später sehen (im Satze 38), dass die Antwort positiv ist, falls es nur überhaupt nichtaxiomatisierbare Systeme gibt, d. i. falls die Klasse 𝔖 unendlich ist; andrerseits wird sich zeigen, dass es unter derselben Voraussetzung auch nichtkonvergente Systeme gibt. Wir haben also drei Klassen von Systemen: (1) axiomatisierbare Systeme, die einzelnen Aussagen zugeordnet sind; (2) nichtaxiomatisierbare konvergente Systeme, die konvergenten unendlichen Folgen von (in strengem Sinne) wachsenden Aussagen entsprechen, und endlich (3) divergente Systeme, die unendlichen divergenten Folgen entsprechen. Die Systeme der ersten beiden Klassen haben im allgemeinen einander ähnliche Eigenschaften; einer der wesentlichen Unterschiede besteht darin, dass kein System der zweiten Klasse den Satz von der doppelten Negation erfüllt.

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# Non-separable metric spaces.

By

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Separability is a property which greatly facilitates work in metric spaces, but it may be of some interest to point out that this property has been unnecessarily assumed in the proofs of certain theorems concerning such spaces and concerning functions defined on them. In particular it may be shown that if a subset of an arbitrary metric space is analytic at each of its points, it is an analytic set in the space. This problem was mentioned recently by Sierpiński in his discussion of locally separable spaces [1]). It is known that a function $f(x, y)$ continuous in $x$ and of class $\alpha$ in $y$ is of class $\alpha+1$ in $(x, y)$ under certain restrictions on the spaces in question. Kuratowski [2]) has asked whether or not this theorem remains true when $x$ and $y$ range over non-separable metric spaces. It is shown in this note that the theorem does remain true in this case. The proofs rest on simple lemmas which for some purposes replace the classical theorem, true only in separable spaces, that a decreasing series of closed or open sets is enumerable.

**1.** The fundamental space to be considered here will be denoted by $M$; it is subject to no conditions except that it be metric unless otherwise specified.

***Lemma 1.*** *If*

(1) $$O^1,\ O^2, \dots,\ O^\lambda,\ O^{\lambda+1}, \dots$$

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1) *Fundamenta Mathematicae*, Vol. 21 (1933) p. 112, problem of Banach.

2) Kuratowski, *Topologie I* (Monografje Matematyczne t. III, Warszawa-Lwów, 1933) p. 181 (footnote).

*is a well ordered family of increasing open sets and* $O^{\lambda}-\sum_{\mu<\lambda} O^{\mu}$ *contains (for each* $\lambda$*) a closed set* $E^{\lambda}$*, then* $H=\sum_{\lambda} E^{\lambda}$ *is an* $F_{\sigma}$.

Let $E_n^{\lambda}$ be all points of $E^{\lambda}$ whose distances from $M-O^{\lambda}$ are $\leqslant 1/n$. The set $E_n^{\lambda}$ is closed. The set $E_n=\sum_{\lambda} E_n^{\lambda}$ is also closed a fact which may be shown in the following manner.

Suppose

(2) $$p_1, p_2, \ldots$$

is a sequence of points which are in $E_n$ and suppose the sequence converges to a point $p$ not in $E_n$. At most a finite number of points of (2) can be in a given $E_n^{\lambda}$ for if there were an infinite number this infinite subset of (2) would converge to $p$ and $E_n^{\lambda}$ would not be closed. Therefore, for every $m$ there is a $k>m$ such that $p_m$ and $p_k$ are not in the same set $E_n^{\lambda}$. But if $\lambda \neq \mu$ the distance between the sets $E_n^{\lambda}$ and $E_n^{\mu}$ is at least $1/n$. Hence[1] $d(p_m, p_k) \geqslant 1/n$ and it has been shown that there are pairs of points indefinitely far along in the sequence (2) which are at a distance at least $1/n$ from each other. The sequence (2) can not be convergent and from this contradiction it follows that $E_n$ is closed. It is clear that $H=\sum_{\lambda} E^{\lambda}=\sum_{n} E_n$ and therefore $H$ is an $F_{\sigma}$.

