying Abel's transformation to the right-hand side of the equation

(12) $$\tfrac12\{s_n(\xi+\lambda_n)+s_n(\xi-\lambda_n)\}=\sum_{m=0}^{n}A_m(\xi)\cos m\lambda_n,$$

and arguing as in Lemma A, we obtain that the right-hand side of (12) is equal to $s_n(\xi)\cos n\lambda_n+o(1)$. In view of (11) and the continuity of $s^*(x)$ over $\mathcal{E}$, we have

$$s_n(\xi)\cos n\lambda_n\leqslant\tfrac12\{s^*(\xi+\lambda_n)+s^*(\xi-\lambda_n)\}+o(1)\leqslant s^*(\xi)+o(1).$$

Since $\cos n\lambda_n\to -1$, we obtain $s^*(\xi)\geqslant -s_*(\xi)$, which gives the first relation in (10). In the general case $(s(\xi)\neq 0)$, the inequality $s^*(\xi)\geqslant -s_*(\xi)$ may be written

(13) $$\tfrac12\{s^*(\xi)+s_*(\xi)\}\geqslant s(\xi).$$

If we started with the function s_*, which we now know to be finite almost everywhere in E, we should obtain, instead of (13), the converse inequality $s^*+s_*\leqslant 2s$, which proves the second equation in (10).

§ 4. The following result completes Theorems A and B.

Theorem C. *Suppose that the series* (4) *and* (5) *are summable* $(C, 1)$ *for* $x\in E$, $|E|>0$, *to sums* $s(x)$ *and* $\bar s(x)$ *respectively. Suppose also that*

(14) $$s^*(x)<+\infty \qquad \textit{for } x\in E.$$

Then the four functions $s^*(x)$, $s_*(x)$, $\bar s^*(x)$, $\bar s_*(x)$ [1]) *are finite almost everywhere in* E, *and satisfy the relations*

(15) $$\tfrac12\{s^*(x)+s_*(x)\}=s(x), \qquad \tfrac12\{\bar s^*(x)+\bar s_*(x)\}=\bar s(x),$$

(16) $$s^*(x)-s_*(x)=\bar s^*(x)-\bar s_*(x).$$

Let $\mathcal{E}\subset E$, $|\mathcal{E}|>0$, be such that the functions $s^*(x)$, $s_*(x)$, and $s(x)$ are continuous over $\mathcal{E}$ (that $s_*(x)$ is finite almost everywhere in $\mathcal{E}$ follows from Theorem B), and let $s_n(x)$ satisfy the inequalities

(17) $$s_*(x)-\varepsilon_n\leqslant s_n(x)\leqslant s^*(x)+\varepsilon_n, \qquad x\in\mathcal{E},$$

where ε_n is a numerical sequence tending to 0. Let $\xi\in\mathcal{E}$ be a point of density of $\mathcal{E}$. There is a sequence $\delta_n\to 0$ such that, if $\lambda_n=(3\pi+\delta_n)/2n$, the points $\xi\pm\lambda_n$ belong to $\mathcal{E}$. Then

(18) $$\tfrac12\{s_n(\xi+\lambda_n)-s_n(\xi-\lambda_n)\}=-\sum_{\nu=1}^{n}\bar A_\nu(\xi)\sin\nu\lambda_n.$$

[1]) The last two functions denote $\overline{\lim}\,\bar s_n(x)$ and $\underline{\lim}\,\bar s_n(x)$ respectively.

Supposing that $\bar s(\xi)=0$, and applying Abel's transformation twice to the right-hand side of (18), we obtain

(19) $$\tfrac12\{s_n(\xi+\lambda_n)-s_n(\xi-\lambda_n)\}=-\bar s_n(\xi)\sin\lambda_n n+o(1).$$

Since $n\lambda_n\to 3\pi/2$, (17) and (19) give

(20) $$\bar s^*(\xi)\leqslant\tfrac12\{s^*(\xi)-s_*(\xi)\}.$$

This inequality shows that $\bar s^*(\xi)<\infty$ almost everywhere in E, so that the equations (15) follow from Theorem B.

Now let $\delta_n'\to 0$ be a sequence such $\xi\pm\lambda_n'\in\mathcal{E}$, where $\lambda_n'=(\pi+\delta_n')/2n$. Replacing λ_n by λ_n' in (19), we obtain

(21) $$\bar s_*(\xi)\geqslant -\tfrac12\{s^*(\xi)-s_*(\xi)\},$$

and, subtracting (21) from (20),

(22) $$\bar s^*(\xi)-\bar s_*(\xi)\leqslant s^*(\xi)-s_*(\xi).$$

In view of the symmetric rôle of the series (4) and (5), the inequality opposite to (22) is also true, and this gives (16). From (19) follows also that *if, for almost every* $x\in E$, *the series* (4) *is strongly summable* $(C, 1)$, *with index* $q>0$, *the same may be said of the series* (5).

§ 5. We recall the following definitions. A series $u_0+u_1+\dots$ is said to be summable (C, k), $k>-1$, to sum s, if

$$\sigma_n^k=S_n^k/C_n^k\to s,$$

where the numbers S_n^k and C_n^k are defined by the equations

(23) $$\sum_{n=0}^{\infty}S_n^k z^n=(1-z)^{-k-1}\sum_{0}^{\infty}u_n z^n, \qquad \sum_{0}^{\infty}C_n^k z^n=(1-z)^{-k-1}.$$

If $u_0+u_1+\dots$ is summable (C, k) to s, then it is also summable A to s, i. e. $u_0+u_1 r+u_2 r^2+\dots\to s$ as $r\to 1$.

The expressions S_n^k and σ_n^k formed for the series (4) and (5), will be denoted by $S_n^k(x)$, $\sigma_n^k(x)$, $\bar S_n^k(x)$, $\bar\sigma_n^k(x)$ respectively. The expressions $\overline{\lim\limits_{n\to\infty}}\,\sigma_n^k(x)$ and $\underline{\lim\limits_{n\to\infty}}\,\sigma_n^k(x)$ will be denoted by $S^k(x)$ and $s^k(x)$ respectively. Similarly we define the functions $\bar S^k(x)$ and $\bar s^k(x)$ for the series (5). If the series (4) and (5) are summable by *some* method of summation, the respective sums will be denoted by $s(x)$ and $\bar s(x)$.

The chief results of Chapter 2 may be formulated as follows.

Theorem 2. *If* (4) *is summable* A *for* $x\in E$, $|E|>0$, *to* $s(x)$, *and if*

(24) $$S^k(x)<\infty \qquad (x\in E,\ k>-1)$$

then (i) *the series* (5) *is summable* A *almost everywhere in* E, (ii) *the* *four functions* $S^k(x)$, $s^k(x)$, $\bar{S}^k(x)$, $\bar{s}^k(x)$ *are finite almost everywhere in* E, *and satisfy the relations*

$$\tfrac{1}{2}\{s^k(x)+S^k(x)\}=s(x), \qquad \tfrac{1}{2}\{\bar{s}^k(x)+\bar{S}^k(x)\}=\bar{s}(x), \tag{25}$$

$$S^k(x)-s^k(x)=\bar{S}^k(x)-\bar{s}^k(x). \tag{26}$$

Remark. In view of the well-known theorem [1]) that the series summable A and finite (C, k) is summable $(C, k+\varepsilon)$ $(\varepsilon>0, k>-1)$, we see that, under the hypotheses of Theorem 2, the series (4) and (5) are summable $(C, k+\varepsilon)$ almost everywhere in E. From Theorem 2 it follows that, if $S^k(x)=+\infty$ and (4) is summable A, for $x \in E$, then $s^k(x)=-\infty$ almost everywhere in E.

Theorem 3. *If the series* (4) *satisfies the condition*

$$-\infty<s^k(x)\leqslant S^k(x)<+\infty \qquad (x \in E,\ |E|>0,\ k>-1) \tag{27}$$

then the series (4) *and* (5) *are summable* $(C, k+\varepsilon)$, $\varepsilon>0$, *almost everywhere in* E, *and satisfy the relations* (25) *and* (26).

The most interesting special case of Theorem 3 is

(α) *If the series* (4) *is summable* (C, k) *almost everywhere in* E, *so is the series* (5).

It must be added that proposition (α) is not entirely new. In a letter [2]) to one of us, Prof. Kolmogoroff mentions that the case $k=0$ of Theorem α was established by A. Plessner. The proof has not so far been published.

The chief idea of the proof of Theorem 3 consists in combining Theorem 1 with Kuttner's argument.

§ 6. Chapter 3 gives a generalisation of the following theorem due to Lusin.

Theorem D. *If the series* (4) *converges in a set* E, $|E|>0$, *to sum* $s(x)$, *the sum* $S(x)$ *of the formally integrated series*

$$\frac{a_0 x}{2}+\sum_{n=1}^{\infty}\frac{a_n \sin nx-b_n \cos nx}{n}=\frac{a_0 x}{2}+\sum_{n=1}^{\infty}\frac{\bar{A}_n(x)}{n} \tag{28}$$

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[1]) Andersen [1].

[2]) dated August 1934. Theorem α, in the case $k=0$, was first enunciated as probable by Privaloff [1], p. 94.

is approximately differentiable at almost every $x \in E$, *and*

$$(d/dx)_a\, S(x)=s(x)\ ^{1)}.$$

We shall prove the following

Theorem 4. *If* (4) *is summable* (C, k), $k\geqslant 0$, *for* $x \in E$, $|E|>0$, *then, for almost every* $x \in E$, *the series* (28) *is summable* $(C, k-1)$ *to sum* $S(x)$, *and* $s(x)=(d/dx)_a\, S(x)$ [2]).

We add that in the case $-1<k<0$ the function $S(x)$ has an ordinary derivative $S'(x)=s(x)$ almost everywhere in E. For, by a theorem of Hardy and Littlewood [2], for every $x \in E$ we have $[S(x+h)-S(x-h)]/2h \to s(x)$ as $h\to 0$. Applying Khintchine's theorem mentioned in § 1, we obtain the desired result. It is plain that, if $-1<k<0$, the function $S(x)$ exists everywhere and is continuous.

§ 7. In Chapter 4 we prove an extension of the following result due to Titshmarch:

Theorem E. *Let* $f(x)$, $0\leqslant x\leqslant 2\pi$, *be an L-integrable function, and* $F(x)$ *its integral. Then the function* $\bar{F}(x)$ *conjugate to* $F(x)$ *is approximately differentiable almost everywhere* [3]).

More generally, we shall prove the following

Theorem 5. *Let* $F(x)$, $0\leqslant x\leqslant 2\pi$, *be an L-integrable function,* $\Phi(x)$ *the integral of* $F(x)$, *and* $\bar{\Phi}(x)$ *the function conjugate to* $\Phi(x)$. *If* $F_{(k)}(x)$ *exists in a set* E, $|E|>0$, *then* $\bar{\Phi}_{(k+1)}(x)$ *exists almost everywhere in* E.

To deduce Theorem E from Theorem 5, we observe that, if $f(x)$ is integrable (even in the special Denjoy sense), $F'(x)$ exists almost everywhere. Hence, by Theorem 5, $\bar{\Phi}_{(2)}(x)$ exists almost everywhere. In view of the second part of Theorem 1, we have the existence of

$$(d/dx)_a\,(d/dx)\,\bar{\Phi}(x)=(d/dx)_a\,\bar{F}(x).$$

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[1]) Lusin [1]; it seems that the proof has never been published. We add that at *every* point x where (4) converges, $S(x)$ possesses an approximate *symmetric* derivative equal to $s(x)$; cf. Rajchman and Zygmund [2].

[2]) Following Hardy and Littlewood, we say that a series $u_0+u_1+\dots$ is summable $(C, -1)$, if it converges and $u_n=o(1/n)$. Summability $(C, -1)$ implies summability (C, k) for any $k > -1$; Hardy and Littlewood [1].

[3]) Titchmarsh [1].

§ 8. In Chapter 5 we establish a number of results about Borel derivatives. We say that a function $f(x)$, $a\leqslant x\leqslant b$, has, at a point x, a right-hand Borel derivate $B_r f(x)$, if the integral

$$(29)\qquad I(h,x)=I(h,x;f)=\int_0^h \frac{f(x+u)-f(x)}{u}\,du=\lim_{\varepsilon\to 0}\int_\varepsilon^h$$

exists, and $I(h,x)/h\to B_r f(x)$ for $h\to+0$. Similarly, replacing, in (29) $\{f(x+u)-f(x)\}/u$ by $\{f(x)-f(x-u)\}/u$ or $\{f(x+u)-f(x-u)\}/2u$, we define the left-hand and the symmetric Borel derivatives, which will be denoted by $B_l f(x)$ and $B_s f(x)$ respectively. If $B_r f(x)=B_l f(x)$, the common value will be denoted by $B f(x)$. To fix ideas, we assume that $f(x)$ is integrable L, but no difficulties would arise if we supposed e. g. that $f(x)$ is integrable in the special Denjoy sense. The derivative $B_s f(x)$ deserves some attention, since the integral $\int_0^h\{f(x+u)-f(x-u)\}\frac{du}{2u}$ is known to converge for almost every x. This is not necessarily true (even if f is continuous) for the expression $I(h,x;f)$. It is plain that, if $Bf(x)$ exists, so does $B_s f(x)$; the converse is not true.

It has recently been shown by Sargent that, *if*

$$\overline{\lim_{h\to+0}}\,|I(h,x;f)/h|<\infty \qquad \textit{for}\quad x\in E,$$

then $Bf(x)$ and $(d/dx)_a f(x)$ exist and are equal almost everywhere in E [1]). In Chapter 5, we prove, besides this result, the following

Theorem 6. *Let $\varphi(h,x;f)$ denote the expression analogous to* (29), *the integrand $\{f(x+u)-f(x)\}/u$ being replaced by $\{f(x+u)-f(x-u)\}/2u$. If $\varphi(h,x;f)=O(h)$ for every $x\in E$, and $h\to 0$, then $Bf(x)$ exists almost everywhere in E.*

The rest of Chapter 5 is devoted to a theory of the trigonometrical integral. It is well-known that the sum of an everywhere convergent trigonometrical series need not be integrable L, and so the series itself need not be a Fourier-Lebesgue series.

The problem arises to find such a definition T of an integral („trigonometrical integral“) that the sums of everywhere convergent trigonometrical series should be integrable T, and the series themselves be the Fourier series of their sums. A solution of this problem was found by Denjoy, who published a series of notes on the subject [1]). Here we shall give another definition, based on the idea of majorants and minorants. The arguments are independent of those of the previous chapters.

Let s be a trigonometrical series convergent everywhere to sum $f(x)$, $0\leqslant x\leqslant 2\pi$. In the first place, we observe that s integrated term by term need not converge everywhere. The sum $F(x)$ of the integrated series exists, in general, only almost everywhere. It is natural to expect that the integrated series represents, in some sense, the „primitive function“ of f, and that the integral of f over an interval (α,β) is defined as $F(\beta)-F(\alpha)$ [2]). Thence we infer that (i) whatever reasonable definition of T-integration we propound, the integral may be neither continuous nor even defined at some points, (ii) the function f may be non-integrable over some intervals (α,β). There will however exist a point-set $P\subset(0,2\pi)$, $|P|=2\pi$, such that, if $\alpha\in P$, $\beta\in P$, then the integral of f over (α,β) will be defined as $F(\beta)-F(\alpha)$. For details we refer the reader to Chapter 5.

[1]) Sargent [1]. If $Bf(x)$ exists for $x\in E$, then $(d/dx)_a f(x)$ exists almost everywhere in E, and is equal to $Bf(x)$; this result is due to Khintchine [1] and Sargent [1].

CHAPTER I.

§ 9. Besides the expressions $\Delta_k(x,u;f)$ considered in § 1, we shall also introduce differences $\tilde\Delta_k(x,u;f)$ defined by the equations

$$(30)\qquad \tilde\Delta_1(x,u;f)=f(x+u)-f(x),\quad \tilde\Delta_k(x,u;f)=\tilde\Delta_{k-1}(x,2u;f)-2^{k-1}\tilde\Delta_{k-1}(x,u;f),$$

for $k=2,3,\ldots$ Wherever it will cause no misunderstanding, we shall write $\tilde\Delta_k(x,u)$, or even $\tilde\Delta_k(u)$ simply. Similarly for the differences $\Delta_k(x,u;f)$. To grasp the meaning of the second equation in (30), observe that $\tilde\Delta_k(u)/u^k=2^{k-1}\{\tilde\Delta_{k-1}(2u)/(2u)^{k-1}-\tilde\Delta_{k-1}(u)/u^{k-1}\}/u$. It is plain that

$$(31)\qquad \tilde\Delta_k(u)=\alpha_k f(x+2^{k-1}u)+\alpha_{k-1}f(x+2^{k-2}u)+\ldots+\alpha_1 f(x+u)+\alpha_0 f(x),$$

where α_j depends on j and k only ($\alpha_k=1$). We shall show that the α's satisfy the conditions

$$(32)\qquad \sum_{i=0}^k \alpha_i=0,\qquad \sum_{i=1}^k 2^{is}\alpha_i=0\qquad (s=1,2,\ldots,k-1).$$

[1]) see e. g. Denjoy [3], [4]; the notes mostly contain statements of results. A detailed account has not yet been published. The case of power series is treated by Verblunsky [1].

[2]) These remarks are due to Lusin [1].

In the first place we observe that, *if the k-th de la Vallée-Poussin derivative $f_{(k)}(x)$ of a function $f(x)$ exists, so does* $\lim\limits_{u\to 0}\tilde{\Delta}_k(x,u;f)/u^k$. For $k=1$ this is obvious. Supposing that the result is true for $k-1$, and that $f_{(k)}(x)$ exists, we obtain

$$(33)\qquad \tilde{\Delta}_k(u)=\tilde{\Delta}_{k-1}(2u)-2^{k-1}\tilde{\Delta}_{k-1}(u)=o(u^{k-1}).$$

Substituting the equation $f(x+u)=f(x)+uf_{(1)}(x)+\ldots+u^kf_{(k)}(x)/k!+o(u^k)$ into (31), we have

$$(34)\qquad \tilde{\Delta}_k(x,u;f)=\sum_{s=0}^{k}\alpha_s f(x)+\sum_{s=1}^{k}\frac{1}{s!}\sum_{i=0}^{k-1}2^{is}\alpha_{i+1}u^s f_{(s)}(x)+o(u^k).$$

From this and (33) we deduce (32). The second equation (32) is certainly false for $s=k$, for the Vandermonde determinant $|2^{is}|$ does not vanish, and $\alpha_k=1\neq 0$. Hence from (34) follows the existence of a number λ_k depending on k only, and such that

$$(35)\qquad \lambda_k\lim_{u\to 0}\tilde{\Delta}_k(x,u;f)/u^k=f_{(k)}(x),$$

provided the right-hand side exists. Thus, the expression $\tilde{D}_kf(x)$, equal to the left-hand side of (35), may be considered as a generalisation of the k-th derivative.

Suppose that there exist $k+1$ numbers $\bar{\alpha}_j$, $0\leqslant j\leqslant k$, $(\alpha_k=1)$ such that, for every function $f(x)$ for which $f_{(k)}(x)$ exists, we have

$$\bar{\alpha}_kf(x+2^{k-1}u)+\bar{\alpha}_{k-1}f(x+2^{k-2}u)+\ldots+\bar{\alpha}_1f(x+u)+\bar{\alpha}_0f(x)=O(u^k).$$

Then $\bar{\alpha}_i=\alpha_i$ for $i=0,1,\ldots,k$. This follows from the fact that the $\bar{\alpha}$'s satisfy the equations (32).

§ 10. ***Lemma 1.*** *If the derivatives $\tilde{D}_kf(x)$ and $f_{(k-1)}(x)$ exist at a point x, so does $f_{(k)}(x)$.*

Suppose that $f_{(i)}(x)=0$ for $i=0,1,\ldots,k-1$, and that $\tilde{D}_kf(x)=0$. If $\tilde{\Delta}_k(u)=o(u^k)$, i. e. $|\tilde{\Delta}_k(u)|\leqslant\varepsilon|u|^k$ for $|u|<\delta=\delta(\varepsilon)$, then

$$|\tilde{\Delta}_{k-1}(2u)-2^{k-1}\tilde{\Delta}_{k-1}(u)|\leqslant\varepsilon|u|^k,\qquad \left|\tilde{\Delta}_{k-1}(u)-2^{k-1}\tilde{\Delta}_{k-1}\left(\frac{u}{2}\right)\right|\leqslant\varepsilon\left|\frac{u}{2}\right|^k,\ldots$$
$$\ldots,\left|\tilde{\Delta}_{k-1}\left(\frac{u}{2^{n-1}}\right)-2^{k-1}\tilde{\Delta}_{k-1}\left(\frac{u}{2^n}\right)\right|\leqslant\varepsilon\left|\frac{u}{2^n}\right|^k.$$

Multiplying these inequalities by 1, 2^{k-1}, $2^{(k-1)2}$,... $2^{(k-1)n}$ respectively, we obtain, by addition, $|\tilde{\Delta}_{k-1}(2u)-2^{(k-1)(n+1)}\tilde{\Delta}_{k-1}(u/2^n)|\leqslant\leqslant 2\varepsilon|u^k|$.

Hence, making $n\to\infty$, and observing that $f_{(k-1)}(x)=\tilde{D}_{k-1}f(x)=0$, we see that

$$|\tilde{\Delta}_{k-1}(2u)|\leqslant 2\varepsilon|u|^k,\quad\text{i. e.}\quad \tilde{\Delta}_{k-1}(u)=o(u^k).$$

From this we similarly deduce $\tilde{\Delta}_{k-2}(u)=o(u^k)$, $\tilde{\Delta}_{k-3}(u)=o(u^k)$,... and finally $\tilde{\Delta}_1(u)=o(u^k)$, i. e. $f(x+u)=o(u^k)$. This proves the lemma.

§ 11. Let us put

$$\Delta_1^*(x,u;f)=f(x+u)-f(x),\quad \Delta_k^*(x,u;f)=\Delta_{k-1}^*(x+u,u;f)-\Delta_{k-1}^*(x,u;f)$$

for $k=2,3,\ldots$ In other words,

$$\Delta_k^*(u)=\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}f(x+iu),\qquad \Delta_k^*(x,u)=\Delta_k\left(x+\frac{k}{2}u,u\right).$$

It is not difficult to see that, if *$f_{(k)}(x)$ exists, so does the limit $D_k^*f(x)=$ $=\lim\Delta_k^*(x,u)/u^k$ (for $u\to 0$), and is equal to $f_{(k)}(x)$*. More generally, *for any fixed λ, the existence of $f_{(k)}(x)$ implies*

$$(36)\qquad \lim_{u\to 0}\frac{\Delta_k(x+\lambda u,u)}{u^k}=f_{(k)}(x).$$

§ 12. We shall now express $\tilde{\Delta}_k$ by means of Δ_k.

Lemma 2. *There are constants $C_0, C_1,\ldots C_{2^{k-1}-k}$ such that*

$$(37)\qquad \tilde{\Delta}_k(x,u)=\sum_{i=0}^{2^{k-1}-k}C_i\Delta_k(x+\tfrac{1}{2}ku+iu,u).$$

In view of the remark concerning (36), and the last remark of § 9, it is sufficient to prove the existence of constants $C_0, C_1,\ldots$, such that the sum on the right of (37) is equal to

$$(38)\qquad \bar{\alpha}_kf(x+2^{k-1}u)+\bar{\alpha}_{k-1}f(x+2^{k-2}u)+\ldots+\bar{\alpha}_0f(x),$$

with $\bar{\alpha}_k=1$.

The expression $\Delta_k(x+\frac{1}{2}ku+iu,u)$ is a linear form L_i in the variables $\xi_p=f(x+up)$, $p=0,1,\ldots,2^{k-1}$. We write $L_i=L_i'+L_i''$, where L_i' is the group of terms containing the variables ξ_p with $p=0,1,2,2^2,\ldots,2^{k-1}$. Hence L_i'' contains $2^{k-1}-k$ variables only. The number of the forms L_i'' is $2^{k-1}-k+1>2^{k-1}-k$. Therefore we can find constants $C_0, C_1,\ldots$ not all of which are equal to 0, and such that $C_0L_0''+C_1L_1''+\ldots$, $\ldots+C_{2^{k-1}-k}L_{2^{k-1}-k}''\equiv 0$. Hence $C_0L_0+C_1L_1+\ldots=C_0L_0'+C_1L_1'+\ldots$ is of the form (38). A moment's consideration shows that $\bar{\alpha}_k\neq 0$ (if $\bar{\alpha}_k=0$ then, as (32) shows, $\bar{\alpha}_{k-1}=\ldots=\bar{\alpha}_1=\alpha_0=0$, which is easily seen to be impossible), so that we may suppose that $\bar{\alpha}_k=1$.

§ 13. ***Lemma 3.*** *If the Riemann derivative* $D_k f(x)$ *exists for every* $x \in E$, $|E|>0$, *then, at almost every point of* E *the derivative* $\widetilde{D}_k f(x)$ *exists and is equal to* $D_k f(x)$.

Let $\mathcal{E} \subset E$, $|\mathcal{E}|>0$, be a set over which $D_k f(x)$ is continuous, and the relation

$$\lim_{u\to 0} \Delta_k(x,u)/u^k = D_k f(x)$$

is satisfied uniformly. The lemma will have been established when we have shown that, if $x \in \mathcal{E}$ is a point of density of $\mathcal{E}$, then $\widetilde{D}_k f(x)$ exists and is equal to $D_k f(x)$. Let $x=0$ be such a point, and suppose that $D_k f(0)=0$.

If the points $\frac{1}{2}ku+iu$, $i=0,1,..,2^{k-1}-k$, belong to $\mathcal{E}$, then, in view of (37), we shall have

$$(39)\qquad \widetilde{\Delta}_k(0,u)=o(u^k).$$

To fix ideas we assume that $u>0$. First of all we shall show that there is a set P of numbers u, for which $u=0$ is a point of density, and along which we have (39). For, since $u=0$ is a point of density of $\mathcal{E}$, the set $\mathcal{E}_i$ of points u such that $u(\frac{1}{2}k+i)$ belongs to $\mathcal{E}$ has also this property. The same may be said of the product

$$P=\mathcal{E}_0\,\mathcal{E}_1 \ldots \mathcal{E}_{2^{k-1}-k},$$

along which we have (39).

Next we shall show that, if $u>0$ is small enough, we can find a number v, $u \leqslant v \leqslant 2u$, such that all the points

$$u+i\frac{v-u}{k}\quad (i=1,2,\ldots,k), \qquad 2^j\frac{u+v}{2}\quad (j=0,1,\ldots,k-1)$$

belong to P. For let $\varphi(t)$ be the characteristic function of P. The set $P_i(u)$ of numbers v belonging to $(u,2u)$, and such that the point $u+(v-u)\,i/k$ belongs to P, is of measure

$$(40)\qquad \int_u^{2u} \varphi\left(u+i\frac{v-u}{k}\right)dv=\frac{k}{i}\int_u^{u(1+i/k)}\varphi(w)dw.$$

Similarly the set $P^j(u)$ of numbers v, $u\leqslant v\leqslant 2u$, such that $2^j(u+v)/2 \in P$, is of measure

$$(41)\qquad \int_u^{2u}\varphi\left(2^j\frac{u+v}{2}\right)dv=\frac{1}{2^{j-1}}\int_{2\cdot 2^{j-1}u}^{3\cdot 2^{j-1}u}\varphi(w)dw.$$

Since $u=0$ is a point of density of P, the integrals (40) and (41) are asymptotically equal to u, i. e. to the length of the interval $(u,2u)$. Thence we deduce that, if u is small enough, all the sets $P_i(u)$ and $P^j(u)$ $(i=1,2,..,k,\ j=0,1,..,k-1)$ have a point v in common, and our assertion is established.

Since $2^j(u+v)/2 \in P$, we have

$$(42_j)\qquad \Delta_k\left(2^j\frac{u+v}{2},\ 2^j\frac{v-u}{k}\right)=\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}f\left(2^j u+2^j i\frac{v-u}{k}\right)=o(u^k)$$

for $j=0,1,..,k-1$. Multiplying (42_j) by α_j, we obtain, by addition,

$$(43)\qquad \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\widetilde{\Delta}_k\left(u+i\frac{v-u}{k}\right)=o(u^k).$$

Observing that $u+i(v-u)/k \in P$, and so $\widetilde{\Delta}_k(u+i(v-u)/k)=o(u^k)$ for $i=1,2,..,k$, we deduce from (43) that $\widetilde{\Delta}_k(u)=o(u^k)$, and the lemma is established.

Lemma 3₁. *Lemma 3 holds if we replace in it* $D_k f(x)$ *by* $D_k^* f(x)=\lim \Delta_k^*(x,u,f)/u^k$.

The proof is similar to that of Lemma 3.

§ 14. ***Lemma 4₁.*** *If* $f_{(k-1)}(x)$ *and* $D_k f(x)$ *exist almost everywhere in* E, *so does* $f_{(k)}(x)$.