**Lemma 2.** *If the sequence* (1) *is such that for every* $\lambda$, $O^{\lambda}-\sum_{\mu<\lambda} O^{\mu}$ *contains a set* $E^{\lambda}$ *which is a* $G_{\alpha}(F_{\alpha})$, $(\alpha>0)$, *then* $H=\sum_{\lambda} E^{\lambda}$ *is a* $G_{\alpha}(F_{\alpha})$[2].

The proof will first be given for the case in which every $E^{\lambda}$ is an $F_1$. An $F_1$ is the sum of an enumerable number of closed sets, so that for each $\lambda$, $E^{\lambda}=\sum_{n} E_n^{\lambda}$ where $E_n^{\lambda}$ is closed. The set $E_n=\sum_{\lambda} E_n^{\lambda}$ is an $F_1$ by lemma 1, and since $H=\sum_{\lambda} E^{\lambda}=\sum_{n} E_n$, $H$ is also an $F_1$. The proof is thus completed for this case.

Consider next the case where each $E^{\lambda}$ is a $G_1$. Let $A^{\lambda}= O^{\lambda}-\sum_{\mu<\lambda} O^{\mu}-E^{\lambda}$. The set $A^{\lambda}$ is an $F_1$ and from what has already been proved $\sum_{\lambda} A^{\lambda}$ is an $F_1$. Since $H=\sum_{\lambda} E^{\lambda}=\sum_{\lambda} O^{\lambda}-\sum_{\lambda} A^{\lambda}$, $H$ is a $G_1$. This concludes the proof in case $\alpha=1$. The remainder of the proof is by transfinite induction and is so entirely analogous to the case already considered that details need not be given.

[1]) The symbol $d(x, y)$ denotes the distance from $x$ to $y$.

[2]) The classification of Borel sets used here is the one given by Kuratowski loc. cit. p. 160. See also Hausdorff, *Mengenlehre*, p. 178. An $F_0$ is a closed set, an $F_1$ is an $F_{\sigma}$ an $F_2$ is an $F_{\sigma\delta}$, a $G_0$ is an open set, a $G_1$ is a $G_{\delta}$ and so on.



The lemma obviously remains true when $\alpha=0$ for the $G_0$'s but simple examples show that it is not true for the $F_0$'s.

A set is said to be **additive** (**multiplicative**) of class $\alpha$ $(\alpha>0)$ if it is the sum (product) of an enumerable number of sets of class less than $\alpha$. When $\alpha=0$, the open and closed sets are additive and multiplicative of class $\alpha$ (For these definitions see Kuratowski loc. cit. p. 160). As a corollary to the preceding lemma, there is the following.

***Corollary.*** *If the sequence* (1) *is such that for each* $\lambda$, $O^{\lambda}-\sum_{\mu<\lambda} O^{\mu}$ *contains a set* $E^{\lambda}$ *which is additive (multiplicative) of class* $\alpha$ $(\alpha>0)$ *then the set* $H=\sum_{\lambda} E^{\lambda}$ *is additive (multiplicative) of class* $\alpha$.

The proof can be made by methods already used.

**Lemma 3.** *If the sequence* (1) *is such that for every* $\lambda$, $O^{\lambda}-\sum_{\mu<\lambda} O^{\mu}$ *contains a set* $E^{\lambda}$ *which is analytic, then* $H=\sum_{\lambda} E^{\lambda}$ *is analytic.*

From the definition of analytic sets,

$$E^{\lambda}=\sum_{(n_1, n_2, \ldots)} F_{n_1}^{\lambda} \cdot F_{n_1 n_2}^{\lambda} \cdot \ldots$$

where the summation is extended over all sequences of integers and where $F_{n_1 n_2 \ldots n_k}^{\lambda}$ is a closed set for every finite set of integers. Define the set $F_{n_1 n_2 \ldots n_k}$ as follows:

$$F_{n_1 n_2 \ldots n_k}=\sum_{\lambda}\Big[F_{n_1 n_2 \ldots n_k}^{\lambda} \cdot O^{\lambda}-\sum_{\mu<\lambda} O^{\mu}\Big].$$

From lemma 1 each of the sets defined in this manner is an $F_{\sigma}$. It can be verified that

$$H=\sum_{\lambda} E^{\lambda}=\sum_{(n_1, n_2, \ldots)} F_{n_1} \cdot F_{n_1 n_2} \cdot \ldots$$

and therefore $H$ is analytic[1].