Lemma 4₂. *If* $\overline{\lim\limits_{u\to 0}}|\Delta_k(x,u)/u^k|<\infty$, $x\in E$, *then* $\overline{\lim\limits_{u\to 0}}|\widetilde{\Delta}_k(x,u)/u^k|<\infty$ *almost everywhere in* E.

Lemma 4₃. *If* $f_{(k-1)}(x)$ *exists and* $\overline{\lim\limits_{u\to 0}}|\Delta_k(x,u)/u^k|<\infty$ *for* $x\in E$, *then*

$$(44)\qquad f(x+t)=f(x)+tf_{(1)}(x)+\ldots+\frac{t^{k-1}}{(k-1)!}f_{(k-1)}(x)+\frac{\omega(x,t)}{k!}t^k,$$

for almost every $x\in E$, *where* $\omega(x,t)=O(1)$ *for* $t\to 0$.

Lemma 4_1 is a corollary of Lemmas 1 and 3. Lemmas 4_2 and 4_3 follow by exactly the same argument as Lemmas 3 and 1 respectively.

§ 15. ***Lemma 5.*** *If* $\overline{\lim\limits_{u\to 0}}|\widetilde{\Delta}_k(x,u;f)|<\infty$ *for* $x\in E$, *the function* f *is bounded in a neighbourhood of almost every point* $x\in E$.

Let E_m be the set of points $x\in E$ such that

$$(45)\qquad |f(x)|<m,\quad \text{and}\quad |\widetilde{\Delta}_k(x,u)|<m\quad \text{for}\quad 0<|u|<1/m.$$

Since $|E-E_m|\to 0$, it is sufficient to show that f is bounded in a neighbourhood of every point $x\in E_m$ which is a point of density of E_m. Let $x=0$ be such a point. We fix m, and write $\mathcal{E}$ for E_m. By an argument similar to that of § 13, we show, that, if $u>0$ is small enough (we consider e. g. the right-hand neighbourhood of the point 0), there is a number $v\in E$, $0<v<u$, such that all the points $v+2^j h$, where $h=(u-v)/2^{k-1}$, $j=0,1,2,\dots,k-2$, belong to $\mathcal{E}$ also. Hence, if $h<1/m$,

$$(46)\qquad |\tilde{\Delta}_k(v,h)|\leqslant m,\quad |f(v)|\leqslant m,\quad |f(v+2^j h)|\leqslant m\quad (j=0,1,\dots,k-2).$$

From (45) and (46) we obtain $|f(u)|\leqslant m\,(|\alpha_0|+\dots+|\alpha_{k-1}|)$, and the lemma follows.

§ 16. ***Lemma 6.*** *If* $\overline{\lim}_{u\to 0}|\tilde{\Delta}_k(x,u)/u^k|<\infty$ *for* $x\in E$, *then* $\overline{\lim}_{u\to 0}|\tilde{\Delta}_{k-1}(x,u)/u^{k-1}|<\infty$ *for almost every* $x\in E$.

In view of Lemma 5, we may suppose that f is bounded. Under this assumption we shall prove that $\overline{\lim}_{u\to 0}|\tilde{\Delta}_{k-1}(x,u)/u^{k-1}|<\infty$ for *every* $x\in E$. If $x\in E$, and $\delta>0$ is small enough, we have $|\tilde{\Delta}_k(x,u)|\leqslant M|u|^k$ for $|u|\leqslant\delta$. Thence, arguing as in § 10, we obtain

$$|\tilde{\Delta}_{k-1}(x,u)-2^{n(k-1)}\tilde{\Delta}_{k-1}(x,u/2^n)|\leqslant 2M|u|^k,$$
$$|2^{n(k-1)}u^{-(k-1)}\tilde{\Delta}_{k-1}(x,u/2^n)|\leqslant 2M\delta+|\tilde{\Delta}_{k-1}(x,u)/u^{k-1}|.$$

If $\frac{1}{2}\delta\leqslant |u|\leqslant\delta$, the right-hand side of the last inequality is less than a constant M'. Hence, writing $\xi=u/2^n$, we have

$$|\xi^{-(k-1)}\tilde{\Delta}_{k-1}(x,\xi)|\leqslant M'\quad\text{for}\quad \delta/2^{n+1}\leqslant|\xi|\leqslant\delta/2^n,\qquad n=0,1,\dots,$$

and the lemma is established.

Repeating the above argument, we obtain $|\tilde{\Delta}_1(x,u)|\leqslant\overline{M}_x|u|$ for $|u|\leqslant\delta_x$. Therefore, using the well-known Denjoy theorem[1]), we see that, *under the hypothesis of Lemma 6, $f'(x)$ exists almost everywhere in E.*

§ 17. It is now easy to prove the special case of Theorem 1 enunciated at the end of § 1. If $D_2 f(x)$ exists for $x\in E$, then, in view of Lemma 3 and the last remark of § 16 the derivative $f'(x)$ exists almost everywhere in E. It remains to apply Lemma 4_1.

[1]) If, for every $x\in E$, the ratio $(f(x+u)-f(x))/u$ is finite when $u\to 0$, then $f'(x)$ exists almost everywhere in E; Denjoy [4]. Cf. e. g. Saks [1], p. 168 sqq.

§ 18. ***Lemma 7.*** *If, for every $x\in E$, we have (44), where $\omega(x,t)=O(1)$ for $t\to 0$, then the derivative $f_{(k)}(x)$ exists almost everywhere in E, and satisfies the equation $f_{(k)}(x)=(d/dx)_a f_{(k-1)}(x)$ for almost every $x\in E$.*

Assuming that this lemma is established, we shall be able to prove Theorem 1. Lemma 4_2 shows that the condition

$$\overline{\lim}_{u\to 0}|\Delta_k(x,u)/u^k|<\infty,\qquad x\in E,$$

may be replaced by $\overline{\lim}_{u\to 0}|\tilde{\Delta}_k(x,u)/u^k|<\infty$. In this new form, the first part of Theorem 1 is certainly true for $k=1$. Let us suppose it to be true for $k-1$. If $\overline{\lim}_{u\to 0}|\tilde{\Delta}_k(x,u)/u^k|<\infty$ for $x\in E$, then $\overline{\lim}_{u\to 0}|\tilde{\Delta}_{k-1}(x,u)/u^{k-1}|<\infty$ almost everywhere in E (Lemma 6), and, by the case already established, $f_{(k-1)}(x)$ exists almost everywhere in E. By Lemma 4_3, $f(x+t)$ is of the form (44), and it is sufficient to apply Lemma 7.

§ 19. It remains to prove Lemma 7. We begin with its second part. Suppose that, for every $x\in E$, we have an equation

$$(47)\qquad f(x+t)=f(x)+f_{(1)}(x)t+\dots+\frac{f_{(k)}(x)}{k!}t^k+\varepsilon_t(x)\,t^k$$

where $\varepsilon_t\to 0$ with t. Let $\mathcal{E}$, $|\mathcal{E}|>0$, be a subset of E, over which $f_{(k)}(x)$ is continuous and $\varepsilon_t\to 0$ uniformly. Let $x\in\mathcal{E}$ be a point of density of $\mathcal{E}$, and suppose for simplicity that $f_{(k)}(x)=0$. In view of (47), we have

$$(48)\qquad \Delta^*_{k-1}(x+u,u)-\Delta^*_{k-1}(x,u)=o(u^k),$$

since the difference on the left is equal to $\Delta^*_k(x,u)$. Since $x\in\mathcal{E}\subset E$, (47) gives $\Delta^*_{k-1}(x,u)=u^{k-1}f_{(k-1)}(x)+o(u^k)$. If $x+u\in\mathcal{E}$, then also $\Delta^*_{k-1}(x+u,u)=u^{k-1}f_{(k-1)}(x+u)+o(u^k)$. Hence, from (48),

$$(48a)\qquad f_{(k-1)}(x+u)-f_{(k-1)}(x)=o(u),$$

i. e. $f_{(k)}(x)=(d/dx)_a f_{(k-1)}(x)$ for $x\in\mathcal{E}$. This establishes the second part of Lemma 7 and, by induction, the equations (3),

§ 20. We shall now prove a result containing that of § 19 as a special case.

Lemma 8. *If we have* (47) *for* $x \in E$, *and if* $0<s<k$, *then, for almost every* $x \in E$,

$$\lim_{u\to 0} {}_a \Delta_s^*(x, u, f_{(k-s)})/u^s = f_{(k)}(x), \tag{49}$$

where $\lim_a$ *denotes the approximate limit, i. e. the limit along a set of points having* $u=0$ *as a point of density.*

Suppose, for example, that $s=2$. Let $\mathcal{E}$ and x have the same meaning as in § 19.

Starting with the relation

$$\Delta_k^*(x+u, u) = \Delta_{k-1}^*(x+2u, u) - \Delta_{k-1}^*(x+u, u) = o(u^k),$$

analogous to (48), we obtain

$$f_{(k-1)}(x+2u) - f_{(k-1)}(x+u) = o(u) \qquad (x+u, x+2u \in \mathcal{E}) \tag{50}$$

In view of (48a) and (50),

$$\Delta_2^*(x, u; f_{(k-1)}) = o(u). \tag{51}$$

Now observe that, on account of (47),

$$\Delta_{(k-2)}^*(x, u; f) = u^{(k-2)} f_{(k-2)}(x) + \lambda\, u^{(k-1)} f_{(k-1)}(x) + o(u^k), \tag{52}$$

where λ is a certain constant. If $x+u$ and $x+2u$ belong to $\mathcal{E}$, then, remembering that $f_{(k)}$ is continuous over $\mathcal{E}$, and $f_{(k)}(x)=0$, we may replace x by $x+u$ or $x+2u$ in (52). Hence, since $\Delta_k^*(x, u; f) = \Delta_2^* \Delta_{k-2}^*(x, u; f)$, we obtain from (52)

$$\Delta_k^*(x, u; f) \quad (=o(u^k)) = u^{k-2} \Delta_2^*(x, u; f_{(k-2)}) + \lambda\, u^{k-1} \Delta_2^*(x, u; f_{(k-1)}) + o(u^k)$$

when $x+u \in \mathcal{E}$, $x+2u \in \mathcal{E}$. From this and (51) we deduce that $\Delta_2(x, u; f_{(k-2)}) = o(u^2)$, for $x+u$ and $x+2u$ belonging to $\mathcal{E}$, i. e. the equation (49) for $s=2$. In the same way we obtain, by turns, the cases $s=3, 4, .., k-1$. The last of them may be stated as follows

$$\lim_{u\to 0} {}_a \Delta_{k-1}^*(x, u, f_{(1)})/u^{k-1} = f_{(k)}(x) \quad \textit{almost everywhere in } E. \tag{53}$$

§ 21. ***Lemma 9.*** *If* (α) $\lim_{u\to 0} {}_a \Delta_k^*(x, u, f)/u^k$ *exists and* (β) $\overline{\lim_{u\to 0}} |\Delta_k^*(x, u, f)/u^k| < \infty$, *for every* $x \in E$, *then* $\lim_{u\to 0} \Delta_k^*(x, u, f)/u^k$ *exists almost everywhere in* E.

In the first place, there is a set $\mathcal{E} \subset E$, with $|E - \mathcal{E}|$ as small as we please, and such that condition (β) is satisfied uniformly over $\mathcal{E}$, i. e.

$$|\Delta_k^*(x, u)| \leqslant M|u|^k \quad \text{for} \quad x \in \mathcal{E}, \quad |u| < \delta. \tag{54}$$

Let $x \in \mathcal{E}$ be a point of density of $\mathcal{E}$, and suppose for simplicity that $x=0$. Let G be a subset of $\mathcal{E}$ having $x=0$ as a point of density and such that

$$\lim_{u\to 0,\, u\in G} \Delta_k^*(0, u; f)/u^k = \lim{}_a \Delta_k^*(0, u, f)/u^k. \tag{55}$$

Suppose, moreover, that the last expression is equal to 0. Let $0<\varepsilon<1$ be a small but fixed number, and let $u>0$ (for $u<0$ the argument is similar). Arguing as in § 13, we show that, if u is small enough, we can find a v, $(1-\varepsilon)\, u \leqslant v \leqslant u$, such that the points

$$v + \frac{u-v}{k} j \quad (j=0, 1, 2, ..., k-1), \quad \text{and} \quad iv \quad (i=1, ..., k) \tag{56}$$

all belong to G. Denoting by $\theta, \theta_1, \theta_2, ...$ numbers not exceeding 1 in absolute value, we have the relations

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(0) = 0$$

$$\Delta_k^*\left(v, \frac{u-v}{k}\right) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f\left(v + \frac{u-v}{k} j\right) = \theta_1 M \left(\frac{u-v}{k}\right)^k$$

. . . .

$$\Delta_k^*\left(iv, i\frac{v-u}{k}\right) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f\left(iv + i\frac{u-v}{k} j\right) = \theta_i M \left(\frac{u-v}{k} i\right)^k$$

for $i=1, 2, .. k$. Multiplying the i-th equation by $(-1)^{k-i}\binom{k}{i}$, we obtain by addition

$$\sum_{j=0}^{k}{}' (-1)^{k-j} \binom{k}{j} \Delta_k^*\left(0, v + \frac{u-v}{k} j\right) = \theta M k \varepsilon^k u^k. \tag{57}$$

Observing that the points $v + j\,(u-v)/k$ belong to G for $j=0, 1, .., k-1$, we deduce from (57) that

$$\Delta_k^*(0, u) = o(u^k) + \theta M k \varepsilon^k u^k, \quad \text{i. e.} \quad \Delta_k^*(0, u) = o(u^k),$$

since ε may be arbitrarily small. This proves the lemma.

§ 22. ***Lemma 10.*** *Let $f(x)$ be a function integrable L over (a, b), and let $F(x)$ be an integral of f, and suppose that $F_{(k+1)}(x)$ exists for $x \in E$. If, for $x \in E$, f satisfies the equation* (44), *where $\omega(x, t) = O(1)$ for $t \to 0$, then $f_{(k)}(x)$ exists almost everywhere in E.*

In view of Lemma 8 (cf. (53)), $\lim_{u \to 0} {}_a \Delta_k^*(x, u; f)/u^k$ exists almost everywhere in E. The equation (44) implies $\overline{\lim}_{u \to 0} |\Delta_k^*(x, u; f)/u^k| < \infty$ for $x \in E$. Hence, by Lemma 9, $\lim_{u \to 0} \Delta_k^*(x, u; f)/u^k$ exists for almost every $x \in E$, and it is sufficient to apply Lemmas 3_1 and 1. We add that the condition concerning the integrability L of f is not very restrictive, since, in view of Lemma 5, f is bounded in the neighbourhood of almost every $x \in E$.

§ 23. ***Lemma 11.*** *Let $f(x)$ be integrable L over (a, b) and let $F^p(x)$ denote a p-th integral of f. Suppose that $F^p_{(p+k)}(x)$ exists for $x \in E$, and that f satisfies* (44) *with $\omega(x, t) = O(1)$ for $x \in E$ and $t \to 0$. Then $f_{(k)}(x)$ exists almost everywhere in E.*

This follows by repeated application of Lemma 10, if we observe that (44) implies

$$F^j(x+t) = F^j(x) + F^j_{(1)}(x)t + \ldots + \frac{F^j_{(j+k-1)}(x)}{(j+k-1)!} t^{j+k-1} + \frac{\omega_j(x, t)}{(j+k)!} t^{k+j},$$

$j = 1, 2, \ldots$, where $\omega_j(x, t) = O(1)$ for $x \in E$, $t \to 0$.

§ 24. We require now a number of lemmas from the theory of Fourier series and analytic functions. By $\mathfrak{S}[f]$ we shall denote the Fourier series of a function $f(x)$, $0 \leqslant x \leqslant 2\pi$, and by $\mathfrak{S}^{(k)}[f]$ the series $\mathfrak{S}[f]$ differentiated term by term k times.

Lemma 12. *If f is integrable over $(0, 2\pi)$, and $f_{(k)}(x)$ exists, then $\mathfrak{S}^{(k)}[f]$ is summable $(C, k+2)$, to $f_{(k)}(x)$, at the point x. If f satisfies* (44) *with $\omega(x, t) = O(1)$ for $t \to 0$, then $\mathfrak{S}^{(k)}[f]$ is finite $(C, k+2)$ at x.*

This lemma is known [1]).

Let

$$P(r, \theta) = \tfrac{1}{2} \cdot \frac{1 - r^2}{1 - 2r\cos\theta + r^2}, \qquad u(r, \theta) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) P(r, t - \theta)\, dt.$$

[1]) See e. g. Zygmund [1], p. 2[illegible].

Lemma 13. *If f satisfies the conditions of the second part of Lemma 12, then $d^k u(r, \theta)/d\theta^k$ is bounded in a region*

$$(58) \qquad |\theta - x| < \alpha(1 - r) \qquad (\beta \leqslant r < 1),$$

where α is an arbitrary but fixed constant, and $\beta = \beta(\alpha)$.

It is difficult to say whether this result has ever been explicitly stated, and so we prefer to give a proof here [1]). We may assume that $x = 0$, $f(0) = f_{(1)}(0) = \ldots = f_{(k-1)}(0) = 0$. Then, all we have to show is that

$$(59) \qquad \int_{-\pi}^{\pi} \left| t^k \frac{d^k}{dt^k} P(r, \theta - t) \right| dt$$

is bounded for $|\theta| < \alpha(1 - r)$ and $r \to 1$. Let $y = y(x)$, $x = x(u)$. Then

$$(60) \qquad \frac{d^k}{du^k} y(x(u)) = \sum_{s=1}^{k} \Phi_s \frac{d^s}{dx^s} y,$$

where $\Phi_1, \Phi_2, \ldots$ are polynomials in dx/du, $d^2x/du^2, \ldots$ On the other hand, a simple induction proves the formula

$$(61) \qquad \frac{d^k}{dx^k} \frac{1}{a^2 + x^2} = \sum_{2q - p = k+2} \frac{A_i x^p}{(a^2 + x^2)^q} \qquad (q \leqslant k+1)$$

where the A's are independent of x and a. Putting $x = 2\sqrt{r} \sin \frac{1}{2}\theta$, $y = 1/[(1-r)^2 + x^2]$, we obtain from (60) and (61)

$$\left| \frac{d^k}{d\theta^k} P(r, \theta) \right| \leqslant M \cdot (1 - r) \sum_{2q - p \leqslant k+2} \frac{|2\sqrt{r} \sin \frac{1}{2}\theta|^p}{[(1-r)^2 + 4r \sin^2 \frac{1}{2}\theta]^q},$$

M denoting a constant. In order to show, therefore, that (59) is bounded, it is sufficient to prove that $(1 - r) \int_{-\pi}^{\pi} |t|^k \frac{|\theta - t|^p}{[(1-r)^2 + (\theta - t)^2]^q} dt$ is bounded for $|\theta| < \alpha(1 - r)$, $r \to 1$, $2q - p \leqslant k + 2$, which presents no difficulty.

[1]) A similar argument gives the following result, which, for the general theory of trigonometrical series, is slightly more interesting than Lemma 13: *If $f_{(k)}(x)$ exists, then $d^k u(r, \theta)/d\theta^k \to f_{(k)}(x)$, provided that $(r, \theta) \to (1, x)$ throughout the region* (58).

§ **25.** *Lemma 14. If a series $u_0+u_1+\dots$ is summable (C, k), $k=0,1,\dots$, to sum s, then it is also summable by the method of Riemann of order $k+2$, to s, that is*

$$\lim_{h\to 0}\left[u_0+\sum_{n=1}^{\infty}u_n\left(\frac{\sin nh}{nh}\right)^{k+2}\right]=s. \tag{62}$$

If $u_0+u_1+\dots$ is finite (C, k), the expression in square brackets in (62) is bounded for $h\to 0$ [1]).

We recall the meaning of the Riemann method of summation. The series $u_0+u_1+\dots$ may be considered as the series $u_0+u_1\cos x+ +u_2\cos 2x+..$ for $x=0$. Let $F(x)$ denote the sum of the last series integrated term by term $k+2$ times. Then it is easy to verify that the existence of the Riemann derivative $D_{k+2}F(0)$ is equivalent to the relation (62), with $s=D_{k+2}F(0)$.

Lemma 15. If a series $u_0+u_1+\dots$ is finite (C, α) and summable A, then it is also summable (C, γ) for any $\gamma>\alpha$.

This is Andersen's well-known theorem [2])

Lemma 16. Let $u(r,\theta)$ be a function harmonic for $r<1$, and let E, $|E|>0$, be a point-set on the circumference $r=1$. Suppose that, for every $x\in E$, there is a region of the form (58) in which $u(r,\theta)$ is bounded. Then, for almost every $x\in E$, the function $u(r,\theta)$ tends to a finite limit as $(r,\theta)\to(1,x)$ along any non-tangential path [3]).

§ **26.** We are now in a position to prove the first part of Lemma 7 (the second was established in § 19). Without loss of generality we may suppose that $f(x)$, $0\leqslant x\leqslant 2\pi$, is bounded (cf. Lemma 5). If $u(r,\theta)$ is the Poisson integral of f, then, by Lemma 13, $d^k u(r,\theta)/d\theta^k$ is bounded in every region of the form (58) $(x\in E,\ \alpha=\alpha(x),\ \beta=\beta(x))$.

Hence, on account of Lemma 16, $\mathfrak{S}^{(k)}[f]$ is summable A almost everywhere in E. Being finite $(C, k+2)$ for $x\in E$ (Lemma 12), $\mathfrak{S}^{(k)}[f]$ must be summable $(C, k+3)$ almost everywhere in E (Lemma 15). Let $F(x)$ be the function obtained by integrating $\mathfrak{S}^{(k)}[f]$ term by

[1]) Kogbetliantz [1]; see also Rajchman and Zygmund [1]. A different proof will be found e. g. in Zygmund [1], p. 304.

[2]) Andersen [1]. The proof will also be found in Zygmund [1] p. 262.

[3]) This result is due to Privaloff [1]. A more general result will be found in Plessner [2].

term $k+5$ times. $F(x)$ is the fifth integral of $f(x)$, and, by Lemma 14, $D_{k+5}F(x)$ exists for almost every $x\in E$. Moreover, integrating (44) five times, we see that $F_{(k+4)}(x)$ exists almost everywhere in E. Applying Lemma 4_1, we deduce the existence of $F_{(k+5)}(x)$, and so also that of $f_{(k)}(x)$ (Lemma 11) for almost every $x\in E$.

This completes the proof of Lemma 7, and so also of Theorem 1.

§ **27.** We will complete Theorem 1 by a number of remarks.

Theorem 7. Let λ be a fixed real number. The conclusion of Theorem 1 holds, if

$$\overline{\lim_{u\to+0}}\left|\frac{\Delta_k(x+\lambda u, u; f)}{u^k}\right|<\infty \quad \text{for} \quad x\in E. \tag{63}$$

It is sufficient to prove that (63) implies $\Delta_k(x, u; f)=O(u^k)$ for $u\to 0$, and almost every $x\in E$. We break up the proof into three lemmas.

Lemma 17. Under the hypothesis (63), $\Delta_k^*(x, u; f)=O(u^k)$ for $u\to+0$, and almost every $x\in E$.

Lemma 18. If $\Delta_k^*(x, u; f)=O(u^k)$ for $u\to+0$, $x\in E$, then $\Delta_k^*(x, u; f)=O(|u|^k)$ for $u\to\pm 0$, and almost every $x\in E$.

Lemma 19. If $\Delta_k^*(x, u; f)=O(|u|^k)$ for $u\to 0$, $x\in E$, then $\Delta_k(x, u; f)=O(|u|^k)$ for $u\to 0$, and almost every $x\in E$.

To fix ideas, we assume that $\lambda\geqslant 0$; for $\lambda<0$ the proof is similar. Let $\mathcal{E}\subset E$, $|\mathcal{E}|>0$, be any set such that $|\Delta_k(x+\lambda u, u)|\leqslant Mu^k$ for $x\in\mathcal{E}$, $0\leqslant u\leqslant\delta$. Let $x_0\in\mathcal{E}$ be a point of density of $\mathcal{E}$. To simplify the notation we suppose that $x_0=0$. Let $u>0$ be any point of $\mathcal{E}$. We define a number ω by the equation

$$u+\lambda\omega-\tfrac{1}{2}k\omega=0, \quad \text{i. e.,} \quad \omega=u/(\tfrac{1}{2}k-\lambda). \tag{64}$$

We may suppose that $\lambda\neq\frac{1}{2}k$, for otherwise Theorem 7 is obvious. We consider separately the cases (i) $\lambda<\frac{1}{2}k$, (ii) $\lambda>\frac{1}{2}k$. In case (i), ω is positive with u, and we have

$$\Delta_k^*(0,\omega)=\Delta_k(u+\lambda\omega,\omega), \quad \textit{so that} \quad |\Delta_k^*(0,\omega)|\leqslant M\omega^k, \tag{65}$$

if u is sufficiently small $(0\leqslant\omega\leqslant\delta)$. Let $u\in\mathcal{E}$ tend to 0. Since $\omega=u/(\frac{1}{2}k-\lambda)$, the set Ω of the numbers ω for which (65) is true, has 0 as a point of density.

We shall now show that $\Delta_k^*(0,u)=O(u^k)$ for $u\to+0$. Let $u>0$ be an arbitrary number, but so small that the mean density of the sets $\mathcal{E}$ and Ω in $(0,u)$ is sufficiently near to 1. Arguing as in § 13, we can find a point v, $\frac{1}{2}u<v\leqslant u$ such that

(a) $v, 2v, 3v, \ldots, kv \in \mathcal{E}$; (b) $u-k\omega, u-(k-1)\omega, \ldots, u-\omega \in \Omega$,

where ω is given by the equation

$$(66) \qquad v+\lambda\omega+\frac{k}{2}\omega=u, \quad \text{i. e.} \quad \omega=(u-v)/(\lambda+\tfrac{1}{2}k).$$

We then have the following relations

$$(67) \begin{cases} \Delta_k(v+\lambda\omega,\omega)=\sum_{j=0}^{k}(-1)^j\binom{k}{j}f(u-j\omega)=\theta_1 M\omega^k \\ \dots\dots\dots\dots\dots \\ \Delta_k(sv+s\lambda\omega,s\omega)=\sum_{j=0}^{k}(-1)^j\binom{k}{j}f(su-sj\omega)=\theta_s M(s\omega)^k, \end{cases}$$

$s=1,2,\ldots,k$, where the θ's do not exceed 1 in absolute value. Assuming, as we may, that $f(0)=0$, and multiplying the s-th equation (67) by $(-1)^{k-s}\binom{k}{s}$, we obtain by addition

$$(68) \qquad \sum_{j=0}^{k}(-1)^j\binom{k}{j}\Delta_k^*(0,u-j\omega)=\theta Mk^k\omega^k=\theta' Mk^k u^k.$$

In view of conditions (b), $|\Delta_k^*(0,u-j\omega)|\leqslant M(u-j\omega)^k\leqslant Mu^k$ if $j>0$. Hence, by (68), $\Delta_k^*(0,u)=O(u^k)$, and this proves Lemma 17 for $\frac{1}{2}k>\lambda$.

In the case $\frac{1}{2}k<\lambda$ the argument is similar. In the first part of the proof we take $u<0$, so that ω defined by (64) is positive. The second part, and in particular the formulae (67) and (68) are unaffected.

§ 28. Passing to the proof of Lemma 18, let $\mathcal{E}\subset E$, $|\mathcal{E}|>0$, be such that $|\Delta_k^*(x,u)|\leqslant Mu^k$ for $x\in\mathcal{E}$, $0\leqslant u\leqslant\delta$. Let $x_0\in\mathcal{E}$, be a point of density of $\mathcal{E}$; we assume that $x_0=0$. Let $u<0$, $u\in\mathcal{E}$; then $w=-u/k>0$, and $|\Delta_k^*(0,-w)|=|\Delta_k^*(u,w)|$. Thence as before we deduce that $|\Delta_k^*(0,-w)|\leqslant Mw^k$ for w sufficiently small and belonging to a set Ω whose right-hand density at 0 is 1.

Let $u<0$ be small enough. We can then find a v, $2u\leqslant v\leqslant u$, such that

(a) $v, 2v, \ldots, kv \in \mathcal{E}$, (b) $u-j(u-v)/k\in\Omega$, $j=1,2,\ldots,k$.

From this and the inequalities

$$\Delta_k^*(v,(u-v)/k)=\theta_1 Mu^k,\ldots, \quad \Delta_k^*(sv,s(u-v)/k)=\theta_s Mu^k,\ldots,$$

we obtain $|\Delta_k^*(0,u)|=O(|u|^k)$, and the lemma is established.

§ 29. In the case of Lemma 19, $\mathcal{E}\subset E$, $|\mathcal{E}|>0$, is again a set such that $|\Delta_k^*(x,u;f)|\leqslant M|u|^k$ for $x\in\mathcal{E}$, $0\leqslant|u|\leqslant\delta$; $x_0\in\mathcal{E}$ is a point of density of $\mathcal{E}$. Suppose that $x_0=0$. Let $u\in\mathcal{E}$, $u<0$. If ω is defined by the condition $u+\frac{1}{2}k\omega=0$, then $\Delta_k(0,\omega)=\Delta_k^*(u,\omega)$. Arguing as before, we see that $|\Delta_k(0,\omega)|\leqslant M\omega^k$ for $\omega\in\Omega$, Ω denoting a set having 0 as a point of right-hand density (since $|\Delta_k(0,\omega)|=|\Delta_k(0,-\omega)|$, we may restrict ourselves to the case $\omega>0$).