[1]) Hausdorff, loc. cit. p. 92. Cf. W. Sierpiński *Introduction to General Topology*, Toronto 1934, p. 141—142.

2. A set $E$ is said to be *locally a $G_\alpha$ ($F_\alpha$) at a point $p$* if there is some open set including $p$ in which $E$ is a $G_\alpha$ ($F_\alpha$). A similar definition applies to the property of being analytic or the complement of an analytic set. In these various cases $p$ is said to be a $G_\alpha$ ($F_\alpha$) point of $E$, to be an analytic point of $E$ and so on.

**Theorem 1.** *If every point of $E$ is a $G_\alpha$ ($F_\alpha$) ($\alpha > 0$), point of $E$, then $E$ is a $G_\alpha$ ($F_\alpha$).*

Consider first the $F_\alpha$ case and well order all points of $E$ in the sequence (3):

$$p_1, p_2, \ldots, p_\lambda, p_{\lambda+1}, \ldots \tag{3}$$

About $p_1$ there is an open set $O^1$ such that $E \cdot O^1$ is an $F_\alpha$. Let $E \cdot O^1 = E^1$ and let $p_{n_2}$ be the first point of (3) not in $O^1$. About $p_{n_2}$ there is an open set $O^2$ such that $E \cdot O^2$ is an $F_\alpha$. Let $E^2 = E \cdot O^2 - O^1$. Suppose that there has been chosen an open set $O^\mu$ for all $\mu < \lambda$, and let $p_{n_\lambda}$ be the first point of (3) not in $\sum_{\mu<\lambda} O^\mu$. About $p_{n_\lambda}$ choose an open set $O^\lambda$ such that $E \cdot O^\lambda$ is an $F_\alpha$, and let $E^\lambda = E \cdot O^\lambda - \sum_{\mu<\lambda} O^\mu$. Continue in this manner until the points of (3) have been exhausted. Each of the sets $E^\lambda$ is an $F_\alpha$ and lemma 2 may be applied. The increasing series of open sets required by that lemma is here the sets $O^1_* = O^1$, $O^2_* = O^1 + O^2$, $O^\lambda_* = \sum_{\mu\leq\lambda} O^\mu$. It follows from the lemma, (since $E = \sum_\lambda E^\lambda$), that $E$ is an $F_\alpha$. The proof for the $G_\alpha$ case is exactly the same.

**Theorem 2.** *If every point of $E$ is an analytic point of $E$, then $E$ is an analytic set.*

This is proved from lemma 3 in exactly the same manner as the preceding theorem is proved from lemma 2.

A lemma similar to lemma 3 may be proved for sets which are complements of analytic sets and from it one may conclude that *if $E$ is a complement of an analytic set at each of its points, then it is a complement of an analytic set.*

3. Assume now that $X, Y, Z$, are metric spaces subject to no further restrictions and assume that $f(x, y)$ is defined on the product space [1]) $X \times Y$ and that the values of $f(x, y)$ are in the space $Z$.

[1]) Kuratowski, loc. cit. p. 87.



**Theorem 3.** *If $f(x, y)$ is continuous in $x$ and of class $\alpha$ in $y$, it is of class $\alpha + 1$ in $(x, y)$.*

Kuratowski (loc. cit. p. 181) has already considered the case where $\alpha = 0$, so that the proof will be given for the case where $\alpha > 0$. Let (4) be a well ordered series of the points of $X$