If $u>0$ is small enough, but otherwise arbitrary, we can find a v, $0<v<u$, such that

(a) $iv\in\mathcal{E}$, $i=-\frac{1}{2}k, -\frac{1}{2}k+1, \ldots, \frac{1}{2}k$; (b) $v+j\omega\in\Omega$, $j=0,1,\ldots,k-1$,

where $\omega=(u-v)/k$. From the equations

$$\Delta_k^*(iv,i\omega)=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}f(iv+ji\omega)=O(u^k), \quad i=-\tfrac{1}{2}k,\ldots,\tfrac{1}{2}k,$$

we deduce

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\Delta_k(0,v+j\omega)=O(u^k),$$

which, in view of (b) gives $\Delta_k(0,u)=O(u^k)$. This completes the proof of Theorem 7.

Theorem 8. *If, for $x\in E$, we have (44), where $\omega(x,t)=O(1)$ when $t\to+0$, then $f_{(k)}(x)$ exists almost everywhere in E.*

For the hypothesis of Theorem 8 implies that $\Delta_k^*(x,u;f)=O(u^k)$ for $x\in E$, $u\to+0$, and it is sufficient to apply Theorem 7 with $\lambda=\frac{1}{2}k$.

§ 30. ***Theorem 9.*** *For every even $k=2,4,\ldots$ there is a function f and a set E, such that (a) $f_{(k-1)}(x)$ exists for almost every $x\in E$ (b) for $x\in E$ we have (44) with $\omega(x,t)\geqslant 0$, and yet (c) $f_{(k)}(x)$ does not exist in E.*

This result shows that the Denjoy theorem (which asserts that, if $f(x+t)=f(x)+t\omega(x,t)$, where $\omega(x,t)\geqslant -M_x$ for $|t|<\delta_x$, $x\in E$, then $f'(x)$ exists almost everywhere in E) cannot be extended to higher derivatives.

We shall construct a function $f(x)$, $0\leqslant x\leqslant 1$, such that (a) $f_{(k-1)}(x)$ exists almost everywhere in E, (b) (44) is true with $\omega(x,t)\geqslant 0$, $x\in E$, $|E|>0$, (c) $f_{(k)}(x)$ exists for no $x\in E$. The set E will be defined as follows. Let $R_0=(0,1)$, and let $\delta_1, \delta_2, ..$ be a positive sequence tending to 0. From R_0 we remove a concentric open interval S_1, such that $|S_1|/|R_0|=\delta_1$. The remaining set R_1 consists of two intervals, from each of which we remove the interior of a concentric interval; the removed intervals are of equal length, and their sum S_2 satisfies the equation $|S_2|/|R_1|=\delta_2$. With $R_2=R_1-S_1$, which consists of four intervals, we proceed similarly, and so on. We write $S_1+S_2+...=S$, $R_0R_1R_2...=E$. If $\delta_1+\delta_2+...<\infty$, then $|E|>0$.

Let (a,b) be an interval contiguous to R_n. We define $f(x)$ over (a,b) by the conditions; $f(a)=f(b)=0$, $f(\frac{1}{2}a+\frac{1}{2}b)=2^{-n(k-\frac{1}{2})}$, $f(x)$ is linear in $(a, \frac{1}{2}a+\frac{1}{2}b)$ and $(\frac{1}{2}a+\frac{1}{2}b, b)$. If we put $f(x)=0$ in E, f is defined over $(0,1)$.

If $f_{(1)}(x)$, $f_{(2)}(x)$,... $f_{(k-1)}(x)$ exist almost everywhere in E, they must vanish there (observe that, by Theorem 1, $f_{(j)}(x)=(d/dx)_a f_{(j-1)}(x)$) so that (44) reduces to $f(x+t)=\omega(x,t)\,t^k/k!$. Since k is even, we have $\omega(x,t)\geqslant 0$, so that (b) is satisfied. Since the length of no interval of R_n exceeds 2^{-n}, it is easy to see that condition (c) is also satisfied.

If the sequence δ_n decreases sufficiently rapidly, then the length i_n of any of the intervals constituting R_n is $>C2^{-n}$, with $C>0$ independent of n. Let us consider a system R_n^* of intervals, of equal length, concentric with R_n and such that $|R_n^*|/|R_n|=1-\varepsilon_n$, where $\{\varepsilon_n\}$ will be defined presently. Let $R_0^*R_1^*R_2^*...=E^*$. If $x\in E^*$, then

$$(*)\qquad \max_t\left|\frac{f(x+t)}{t^{k-1}}\right|\leqslant \max_n(\varepsilon_n i_n)^{-k-1}2^{-n(k-1/2)}\leqslant\max_n C^{-(k-1)}\varepsilon_n^{-(k-1)}2^{-n/2}.$$

If the sequence $\varepsilon_n^{-(k-1)}\leqslant 2^{n/2}$, then the last term of (*) is bounded, and so, by Lemma 7, $f_{(k-1)}(x)$ exists almost everywhere in E^*. Since $|E-E^*|$ is small with $\varepsilon_1+\varepsilon_2+...$, condition (a) is satisfied.

If k is odd, the above construction gives a function $f(x)$ which, besides (a) and (c) satisfies the condition (b'): $t\,\omega(x,t)\geqslant 0$.

CHAPTER II.

§ 31. We begin the proof of Theorem 2 with the following

Lemma 20. *If for $x\in E$, $|E|>0$, the series (4) is summable $(C,k+1)$, to sum $s(x)$, and if*

$$(69)\qquad S^k(x)<+\infty \qquad (x\in E,\ k>-1)$$

then, for almost every $x\in E$, we have $s^k(x)>-\infty$, and

$$(70)\qquad \tfrac{1}{2}\{S^k(x)+s^k(x)\}=s(x).$$

Suppose first that $k\geqslant 0$ is an integer. Let $\mathcal{E}\subset E$, $|\mathcal{E}|>0$, be such that $s(x)$ and $S^k(x)$ are continuous over E, and

$$(71)\qquad \sigma_n^k(x)<S^k(x)+\varepsilon_n \qquad (x\in\mathcal{E}),$$

with ε_n independent of x, and tending to 0. Let $\xi\in\mathcal{E}$ be a point of density of $\mathcal{E}$, and $\{\lambda_n\}$ a sequence such that $n\lambda_n\to\pi$, $\xi\pm\lambda_n\in\mathcal{E}$. From (71) it follows

$$(72)\qquad \tfrac{1}{2}\{\sigma^k(\xi+\lambda_n)+\sigma^k(\xi-\lambda_n)\}<\tfrac{1}{2}\{S^k(\xi+\lambda_n)+S^k(\xi-\lambda_n)\}+\varepsilon_n.$$

The left-hand side of (72) is equal to

$$(73)\qquad \frac{1}{C_n^k}\sum_{\nu=0}^{n}C_{n-\nu}^k A_\nu(\xi)\cos\nu\lambda_n=$$

$$=\frac{1}{C_n^k}\sum_{\nu=0}^{n-k-2}S_\nu^{k+1}(\xi)\,\Delta_\nu^{k+2}(C_{n-v}^k\cos v\lambda_n)+\frac{1}{C_n^k}\sum_{j=0}^{k+1}S_{n-j}^j(\xi)\,\Delta_{n-j}^j(C_{n-v}^k\cos v\lambda_n)=$$
$$=H_n+K_n,$$

say (it must be remembered that the differences are taken with respect to the suffix v). Using the formulae

$$(74)\qquad \Delta_\nu^{k+2}\{c_v d_v\}=\sum_{j=0}^{k+2}\binom{k+2}{j}\Delta_\nu^j\{c_v\}\,\Delta_{\nu+j}^{k+2-j}\{d_v\},$$

$$(75)\qquad \Delta_\nu^j\{C_{n-v}^k\}=C_{n-\nu}^{k-j},\qquad |\Delta_{\nu+j}^{k+2-j}\{\cos v\lambda_n\}|\leqslant\lambda_n^{k+2-j},$$

$$(76)\qquad C_{n-\nu}^{k-j}=O(n-\nu)^{k-j}=O(n^{k-j}),\ j=0,1,...,k;\ C_m^{-1}=C_m^{-2}=0 \text{ for } m>1,$$

we obtain, for $0\leqslant\nu\leqslant n-k-2$,

$$(77)\qquad |\Delta_\nu^{k+2}(C_{n-v}^k\cos v\lambda_n)|\leqslant\sum_{j=0}^{k}O(n^{k-j})\,\lambda_n^{k+2-j}=O(n^{-2}).$$

Hence, assuming that $s(\xi)=0$, i. e. $S_\nu^{k+1}(\xi)=o(\nu^{k+1})$, we get $H_n=o(1)$. Now observe that the coefficient of $S_{n-j}^j(\xi)$ in K_n is equal to

$$(78)\qquad C_j^{k-j}\cos n\lambda_n+\sum_{s=0}^{j-1}\binom{j}{s}C_j^{k-s}\Delta_{n-j+s}^{j-s}\{\cos v\lambda_n\}=C_j^{k-j}\cos n\lambda_n+O(n^{-1}),$$

and so, since the condition $S_m^{k+1}(\xi)=o(m^{k+1})$ implies $S_m^j(\xi)=\mathring{o}(m^{k+1})$, $j=0,1,..,k$, we obtain from (72), (73), and (78)

$$\overline{\lim_{n\to\infty}}\frac{\cos n\lambda_n}{C_n^k}\sum_{j=0}^{k+1} S_{n-j}^j(\xi)\, C_j^{k-j}\leqslant S^k(\xi). \tag{79}$$

Noting that $\cos n\lambda_n\to -1$, $C_{k+1}^{-1}=0$, (79) may be written

$$\underline{\lim_{n\to\infty}}\frac{1}{C_n^k}\sum_{j=0}^{k} S_{n-j}^j(\xi)\, C_j^{k-j}\geqslant -S^k(\xi). \tag{80}$$

Consider now the difference

$$\sigma_n(\xi)-\sigma_n^{k+1}(\xi)=\sum_{\nu=0}^{n} A_\nu(\xi)\left\{\frac{C_{n-\nu}^k}{C_n^k}-\frac{C_{n-\nu}^{k+1}}{C_n^{k+1}}\right\}= \tag{81}$$

$$=\frac{1}{(n+k+1)\,C_n^k}\sum_{\nu=0}^{n}\nu\, C_{n-\nu}^k A_\nu(\xi).$$

If to the last sum we apply an argument similar to (73), we obtain

$$\sigma_n^k(\xi)-\sigma_n^{k+1}(\xi)=\frac{n}{n+k+1}\,\frac{1}{C_n^k}\sum_{j=0}^{k} S_{n-j}^j(\xi)\, C_j^{k-j}+o(1)\,^{1}). \tag{82}$$

From (80) and (82), we see that $\underline{\lim}\,\sigma_n^k(\xi)\geqslant -S^k(\xi)$, or $s^k(\xi)\geqslant -S^k(\xi)$. This inequality was established under the assumption $s(\xi)=0$. In the general case it may be written

$$s^k(\xi)+S^k(\xi)\geqslant 2s(\xi), \tag{83}$$

which relation, therefore, holds almost everywhere in E. In particular $s^k(\xi)>-\infty$ for almost every $\xi\in E$. Hence, changing the signs of all the coefficients in (4), and applying (83) to the new series, we obtain the inequality opposite to (83), and (70) is established.

§ 32. It remains to consider the case of non-integral k. Let $l=[k]$, so that $l<k<l+1$. Suppose first that (4) is summable in E

[1]) The proof is even simpler, if we use the equation
$$\nu\, C_{n-\nu}^k=-(k+1)\, C_{n-\nu-1}^{k+1}+n\, C_{n-\nu}^k.$$

not only $(C,k+1)$ but also $(C,l+1)$. The left-hand side of (73) is equal to

$$\frac{1}{C_n^k}\sum_{\nu=0}^{n-l-2} S_\nu^{l+1}(\xi)\,\Delta_\nu^{l+2}(C_{n-\nu}^k\cos\nu\lambda_n)+\frac{1}{C_n^k}\sum_{j=0}^{l+1} S_{n-j}^j(\xi)\,\Delta_{n-j}^j\,(C_{n-\nu}^k\cos\nu\lambda_n)=$$
$$=P_n+Q_n,$$

say. Using the formula (74) for $\Delta_\nu^{l+2}(C_{n-\nu}^k\cos\nu\lambda_n)$, and observing that C_m^k is asymptotically equal to $m^k/\Gamma(k+1)$ (as $m\to\infty$), we find without difficulty that

$$P_n=\frac{1}{C_n^k}\sum_{\nu=0}^{n-l-2} S_\nu^{l+1}(\xi)\, C_{n-\nu}^{k-l-2}\cos(\nu+l+2)\lambda_n+o(1)= \tag{84}$$
$$=\frac{\cos n\lambda_n}{C_n^k}\sum_{\nu=0}^{n-l-2} S_\nu^{l+1}(\xi)\, C_{n-\nu}^{k-l-2}+o(1)\,^{1}),$$

assuming, as we may, that $\sigma_m^{l+1}(\xi)\to 0$. Similarly,

$$Q_n=\frac{1}{C_n^k}\sum_{j=0}^{l+1} S_{n-j}^j(\xi)\sum_{s=0}^{j-1}\binom{j}{s} C_j^{k-s}\, O(\lambda_n)^{j-s}+$$
$$+\frac{1}{C_n^k}\sum_{j=0}^{l+1} S_{n-j}^j(\xi)\, C_j^{k-j}\cos n\lambda_n=o(1)+\frac{\cos n\lambda_n}{C_n^k}\sum_{j=0}^{l+1} S_{n-j}^j(\xi)\, C_j^{k-j}.$$

Hence the left-hand side of (73), and so also of (72) is equal to

$$\frac{\cos n\lambda_n}{C_n^k}\left\{\sum_{\nu=0}^{n-l-2} S_\nu^{l+1}(\xi)\, C_{n-\nu}^{k-l-2}+\sum_{j=0}^{l+1} S_{n-j}^j(\xi)\, C_j^{k-j}\right\}+o(1),$$

whence, since $n\lambda_n\to\pi$, we obtain

$$\underline{\lim_{n\to\infty}}\frac{1}{C_n^k}\left\{\sum_{\nu=0}^{n-l-2} S_\nu^{l+1}(\xi)\, C_{n-\nu}^{k-l-2}+\sum_{j=0}^{l+1} S_{n-j}^j(\xi)\, C_j^{k-j}\right\}\geqslant -S^k(\xi). \tag{85}$$

Now consider the equation (81) and apply, to the last sum in it, Abel's transformation $l+2$ times. Essentially the same argument

[1]) Since $\cos(\nu+l+2)\lambda_n-\cos n\lambda_n=O\{(n-\nu)\lambda_n\}$, and $(n-\nu)\, C_{n-\nu}^{k-l-2}=$ $=O(n-\nu)^{k-l-1}$.

which led to (85), shows that $\sigma_n^k(\xi)-\sigma_n^{k+1}(\xi)$ is equal to

$$\frac{n}{n+k+1}\frac{1}{C_n^k}\left\{\sum_{\nu=0}^{n-l-2} S_\nu^{l+1}(\xi)\, C_{n-\nu}^{k-l-2}+\sum_{j=0}^{l+1} S_{n-j}^{j}(\xi)\, C_j^{k-j}\right\}+o(1).$$

Comparing this with (85), we obtain (83) (with $s(\xi)=0$), and so also (70).

We add that the foregoing argument shows that, if $\sigma_n^{l+1}(\xi)=O(1)$, and $S^k(\xi)<\infty$, then $\sigma_n^k(\xi)=O(1)$.

§ 33. To remove the assumption concerning summability $(C, l+1)$, we argue as follows. From the hypothesis of Lemma 20 we see that (4) is summable $(C, l+2)$, and $S^{l+1}(x)<\infty$, for $x\in E$. Hence, $\sigma_n^{l+1}(x)=O(1)$ almost everywhere in E. In view of the last remark of § 32, we have $\sigma_n^k(x)=O(1)$ almost everywhere in E. Since $l<k<l+1$, an application of the Andersen theorem stated in § 25, shows that (4) is summable $(C, l+1)$ almost everywhere in E. This, in connection with the result of § 32, completes the proof of Lemma 20.

§ 34. ***Lemma 21.*** *If* (4) *is summable A in a set E, $|E|>0$, to sum $s(x)$, and if* (69) *is true, then, at almost every point $x\in E$, we have $s^k(x)>-\infty$ and the equation* (70).

For the proof it is sufficient to apply Lemma 20, and the following

Lemma 22. *If a series $u_0+u_1+\dots$ is summable A, and its k-th Cesàro means $(k>-1)$ are bounded above, the series is summable $(C, k+1)$* [1]).

[1]) Littlewood [1]; the result is proved there for $k=0$ only, but the general theorem can be deduced from this special case by a comparatively simple argument; cf. Zygmund [2], p. 329. It must however be added that in the last paper only the case $k\geqslant 0$ is explicitly dealt with. Although the proof may be so modified as to cover the case $-1<k<0$ also, this is not required for our purposes, for in this case we may argue as follows. The condition $S^k(x)<\infty$, $x\in E$, $k<0$, implies $S^0(x)<\infty$, and so summability $(C, 1)$. Hence $s_n(x)$ is finite almost everywhere in E, and, in view of the last remark of § 32, the same is true for $s_n^k(x)$. By Andersen's theorem, $s_n(x)$ converges almost everywhere in E, and it is sufficient to apply the result of § 32 (with $l=-1$).

§ 35. ***Lemma 23.*** *Under the hypothesis of Theorem 3, the series* (4) *and* (5) *are summable $(C, k+6)$ almost everywhere in E.*

For let $\Phi(x)$ denote the sum of the series (4) integrated term by term $(l+2)$ times $(l=[k]+1)$. From the second part of Lemma 14, and Theorem 1, follows the existence of $\Phi_{(l+2)}(x)$ for $x\in\mathcal{E}\subset E$, $|\mathcal{E}|=|E|$. We may assume that $a_0=0$, and that $\Phi(x)$ is periodic. Differentiating the Fourier series of $\Phi(x)$ $(l+2)$ times, we obtain the series (4), which in view of Lemma 12, is summable $(C, l+2+2)$, and so also $(C, k+5)$, for $x\in\mathcal{E}$. In order to prove the second part of the lemma, we need the following

Lemma 24. *Let $\Phi(x)$ be a periodic function, $\Psi(x)$ its integral, and $\bar{\Psi}(x)$ the function conjugate to $\Psi(x)$ (we may assume that Ψ is periodic). Then, if $\Phi_{(\alpha)}(x)$ exists for $x\in E$, the derivative $\bar{\Psi}_{(\alpha+1)}(x)$ exists almost everywhere in E $(\alpha=0, 1,\dots)$.*

Assuming this lemma, which is identical with Theorem 5 (the proof we postpone to Chapter 4), we see that $\bar{\Psi}_{(l+3)}(x)$ exists almost everywhere in E. Hence, the series (5), which is obtained by differentiating the Fourier series of $\bar{\Psi}(x)$ $(l+3)$ times, is summable $(C, l+5)$, and so also $(C, k+6)$, at almost every point of $\mathcal{E}$.

§ 36. ***Lemma 25.*** *If $-\infty<s^k(x)\leqslant S^k(x)<\infty$ for $x\in E$, $|E|>0$, and if* (5) *is summable $(C, k+1)$ in E, then, at almost every point of E,* (5) *is finite (C, k) and satisfies the relations*

$$(86)\qquad \tfrac{1}{2}\{\bar{s}^k(x)+\bar{S}^k(x)\}=\bar{s}(x),\qquad S^k(x)-s^k(x)=\bar{S}^k(x)-\bar{s}^k(x).$$

The proof is analogous to that of Lemma 20, so that we may condense some parts. Let $\mathcal{E}\subset E$, $|\mathcal{E}|>0$, be a set over which the functions $s^k(x)$ and $S^k(x)$ are continous, and such that

$$s^k(x)-\varepsilon_n\leqslant\sigma_n^k(x)\leqslant S^k(x)+\varepsilon_n \qquad (x\in\mathcal{E}),$$

with $\varepsilon_n\to 0$. Let $\xi\in\mathcal{E}$ be a point of density of $\mathcal{E}$, an λ_n a sequence such that $n\lambda_n\to\pi/2$, $\xi\pm\lambda_n\in\mathcal{E}$. Consider the difference

$$(87)\qquad \tfrac{1}{2}\{\sigma_n^k(\xi+\lambda_n)-\sigma_n^k(\xi-\lambda_n)\}=-\frac{1}{C_n^k}\sum_{\nu=1}^{n}\bar{A}_\nu(\xi)\, C_{n-\nu}^k\sin\nu\lambda_n,$$

and, to fix ideas, suppose that k is an integer. We may then argue exactly in the same way as in § 31, the only difference being that now we have $\bar{A}_\nu(\xi)$ and $\sin \nu \lambda_n$ instead of $A_\nu(\xi)$ and $\cos \nu \lambda_n$. Assuming that $\bar{s}(\xi)=0$, we find the following relations, corresponding to (79) and (80).

$$\tfrac{1}{2}\{\sigma_n^k(\xi+\lambda_n)-\sigma_n^k(\xi-\lambda_n)\}=-\frac{\sin n\lambda_n}{C_n^k}\sum_{j=0}^{k}\bar{S}_{n-j}^{j}(\xi)\,C_j^{k-j}+o(1).$$

$$(88)\qquad -\varliminf_{n\to\infty}\frac{1}{C_n^k}\sum_{j=0}^{k}\bar{S}_{n-j}^{j}(\xi)\,C_j^{k-j}\leqslant\tfrac{1}{2}\{S^k(\xi)-s^k(\xi)\}.$$

Arguing as in (81), we obtain for $\bar{\sigma}_n^k(\xi)-\bar{\sigma}_n^{k+1}(\xi)$ an expression analogous to the right-hand side of (82), with $S_{n-j}^j(\xi)$ replaced by $\bar{S}_{n-j}^j(\xi)$. From this and (88) we deduce that $\bar{s}^k(\xi)\geqslant-\tfrac{1}{2}\{S^k(\xi)-s^k(\xi)\}$.

If $\bar{s}(\xi)\neq 0$, the last relation takes the form

$$(89)\qquad \bar{s}^k(\xi)-\bar{s}(\xi)\geqslant-\tfrac{1}{2}\{S^k(\xi)-s^k(\xi)\}.$$

Applying (89), which is true for almost every $\xi\in E$, to the series (5) with coefficients multiplied by (-1), we get the inequality $\bar{S}^k(\xi)-\bar{s}(\xi)\leqslant\tfrac{1}{2}\{S^k(\xi)-s^k(\xi)\}$. From this and (89) we deduce

$$(90)\qquad \bar{S}^k(\xi)-\bar{s}^k(\xi)\leqslant S^k(\xi)-s^k(\xi).$$

Being finite (C,k) and summable $(C,k+6)$ (Lemma 23), the series (4) is summable $(C,k+1)$ almost everywhere in $\mathcal{E}$. Hence, under the hypothesis of Lemma 25, the series (4) and (5) play symmetric rôles. Therefore the inequality opposite to (90) is also true, which proves (86). In the above argument k was an integer. The case of general $k>-1$ may be left to the reader; the argument is analogous that of § 32.

§ 37. We are now in a position to prove Theorems 2 and 3. Let us suppose the hypotheses of Theorem 2 satisfied. In view of Lemma 21, $\sigma_n^k(x)=O(1)$ almost everywhere in E. The hypotheses of Theorem 3 are therefore also satisfied, so that it is sufficient to prove the latter theorem. By Lemma 23, (4) is summable $(C,k+6)$ for almost every $x\in E$, and so (Lemma 15) summable $(C,k+\varepsilon)$.

In particular (4) is summable $(C,k+5)$. Since (5) is summable $(C,k+6)$, it is, by Lemma 25, summable $(C,k+5)$. Repeating this argument, we obtain that (5) is summable $(C,k+1)$ and finite (C,k) for almost every $x\in E$. This implies summability $(C,k+\varepsilon)$ of (5).

To prove (25) and (26), it is sufficient to apply Lemmas 21 and 25 once more. This completes the proof of Theorems 2 and 3.

CHAPTER III.

§ 38. Let us suppose that (4) is summable (C,k), $k\geqslant 0$, for $x\in E$, $|E|>0$. Assuming, for simplicity, that $a_0=0$, we shall consider the two series

$$(91)\qquad \text{a)}\ \sum_{n=1}^{\infty}\frac{a_n\cos nx+b_n\sin nx}{n},\qquad \text{b)}\ \sum_{n=1}^{\infty}\frac{a_n\sin nx-b_n\cos nx}{n}.$$

In view of a very well-known result, at every point x where (4) is summable (C,k), the series (91a) is summable $(C,k-1)$ [1]). Hence, by Theorem 2, (91b) is summable $(C,k-1)$ almost everywhere in E. Let $s(x)$ and $S(x)$ denote the sums of the series (4) and (91b) respectively. Let l be the least integer $\geqslant k$, and $\Phi(x)$ the sum of the series (4) integrated term by term $(l+2)$ times. In view of Lemma 14, $D_{l+2}\Phi(x)$ exists for $x\in E$, whence, applying Theorem 1, we see that $\Phi_{(l+2)}$ exists almost everywhere in E. Since (4) and (91b) are $\mathfrak{S}^{(l+2)}[\Phi]$ and $\mathfrak{S}^{(l+1)}[\Phi]$ respectively (for the notation cf. § 24), using Lemma 12 we obtain that $S(x)=\Phi_{(l+1)}(x)$, $s(x)=\Phi_{(l+2)}(x)$ almost everywhere in E. Applying Theorem 1, we see that $s(x)=(d/dx)_a\,S(x)$, and this completes the proof of Theorem 4.

§ 39. Theorem 4 holds, in an appropriate form, if (4) is finite (C,k) for $x\in E$. For then, by Theorem 3, (4) is summable $(C,k+\varepsilon)$ almost everywhere in E $(\varepsilon>0)$. Arguing as before, we see that (91b) is summable $(C,k-1+\varepsilon)$, $\varepsilon>0$, almost everywhere in E. If $s(x)$ and $S(x)$ denote the sums of (4) and (91b) respectively, we again obtain $s(x)=(d/dx)_a\,S(x)$.

Theorem 4 may be completed by the following

Theorem 10. *Let (4) be summable (C,k), $k\geqslant 0$, to sum $s(x)$, for $x\in E$, $|E|>0$, and let l be any positive integer such that $k-l\geqslant -1$. Then the series (4) integrated term by term l times is summable $(C,k-l)$ almost everywhere in E. If $S(x)$ is the sum of the integrated series, then*

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[1]) See e. g. Hardy and Riesz [1], or Zygmund [2], p. 326.

$s(x)$ is, for almost every $x \in E$, the l-th de la Vallée-Poussin approximate derivative of $S(x)$, that is we have

$$(92)\quad S(x+h)=S(x)+S_1(x)\,h+\dots+S_{l-1}(x)\frac{h^{l-1}}{(l-1)!}+s(x)\frac{h^l}{l!}+\varepsilon_h(x)\frac{h^l}{l!},$$

for almost every $x \in E$, and h tending to 0 along a set $H=H_x$ having $h=0$ as a point of density.

Using the argument of § 38, Theorem 10 is a simple corollary of the following proposition, the proof of which will be given elsewhere [1].

Lemma 26. *If $f_{(k)}(x)$ exists in a set E, $|E|>0$, then $f_{(k)}(x)$ is, almost everywhere in E, the $(k-s)$-th de la Vallée Poussin approximate derivative of $f_{(s)}(x)$ $(s=0,1,2,..,k-1)$.*

We add that, if $k-l<-1$ in Theorem 10, then $D_l S(x)$ exists and is equal to $s(x)$ for every $x \in E$ [2]. Hence, by Theorem 1, $S_{(l)}(x)$ exists and is equal to $s(x)$ almost everywhere in E.

CHAPTER IV.

§ 40. To prove Theorem 5, we may suppose that the constant term of the Fourier series of $F(x)$ vanishes, so that $\Phi(x)$ is periodic, and

$$\bar{\Phi}(x)=-\frac{1}{\pi}\int_{-\pi}^{\pi}\Phi(t)\,\tfrac12\operatorname{ctg}\tfrac12(t-x)\,dt. \quad \text{Let} \quad \tilde{\Phi}(x)=-\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\Phi(t)}{t-x}dt.$$

The difference $\tilde{\Phi}(x)-\bar{\Phi}(x)$ is regular in the interior of $(-\pi,\pi)$, and so it is sufficient to consider the function $\tilde{\Phi}(x)$ instead of $\bar{\Phi}(x)$. We shall show that $D_{k+1}\tilde{\Phi}(x)$ exists almost everywhere in E, for this, in view of Theorem 1, implies the existence of $\tilde{\Phi}_{(k+1)}(x)$. The proof of Theorem 5 will be based on the following

Lemma 27. *Suppose that $f(x)$ is a function of period 2π, integrable L. If $f_{(k)}(x)$ exists for $x \in E$, $|E|>0$, that is, if we have (44)*

1) See Marcinkiewicz [1].