$$x_1, x_2, \ldots, x_\lambda, x_{\lambda+1}, \ldots \tag{4}$$

For $n$ fixed choose about $x_1$ an open set $O^1_n$ of diameter less than $1/n$. Let $x^{n_2}_n$ be the first point of (4) not in $O^1_n$. Continue in this manner until all points of $X$ are covered and suppose this covering made for every $n$. In order to prove the theorem it is necessary to show that all points $(x, y)$, such that $f(x, y)$ is in a set $F$ closed in $Z$, form a set which is multiplicative of class $\alpha + 1$. Let $F^k$ be all points of $Z$ whose distances from $F$ are $< 1/k$, and let $K^k$ be the set of all points $(x, y)$ such that $f(x, y)$ is in $F^k$. Also let $(x^{n_\lambda}_n \times Y) \cdot K^k = A^{nk}_{n_\lambda}$. This set is additive of class $\alpha$. Let $B^{nk}_{n_\lambda}$ be the set of all points $(x, y)$ such that $x$ is in $O^\lambda_n - \sum_{\mu<\lambda} O^\mu_n$ and such that $y$ is in the projection on $Y$ of $A^{nk}_{n_\lambda}$. The set $B^{nk}_{n_\lambda}$ is also additive of class $\alpha$. Since $B^{nk}_{n_\lambda}$ is in $(O^\lambda_n \times Y) - (\sum_{\mu<\lambda} O^\mu_n \times Y)$ and since $O^\mu_n \times Y$ is an open set, the corollary to lemma 2 may be applied, where the open sets required by the corollary are sums of the sets $O^\mu_n$. It follows that $\sum_\lambda B^{nk}_{n_\lambda}$ is additive of class $\alpha$. Let $B^{nk} = \sum_\lambda B^{nk}_{n_\lambda}$. If $(x, y)$ is in $B^{nk}$ then there is an $x^{n_\lambda}_n$ such that the distance from $x^{n_\lambda}_n$ to $x$ is less than $1/n$ and such that $(x^{n_\lambda}_n, y)$ is in $K^k$. Let

$$B^k = (B^{1k} + B^{2k} + B^{3k} + \ldots) \cdot (B^{2k} + B^{3k} + \ldots) \ldots .$$

If $(x, y)$ is in $B^k$ there is some point $x^1$ whose distance from $x$ is arbitrarily small and which is such that $(x^1, y)$ is in $K^k$. That is if $(x, y)$ is in $B^k$, $(x, y)$ is either in $K^k$ or else there is a sequence of $x^1$'s converging to $x$ such that $(x^1, y)$ is in $K^k$. Because of the continuity of $f(x, y)$ in $x$, $(X \times y) \cdot K^k$ is open in $(X \times y)$ for every $y$, so that if $(x, y)$ is in $K^k$ there is for every $N$ an $n > N$ and a $\lambda$ such that $(x^{n_\lambda}_n, y)$ is in $K^k$ and $x$ is in $O^\lambda_n$. Hence whenever $(x, y)$ is in $K^k$, it is in $B^{nk}_{n_\lambda}$ and therefore in $B^{nk}$. Since this is true for infinitely many $n$'s, $(x, y)$ must also be in $B^k$. This proves that $K^k \subset B^k$, and it follows that $\prod_k K^k \subset \prod_k B^k$. The converse relation of inclusion will now be demonstrated.

Suppose $(a, b)$ is in $B^k$. Let $a_i$ be a sequence of points in $X$ approaching $a$ and such that $f(a_i, b)$ is in $F^k$. That such a sequence exists was shown above. From the continuity of $f(x, y)$ in $x$ one concludes that $f(a, b)$ is in $\overline{F^k}$, so that if $(a, b)$ is in $B^k$ for all $k$, $f(a, b)$ is in $\overline{F^k}$ for all $k$. Therefore $f(a, b)$ is in $F$ and it has been demonstrated that $\prod_k K^k \supset \prod_k B^k$. From the two inequalities one concludes that $\prod_k K^k = \prod_k B^k$. But if $(x, y)$ is in $\prod_k K^k$, $f(x, y)$ is in $F$ and conversely. Therefore $\prod_k K^k = \prod_k B^k$ is the set whose nature concerns us. The set $B^{nk}$ has been shown to be additive of class $\alpha$, and by definition $B^k$ is multiplicative of class $\alpha + 1$. Therefore $\prod_k B^k$ is also multiplicative of the same class, and it follows that $f(x, y)$ is of class $\alpha + 1$. This completes the proof.

Assume now that $f(x)$ is a function defined on the space $X$ and having values in the space $Y$. The set of all points $(x, f(x))$ in $X \times Y$ is called the *image of* $f(x)$ and is denoted by $I$. In case $Y$ is separable and $f(x)$ is of class $\alpha$ it is known (Kuratowski loc. cit. p. 183) that $I$ is multiplicative of class $\alpha$. The same result is true when $Y$ is non-separable, as will now be shown.