2) Kogbetliantz [1]; cf. also Zygmund [1], p. 304.

with $\omega(x,t)$ tending to $f_{(k)}(x)$ as $t\to 0$, $x\in E$, then the integral

$$(93)\quad \int_0^{\pi}\frac{\omega(x,t)-\omega(x,-t)}{t}dt=\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\pi}\frac{\omega(x,t)-\omega(x,-t)}{t}dt$$

exists almost everywhere in E.

This result is due to Plessner [1]. Assuming it, let $x\in E$ be any point for which (93) exists. Supposing, as we may, that $x=0$, $F(0)=F_{(1)}(0)=\dots=F_{(k)}(0)=0$, we write (see § 1)

$$(94)\quad \Delta_{k+1}(0,u;\tilde{\Phi})=-\frac{1}{\pi}\int_{-\pi}^{\pi}\Phi(t)\Delta_{k+1}\left(0,u;\frac{1}{t}\right)dt = {}^{2)}$$

$$=-\frac{1}{\pi}\left[\int_{-2ku}^{2ku}+\int_{-\pi}^{-2ku}+\int_{2ku}^{\pi}\right]=-\frac{1}{\pi}\int_{-2ku}^{2ku}-\frac{1}{\pi}\int_{2ku}^{\pi}\{\Phi(t)+(-1)^k\Phi(-t)\}\,\Delta_{k+1}\left(0,u;\frac{1}{t}\right)=$$

$$=A_u+B_u,$$

say. By hypothesis, $F(t)=o(t^k)$ as $t\to 0$, so that $\Phi(t)=o(t^{k+1})$.

Let us observe that, if $-\frac12 k\leqslant s\leqslant\frac12 k$, then

$$(95)\quad \int_{-2ku}^{2ku}\frac{\Phi(t)\,dt}{t-su}=\int_{-2ku}^{2ku}\frac{\Phi(t)-\Phi(su)}{t-su}dt+\Phi(su)\int_{-2ku}^{2ku}\frac{dt}{t-su}\ {}^{3)}.$$

The second term on the right is $o(u^{k+1})$, since the coefficient of $\Phi(su)$ is bounded. By the mean value theorem, the integrand of the second integral on the right is $o(u^k)$. Hence the left-hand side of (95) is $o(u^{k+1})$, and so also $A_u=o(u^{k+1})$.

It can easily be verified by induction that

$$\Delta_{k+1}(0,u;1/t)=\frac{(-1)^{k+1}(k+1)!\,u^{k+1}}{\prod_{i=0}^{k+1}\left(t+\frac{k+1}{2}u-iu\right)}$$

1) See Plessner [2]; an elementary proof will be found in Marcinkiewicz [1].

2) By $\Delta_{k+1}(0,u;1/t)$ we mean $\Delta_{k+1}(x,u;1/(t-x))/_{x=0}$, where the difference is taken with respect to x.

3) The integrals are understood in the Cauchy principal sense.

We have

$$(96)\qquad \pi B_u = \int_{2ku}^{\delta} + \int_{\delta}^{\pi} [\Phi(t)+(-1)^k \Phi(-t)]\, \Delta_{k+1}(0, u; 1/t)\, dt = B'_u + B''_u$$

say, where $\delta > 0$ is a small but fixed number.

Applying the second mean value theorem to B'_u, we find

$$(97)\qquad |B'_u| \leqslant M u^{k+1} \left| \int_{2ku}^{\xi} [\Phi(t)+(-1)^k \Phi(-t)] \frac{dt}{t^{k+2}} \right|,$$

where M is independent of Φ and u, and $2ku < \xi < \delta$. An integration by parts shows the integral on the right of (97) to be equal to

$$-\frac{1}{k+1}\left\{\frac{\Phi(t)+(-1)^k \Phi(-t)}{t^{k+1}}\right\}_{2ku}^{\xi} + \frac{1}{k+1}\int_{2ku}^{\xi} \frac{F(t)-(-1)^k F(-t)}{t^{k+1}}\, dt =$$

$$= -\frac{1}{k+1}\left\{\frac{\Phi(t)+(-1)^k \Phi(-t)}{t^{k+1}}\right\}_{2ku}^{\xi} + \frac{1}{(k+1)!}\int_{2ku}^{\xi} \frac{\omega(0,t)-\omega(0,-t)}{t}\, dt,$$

which expression is small with δ. Hence

$$(98)\qquad \overline{\lim_{u\to 0}}\, |B'_u/u^{k+1}| \leqslant K_\delta, \quad \text{where} \quad K_\delta \to 0 \text{ with } \delta.$$

Since δ id fixed,

$$(99)\quad \left\{\begin{aligned} \frac{B''_u}{u^{k+1}} &\to (-1)^{k+1}(k+1)! \int_{\delta}^{\pi} \frac{\Phi(t)+(-1)^k \Phi(-t)}{t^{k+2}}\, dt = \\ &= (-1)^{k+1}(k+1)! \left\{-\frac{\Phi(t)+(-1)^k \Phi(-t)}{(k+1)\, t^{k+1}}\right\}_{\delta}^{\pi} + \\ &+ (-1)^{k+1}(k+1)! \frac{1}{k+1}\int_{\delta}^{\pi} \frac{\omega(0,t)-\omega(0,-t)}{k!\cdot t}\, dt. \end{aligned}\right.$$

In view of the existence of (93) for $x=0$, and the relation $\Phi(\pm\delta) = o(\delta^{k+1})$, the right-hand side of (99) is equal to $C + K'_\delta$ where C is independent of δ, and $K'_\delta \to 0$ with δ. This, together with (94), (98), (99), and the relation $A_u = o(u^{k+1})$, shows that $D_{k+1}\tilde{\Phi}(0)$ exists, and Theorem 5 is established.

CHAPTER V.

§ 41. Given a function $f(x) \in L$, $a \leqslant x \leqslant b$, we shall denote by $\varphi(h) = \varphi(h, x; f)$ the integral $\int_0^h \{f(x+u) - f(x-u)\} \frac{du}{2u} = \lim_{\varepsilon\to 0} \int_\varepsilon^h$, if it exists. Plainly, $\varphi(h) \to 0$ with h. By $\overline{D}_2 f(x)$ and $\underline{D}_2 f(x)$ we shall denote the upper and lower limits of $\Delta_2(x, h; f)/h^2$ for $h \to 0$.

Lemma 28. *Let $f \in L$, and let $F(x)$ be the integral of f. A necessary and sufficient condition that $B_s f$ should exist at a point x, is the existence of $D_2 F(x)$; then $D_2 F(x) = B_s f(x)$.*

A necessary and sufficient condition that $\varphi(h) = O(h)$ for $h \to 0$, is that $|\underline{D}_2 F(x)| + |\overline{D}_2 F(x)| < \infty$ [1]*.*

Let $0 < k < h$, and suppose that $\varphi(h)$ exists. From the equation

$$\Delta_2(x, h; F) - \Delta_2(x, k; F) = \int_k^h \{f(x+u) - f(x-u)\}\, du = 2\int_k^h \varphi'(u)\, u\, du,$$

integrating by parts and making $k \to 0$, we obtain

$$(100)\qquad \Delta_2(x, h; F) = 2\left[\varphi(h)\, h - \int_0^h \varphi(t)\, dt\right].$$

This shows that, if $\varphi(t)/t \to A$, then $D_2 F(x) = A$. If $\varphi(t) = O(t)$, then $\Delta_2(x, h; F) = O(h^2)$.

Now let $\Delta_2(x, t; F) = \int_0^t \{f(x+u) - f(x-u)\}\, du = \psi(t)$. Then, if $0 < k < h$,

$$\int_k^h \{f(x+u) - f(x-u)\} \frac{du}{2u} = \int_k^h \frac{\psi'(u)}{2u}\, du.$$

Integrating by parts, making $k \to 0$, and supposing that $\psi(t) = O(t^2)$, we obtain

$$(101)\qquad \int_0^h \frac{f(x+t) - f(x-t)}{2t}\, dt = \frac{1}{2}\left[\frac{\psi(h)}{h} + \int_0^h \frac{\psi(t)}{t^2}\, dt\right].$$

[1]) The sufficiency of the conditions was established by Khintchine [1], p. 221.

Hence, if $\psi(h)/h^2 \to A$, then $\varphi(h)/h \to A$; if $\psi(h)=O(h^2)$, then $\varphi(h)=O(h)$. This completes the proof of the lemma.

§ 42. ***Lemma 29.*** *Let f and F have the same meaning as before. A necessary and sufficient condition that, for a given x,*

$$F(x+t)=F(x)+t\,f(x)+\frac{\omega(x,t)}{2}t^2,$$

where $\omega(x, t)$ is bounded for $t\to+0$, is that $I(h, x; f)=O(h)$ for $h\to 0$ (cf. (29)).

A necessary and sufficient condition for the existence of $F_{(2)}(x)$ is the existence of $B\,f(x)$, which, then is equal to $F_{(2)}(x)$.

If we observe that $[F(x+t)-F(x)-tf(x)]'=f(x+t)-f(x)$, the proof becomes similar to that of Lemma 28, and may be left to the reader.

In order to prove Theorem 6, we note that if $\varphi(h, x; f)=O(h)$ for $x\,\epsilon\,E$, $h\to 0$, then, by Lemma 28 and Theorem 1, $F_{(2)}(x)$ exists almost everywhere in E, and it is sufficient to apply Lemma 29. To establish the Sargent theorem enunciated in § 8, we observe that, if $I(h, x; f)=O(h)$ for $x\,\epsilon\,E$, then, by Lemma 29 and Theorem 8, $D_2F(x)$ exists almost everywhere in E. So does (Theorem 1) $(d/dx)_a\,(d/dx)_a\,F(x)=(d/dx)_a\,f(x)$, which, is equal almost everywhere, to $D_2F(x)=B\,f(x)$.

§ 43. Given a function $g\,\epsilon\,L$, by $\overline{B}_s\,g(x)$ and $\underline{B}_s\,g(x)$ we shall mean $\overline{\lim\limits_{h\to 0}}\,\varphi(h, x; g)/h$ and $\underline{\lim\limits_{h\to 0}}\,\varphi(h, x; g)/h$ respectively. In what follows we assume that the integral $\varphi(x, h; f)$ either converges or diverges to $\pm\infty$. In the latter case, $B_s f(x)=\pm\infty$.

We shall say that g is *continuous in mean*, or, simply, *m*-continuous, at a point x, if $\lim\limits_{h\to 0}\frac{1}{h}\int\limits_0^h g(x+t)\,dt=g(x)$.

Let now $f(x)$ be a function defined in an interval (a, b), and P a point set contained in (a, b), $a\,\epsilon\,P$, $b\,\epsilon\,P$, $|P|=b-a$. A function $M(x)$, $a\leqslant x\leqslant b$, will be called a *majorant* of $f(x)$, corresponding to the basis P, if (i) $M(x)\,\epsilon\,L$, (ii) $M(x)$ is *m*-continuous for $x\,\epsilon\,P$, (iii) $M(a)=0$, (iv) $-\infty\neq\underline{B}_s M(x)\geqslant f(x)$, $a<x<b$, except, perhaps, at an enumerable subset E of P. Similarly we define a minorant $m(x)$: (i′) $m(x)\,\epsilon\,L$, (ii′) $m(x)$ is *m*-continuous for $x\,\epsilon\,P$, (iii′) $m(a)=0$, (iv′) $+\infty\neq\overline{B}_s\,m(x)\leqslant f(x)$, $a<x<b$, except, perhaps, at an enumerable subset E of P.

Let $F^*(b)=\operatorname{Inf} M(b)$, $F_*(b)=\operatorname{Sup} m(b)$. If $F^*(b)=F_*(b)$, we shall say that $f(x)$ is integrable T over (a, b), and write

$$F_*(b)=F^*(b)=F(b)=(T)\int\limits_a^b f(x)\,dx. \tag{102}$$

It will often be convenient to say that $f(x)$ is integrable $T(P)$ over (a, b).

§ 44. ***Lemma 30.*** *Let $f(x)$ belong to L, and $F(x)$ be the integral of f. Then $\underline{D}_2F(x)\leqslant\underline{B}_s\,f(x)\leqslant\overline{B}_s f(x)\leqslant\overline{D}_2F(x)$.*

Let us first assume that $\varphi(h, x; f)$ converges at the point considered. To prove e. g. the last inequality, it is sufficient to show that, if $\overline{B}_s f(x)>0$, then $\overline{D}_2F(x)\geqslant 0$. If $\overline{B}_s f(x)>0$, then $\varphi(h)=\varphi(h, x; f)$ takes positive values in every neighbourhood of $h=0$. Since $\varphi(h)\to 0$ with h, we deduce, that there is a sequence of points $\xi_1>\xi_2>\ldots\to 0$, such that $\varphi(\xi_k)\geqslant\varphi(h)$ for $0\leqslant h\leqslant\xi_k$, and so also $\xi_k\,\varphi(\xi_k)\geqslant\int\limits_0^{\xi_k}\varphi(t)\,dt$. In view of (100), $\Delta_2(x, \xi_k; F)\geqslant 0$, i. e. $\overline{D}_2F(x)\geqslant 0$. The inequality $\underline{D}_2F(x)\leqslant\underline{B}_s f(x)$ may be established in the same way.

Let us now suppose, for example, that $\varphi(x, h; f)=+\infty$, so that $\underline{B}_s f(x)=\overline{B}_s f(x)=+\infty$; and let us suppose, contrary to what we intend to prove, that $\overline{D}^2F(x)<\infty$. By subtracting a linear function from f, we may assume that $\overline{D}_2F(x)<0$. We put $\varphi^*(u)=$ $=\int\limits_u^b[f(x+t)-f(x-t)]\frac{dt}{2t}$ $(u>0)$. It is easy to see that $\varphi^*(u)=o(1/u)$ so that the same argument which led to (100) gives

$$\Delta_2(x, h; F)=-2\left[\varphi^*(h)h-\int\limits_0^h\varphi^*(t)\,dt\right]. \tag{102}$$

Since, by hypothesis, $\varphi(h)\to+\infty$ with $1/h$, there is a sequence $\xi_1, \xi_2, \ldots\to 0$ such that $\varphi(t)\geqslant\varphi(\xi_i)$ for $0<t\leqslant\xi_i$. From (102a) we deduce that $\Delta_2(x, \xi_i, F)\geqslant 0$, so that $\overline{D}_2F(x)\geqslant 0$, contrary to assumption.