***Theorem 4.*** *If $f(x)$ is of class $\alpha$, $I$ is multiplicative of class $\alpha$.*

As the theorem is obviously true in case $\alpha = 0$, it will now be assumed that $\alpha > 0$. Let (5) be a well ordered series of the points of $Y$

(5) $$y_1, y_2, \dots, y_\lambda, y_{\lambda+1}, \dots$$

Let $G^n_\lambda$ be the open sphere of center $y_\lambda$ and radius $1/n$, and let $E^n_\lambda$ be the set of points $x$ in $X$ such that $f(x)$ is in $\overline{G^n_\lambda}$. The set $E^n_\lambda$ is multiplicative of class $\alpha$. Let $H^n_\lambda = [E^n_\lambda \times (G^n_\lambda - \sum_{\mu<\lambda} G^n_\mu)]$ and let $O^n_\lambda = X \times \sum_{\mu\leqslant\lambda} G^n_\mu$. The sets $O^n_\lambda$ are increasing with $\lambda$ and are open. For each $\lambda$, $H^n_\lambda \subset O^n_\lambda - \sum_{\mu<\lambda} O^n_\mu$, so that the corollary to lemma 2 may be applied. From this corollary it follows that $H^n = \sum_\lambda H^n_\lambda$ is multiplicative of class $\alpha$; this is because $H^n_\lambda$ as the product of two sets each of which is multiplicative of class $\alpha$ is itself multiplicative of class $\alpha$. The set $I$ is in $H^n$. In order to show this let $(a, b)$ be any point of $I$. Let $\lambda$ be the first ordinal such that $G^n_\lambda$ contains $b$.



Then $G^n_\lambda - \sum_{\mu<\lambda} G^n_\mu$ must also contain $b$ and by definition $(a, b)$ must be in $H^n_\lambda$, and therefore in $H^n$. Therefore $(a, b)$ is in $\prod_n H^n$, and it follows that $I \subset \prod_n H^n$. It will now be shown that $\prod_n H^n \subset I$. Let $(c, d)$ be any point in $\prod_n H^n$. Then for every $n$ the distance from $(c, d)$ to $(c, f(c))$ is $\leqslant 1/n$. Hence $d = f(c)$ and $(c, d)$ is in $I$. It has now been shown that $I = \prod_n H^n$. Since $H^n$ is multiplicative of class $\alpha$ it follows that $I$ is the same. The theorem is thus seen to be true.

4. A set $E$ is said to be developable if it can be written in the form

(6) $$E = F_1 - F_2 + \dots + F_\lambda - F_{\lambda+1} + \dots$$

where the terms of the series are closed and decreasing, and where $F_\lambda$ is assumed to be positive if $\lambda$ is a limit ordinal. Kuratowski (p. 112) shows that *a developable set in a separable space is both an $F_\sigma$ and a $G_\delta$.* This fact remains true when the space is *arbitrary.* In order to show this it may be assumed that the last term of (6) in case a last term exists, is negative for if there is a last term and it is positive another term which is the null set may be added and given a negative sign. Let $\beta_1, \beta_2, \beta_3, \dots$ be the indices of the positive terms in (6) written in increasing order. Let $O_{\beta_\lambda}$ be the complement of $F_{\beta_{\lambda}+1}$ and let $F^*_{\beta_\lambda} = F_{\beta_\lambda} \cdot O_{\beta_\lambda} - \sum_{\mu<\lambda} O_{\beta_\mu}$. To this series of sets lemma 2 may be applied and it follows that $\sum_\lambda F^*_{\beta_\lambda}$, which is equal to $E$, is an $F_\sigma$. Since the complement of $E$ is developable, $CE$ must also be an $F_\sigma$ and therefore $E$ is a $G_\delta$.

Therefore *in complete metric spaces sets which are $F_\sigma$'s and $G_\delta$'s coincide with developable sets* (See Kuratowski loc. cit. p. 207). It now follows that *a function $f(x)$ which is pointwise discontinuous on every closed set is a function of class* 1, because the proof given by Kuratowski (loc. cit. p. 190) is now valid for the more general case.

The following question is an interesting one: If $f(x)$ is of class 1 (as defined by Kuratowski) on an arbitrary complete metric space and has values in an arbitrary metric space, must it have points of continuity? I was unable to answer this question.