§ 45. Returning to majorants and minorants, we shall consider the difference $\delta(x)=M(x)-m(x)$, and its integral $\Delta(x)=\int\limits_a^x \delta(t)dt$. In view of conditions (iv) and (iv'), $\underline{B}_s\delta(x)\geqslant\underline{B}_s M(x)-\overline{B}_s m(x)\geqslant 0$, outside E, and so, by Lemma 30, $\overline{D}_2\Delta(x)\geqslant 0$ outside E. It is well known that, if a continuous function $F(x)$, $a<x<b$, satisfies the inequality $\overline{D}^2F(x)\geqslant 0$, except perhaps at an enumerable set E, where, however, we have $F(x+h)+F(x-h)-2F(x)=o(h)$, then F is convex. In our case $E\subset P$, so that $\Delta(x)$ is m-continuous in E. This gives $\Delta(x+h)+\Delta(x-h)-2\Delta(x)=o(h)$ for $x\in E$. It follows that $\Delta(x)$ is convex.

Now the right-hand derivative $\Delta_r'(x)$ exists for $a\leqslant x<b$, is non-decreasing, and $\Delta_r'(x)=\Delta_r'(x+0)$. Since $\Delta_r'(x)=\delta(x)$ almost everywhere, it follows that $\Delta_r'(x)=\delta(x)$ at every point x where δ is m-continuous. Therefore $\delta(x)$ is non-decreasing over the set P deprived of the point b. To remove the last restriction, we repeat the previous argument using the left-hand derivative $\Delta_l'(x)$, for which $\Delta_l'(x)=\Delta_l'(x-0)$, $a<x\leqslant b$. Since $\delta(x)=M(x)-m(x)$ is non decreasing over P, $a\in P$, $\delta(a)=0$, it follows that $\delta(x)\geqslant 0$ over P. In particular $M(b)\geqslant m(b)$, $F^*(b)\geqslant F_*(b)$.

§ 46. (α) *If $f(x)$ is integrable $T(P)$ over (a,b), then it is also integrable T over any interval (a,x) with $x\in P$.*

For, if $x\in P$, then $0\leqslant M(x)-m(x)\leqslant M(b)-m(b)$. It follows that $F^*(x)\leqslant F_*(x)$. Since also $F_*(x)\leqslant F^*(x)$, we obtain that $F_*(x)=$ $=F^*(x)$.

(β) *If f is integrable $T(P)$ over (a,b), then for any points $\alpha\in P$, $\beta\in P$, $a\leqslant\alpha\leqslant\beta\leqslant b$, f is integrable $T(P)$, and*

$$T(P)\int\limits_\alpha^\beta f(x)\,dx=F(\beta)-F(\alpha).$$

(γ) *If $f(x)$ is integrable $T(P)$ and $T(Q)$ over (a,b), then*

$$T(P)\int\limits_a^b f(x)\,dx=T(Q)\int\limits_a^b f(x)\,dx.$$

To prove (γ), we observe that, from the hypothesis, $a\in P$, $a\in Q$, $b\in P$, $b\in Q$. Let $R=P.Q$. It is sufficient to use the fact that integrability $T(P)$ over (a,b) implies integrability $T(R)$ over (a,b), the value of the integral being the same.

(δ) *If f is integrable L over (a,b), then it is also integrable $T(P)$, where $P=(a,b)$, to the same value.*

(ε) *The function $F(x)$ is integrable L over (a,b).*

(ζ) *If c_1, c_2 are constants, then*

$$(T)\int\limits_a^b (c_1f_1+c_2f_2)\,dx=c_1(T)\int\limits_a^b f_1\,dx+c_2(T)\int\limits_a^b f_2\,dx.$$

(η) *For almost every x, $B\,F(x)=f(x)$.*

Proposition (ε) follows from the inequality $m(x)\leqslant F(x)\leqslant M(x)$ $(x\in P)$. To prove (η), we observe that, since $M(x)-m(x)$ is non-increasing over P, so are $M(x)-F(x)$ and $F(x)-m(x)$. From the relations $F=(F-m)+m$ and $M=(M-F)+F$, we obtain respectively $\overline{B}_sF\leqslant\overline{B}_s(F-m)+\overline{B}_sm\leqslant\overline{B}_s(F-m)+f$ and $f\leqslant\underline{B}_sM\leqslant\underline{B}_sF+\overline{B}_s(M-F)$,

$$(103)\qquad \overline{B}_sF(x)-\{F(x)-m(x)\}'\leqslant f(x)\leqslant\underline{B}_sF(x)+\{M(x)-F(x)\}'$$

for almost every x. Now observe that, if $\varphi(x)$ is a non-decreasing function over (a,b), $\varphi(a)=0$, $\varphi(b)<\varepsilon$, then the set E_η of points x where $\varphi'(x)>\eta$ is of measure $<\varepsilon/\eta$. For

$$\varepsilon>\varphi(b)\geqslant\int\limits_a^b\varphi'(x)\,dx\geqslant\eta\cdot|E_\eta|.$$

Taking $\varphi(x)=M(x)-F(x)$, where $M(b)-F(b)<\varepsilon$, we obtain that $\{M(x)-F(x)\}'\leqslant\sqrt{\varepsilon}$ outside a set of measure $\leqslant\sqrt{\varepsilon}$. This, and the second inequality (103), gives $f(x)\leqslant B_sF(x)$ almost everywhere. In the same way from the first inequality (103) we obtain $\overline{B}_sF(x)\leqslant f(x)$ and so $B_sF(x)=f(x)$. It remains to apply Theorem 6.

Proposition (δ) follows from the fact that every function integrable L is integrable in the sense of Denjoy-Perron, and it is plain that the latter integrability is a special case of integrability T.

Proposition (ζ) is obvious.

(θ) *Let $a<b<c$. If f is integrable T over (a,b) and over (b,c), then it is also integrable T over (a,c). The integral over (a,c) is equal to the sum of the integrals over (a,b) and (b,c).*

Let $M_1(x)$, $m_1(x)$, $M_2(x)$, $m_2(x)$ be majorants and minorants of f over the intervals (a,b) and (b,c) respectively. Let $M(x)=M_1(x)$ for $a\leqslant x\leqslant b$, $M(x)=M_1(b)+M_2(x)$ for $b\leqslant x\leqslant c$, and let $m(x)$ be defined similarly. Then $M(x)$ is a majorant, $m(x)$ a minorant, of f over (a, c), and it is not difficult to see that $M(c)-m(c)$ may be arbitrarily small. Hence f if integrable T over (a, c). The proof of the second part of (θ) is obvious.

§ 47. We shall now suppose that the function $f(x)$, and so also the set P, are continued outside (a, b) by the condition of periodicity: $f(x+k\omega)=f(x)$, $k=\pm 1, \pm 2, \ldots$, where $\omega=b-a$.

From propositions (α) and (θ) we deduce

(ι) *If f is integrable $T(P)$ over (a, b), and if $a+u \in P$, $b+u \in P$, then f is integrable $T(P)$ over $(a+u, b+u)$, and*

$$(T)\int_a^b f\,dx=(T)\int_{a+u}^{b+u} f\,dx.$$

We shall now prove that

($\varkappa$) *If the series (4) converges everywhere to sum $f(x)$, then, for almost every u, $f(x)$ is integrable T over $(u, 2\pi+u)$. Moreover (4) is the Fourier series of f, that is*

$$(104)\qquad \pi a_k=(T)\int_u^{2\pi+u} f(x)\cos kx\,dx, \qquad \pi b_k=(T)\int_u^{2\pi+u} f(x)\sin kx\,dx$$

for $k=0, 1, \ldots$, and almost every u.

Let $F(x)$ denote the sum of the series

$$(105)\qquad \frac{1}{2}a_0 x+\sum_{n=1}^{\infty}\frac{a_n\sin nx-b_n\cos nx}{n},$$

wherever the latter converges. Let P denote the set of points where $F(x)$ exists and is m-continuous (it is obvious that every L-integrable function is m-continuous almost everywhere). Let $u \in P$; we shall show that $G(x)=F(x)-F(u)$ is a majorant and minorant of f in the interval $(u, u+2\pi)$, with respect to the basis P. Conditions (i), (ii), (iii), (i'), (ii'), (iii') are plainly satisfied. Let $\Phi(x)$ denote the sum of the series (4) integrates twice. Since, by Riemann's classical theorem, $D_2\Phi(x)=f(x)$, an application of Lemma 28 gives $B_s G(x)=f(x)$, and proves (iv) and (iv'). Therefore f is integrable $T(P)$ over $(u, u+2\pi)$, and the value of the integral is equal to $F(u+2\pi)-F(u)$. In view of (105), $F(u+2\pi)-F(u)=\pi a_0$, which gives (104)

for $k=0$. In order to prove the formula (104) for a_k, $k>0$, (the proof for b_k is similar), we multiply (4) by $\cos kx$ and replace the products $\cos nx\cos kx$, $\sin nx\cos kx$ by sums of cosines and sines. It is not difficult to see that we obtain a new trigonometrical series, which converges everywhere to $f(x)\cos kx$. (This is a special case of the Rajchman theory of the formal multiplication of trigonometrical series; see Rajchman [1]; Zygmund [1], p. 279 sqq). The constant term of the new series is $\frac{1}{2}a_k$, so that the formulae (104) are established.

The following proposition enables us to simplify the definition of the basis in the case of trigonometrical series.

§ 48. ***Lemma 31.*** *If $|a_n|+|b_n|\to 0$, the sum $F(x)$ of (105) is m-continuous at every point where it exists.*

For let $\Phi(x)$ denote the sum of the series (105) integrated term by term. It is Fatou's well-known result that

$$\{\Phi(x+h)-\Phi(x-h)\}/2h\to F(x),$$

whenever $F(x)$ exists [1]). This implies that $\{\Phi(x+h)-\Phi(x)\}/h\to F(x)$, since, by Riemann's classical theorem, $\Phi(x+h)+\Phi(x-h)-2\Phi(x)=o(h)$ for every x.

Hence, for the basis P we may take the set C of points of convergence of (105). In particular,

(λ) *If (105) converges for $x=\alpha$, $x=\beta$ ($\alpha<\beta$), the sum $f(x)$ of the everywhere convergent series (4) is integrable T over (α, β) to the value $F(\beta)-F(\alpha)$.*

It is not difficult to see that for u in (104) we may take the numbers belonging to C. If $k=0$, this follows from (λ). Now let us consider, for example, the case of a_k, $k>0$. Let us denote by s the series (4), and by s^k the series

$$(106)\qquad \tfrac{1}{2}a_0^k+\sum_{n=1}^{\infty}(a_n^k\cos nx+b_n^k\sin nx)$$

obtained by multiplying (4) by $\cos kx$. Let S be the series (105), and S^k an analogous series formed for (106). By s_n, s_n^k, S_n, S_n^k we denote the n-th partial sums of the series s, s^k, S, S^k respectively. Since $s_n^k(x)-\cos kx\cdot s_n(x)$ tends uniformly to 0, we obtain that

$$(107)\qquad \int_v^x\{s_n^k-\cos kt\cdot s_n\}\,dt=[S_n^k(t)]_v^x-[\cos kt\cdot S_n(t)]_v^x-k\int_v^x\sin kt\cdot S_n(t)\,dt$$

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[1]) See e. g. Zygmund [1], p. 272.

tends to 0. Let v be such that S and S^k converge for $x=v$. Since the last integral on the right of (107) tends to a limit, S^k converges for $x \in C$. Observing that $a_0^k=a_k$ in (106), we obtain the required result.

§ 49. The definition of integrability T raises a number of problems to which we will return in another paper. Here we shall make some additional remarks.

(μ) If $f_1(x)=f_2(x)$ almost everywhere in (a, b), and if f_1 is integrable $T(P)$ over (a, b), so is $f_2(x)$, and both integrals are equal.

The proof is the same as for Perron's integral, and need not be repeated here [1]). Hence, the T-integral of a function f may be defined, even if $f(x)$ is only defined outside a set E of measure 0. For, if e. g. $g(x)=f(x)$ outside E, $g(x)=0$ in E, we may put $(T)\int_a^b f\,dx=(T)\int_a^b g\,dx$, provided the last integral exists.

(ν) If the partial sums of (4), where $|a_n|+|b_n|\to 0$, are finite for every x, the series (4) is summable $(C, 1)$ almost everywhere to a function $f(x)$. If C is the set of points of convergence of (105), $f(x)$ is integrable T over any interval (α, β) with $\alpha \in C, \beta \in C$. Moreover, we have (104) for $u \in C$.

The first part of (ν) follows from Theorem 3. If Φ is the sum of the series integrated term by term twice, then $-\infty<\underline{D}_2\Phi\leqslant\overline{D}_2\Phi<\infty$ and so, by Theorem 1 and Lemma 30, $D_2\Phi(x)=B_sF(x)$ exists outside a set E, $|E|=0$. Let $g(x)=\underline{B}_sF(x)$. Then, if $u \in C$, $M(x)=F(x)-F(u)$ is a majorant of g over $(u, 2\pi+u)$, with respect to the basis C. We put $h(x)=\underline{B}_sF(x)-\overline{B}_sF(x)$; $h(x)$ is finite everywhere and vanishes almost everywhere, so that $h \in L$. Let $\mu(x)$ be a minorant of h over $(u, u+2\pi)$: $\overline{B}_s\mu(x)\leqslant h(x)$, $\mu(u)=0$, $\mu(u+2\pi)>-\varepsilon$ (since $h(x)\leqslant 0$, $\mu(x)$ is non-increasing). A moment's consideration shows that $m(x)=F(x)-F(u)+\mu(x)$ is a minorant of g. For we have $\overline{B}_sm\leqslant\overline{B}_sF+\overline{B}_s\mu\leqslant\overline{B}_sF+h=\underline{B}_sF=g$. Since $M(u+2\pi)-m(u+2\pi)<\varepsilon$, g is integrable T over $(u, u+2\pi)$, and so is, in view of our convention, f. The proof of (104) and of the fact that the integral of f over (α, β), $\alpha \in C, \beta \in C$, is equal to $F(\beta)-F(\alpha)$, is the same as before.

We mention, without proof, that proposition (ν) holds even if the partial sums of (4) are unbounded at an at most enumerable set of points.

It can be shown that (ν) remains valid even if (4) is finite (C, γ), $\gamma<\frac{1}{2}$, for every x. The basis P, however, need not coincide then with the set C.

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[1]) See Saks [1], 134.

26. VI. 1935.

Added in proof (19. I. 36) Theorem 8 of this paper has meanwhile also been established by Denjoy (Fund. Math. **25**, p. 293).

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