est égal à *zero* ou à *un*, on serait amené à considérer des séries telles que les suivantes, pour $s_0 = 1$, en posant $l_k = \frac{1}{\lambda^k}$:

$$\sum l_k (\log \lambda_k)^n,$$

$$\sum l_k (\log \log \lambda_k)^n,$$

et dans le cas où $s_0 = 0$, des séries telles que les suivantes:

$$\sum \frac{1}{(\log \lambda_k)^n},$$

$$\sum \frac{1}{(\log \log \lambda_k)^n},$$

séries qui sont données seulement à titre d'exemple, car la variété des possibilités est considérable. Il y a un très vaste champ de recherches à explorer.

Signalons enfin, pour mémoire, l'étude des cas où la croissance des fonctions considérées ne serait pas *régulière*, c'est-à-dire en relation simple avec la fonction exponentielle.

Les ensembles définis par les intervalles fondamentaux normaux rattachés à des ensembles énumérables classiques sont assurément fort particuliers; ils donnent toutefois des exemples en fait fort généraux et permettent d'étudier très complètement la classification des ensembles de mesure nulle. Je souhaite que les brèves indications qui précèdent encouragent à poursuivre cette étude ceux qui s'intéressent à la théorie des ensembles.

A set of axioms for plane analysis situs.

By

R. L. Moore (Austin).

As far as I know, Veblen was the first to conceive the idea of basing geometry on a set of axioms in terms of what he called „chunks" (of space). He had this idea as early as 1905 and discussed it with me at that time. In 1913 Huntington published a paper [1]) in which he founded geometry on a set of postulates in terms of „sphere" and „inclusion"; and Whitehead and Nicod have given some thought to related questions.

Huntington makes much use of what may be termed the convexity of his undefined spatial elements. It is natural that he should do so in founding a *geometry*. The notion of the convexity of these elements is intimately [2]) related to the notion *straight line* which is one of the fundamental elements of *geometry*.

For many years I have endeavored to found point-set theoretic *analysis situs* on the basis of a set of axioms in terms of the notion of what I shall call „piece" and the relation „embedded in". In so far as I was endeavoring to found *analysis situs*, as opposed to *geometry*, I would naturally avoid postulating that these pieces be convex, the notions convexity and colinearity being foreign to analysis situs. In April, 1930, I delivered [3]), at The University of Texas, a set of five lectures in which I dealt with a set of Axioms in

[1]) *A set of postulates for abstract geometry, expressed in terms of the simple relation of inclusion,* Math. Ann., Bd. LXXIII, 522—559.

[2]) In order that three given points of a Euclidean space should be colinear it is necessary and sufficient that they may be lettered A, B and C in such a way that B is contained in every convex region that contains both A and C.

[3]) As University of Texas research lecturer for the session 1929—1930.

terms of *piece* and *embedded in.* On November 18, 1931, at a meeting of the American Mathematical Society at Pasadena, California, I delivered an address dealing with the same set of axioms except, possibly, for more or less minor modifications. This set included a set [1]) of axioms which (except possibly for minor considerations) was identical with the set consisting of Axioms 2, 3, 4 and 5 and that portion of Axiom 1, of the present paper, which remains on the removal of the statement numbered (6). To these were added additional axioms limiting space to being topologically equivalent either to the plane or to a sphere. I was, however, not well satisfied with some of these additional axioms and it is only recently that I have obtained the formulations given below of Axioms 7 and 8 and the proposition labelled (6) in the statement of Axiom 1. In connection with Axiom 8, I am much indebted to Kuratowski's paper „Une caractérisation topologique de la surface de la sphère" [2]). This axiom is formulated in such a way as to enable me to prove Theorem 4 of Section III by an argument closely related to that given by Kuratowski, on pages 311 and 312 of his paper, to prove the proposition that $\mathcal{S}_1 \rightarrow \mathcal{S}$.

The set of Axioms 1—8 is satisfied if, speaking *roughly*, the word „piece" is interpreted to mean any limited piece (in the ordinary [3]) sense) of the plane and the word „embedded" is interpreted in a natural manner. In such an interpretation, a *piece* of a plane is not regarded as a set of points.

Speaking *accurately*, if S is a plane, Axioms 1—8 are satisfied if the word „piece" is interpreted to mean any [4]) connected and limited domain in S (regardless of what sort of boundary it has) and the piece x is said to be embedded in the piece y if, and only if, y contains x together with its boundary.

If a space satisfies Axioms 1—8 and the terms *point* and *limit point* are defined as described in this paper then the set of all points is topologically equivalent either to a plane or to the surface of a sphere.

[1]) Every space satisfying these axioms is topologically equivalent to a connected, locally connected and complete metric space having no cut point.

[2]) Fund. Math. XIII, 1929, pp. 307—318.

[3]) It would seem to be accordance with ordinary usage to refrain from applying the term „piece of space" to a point or a straight line or anything else which has no interior.

[4]) These axioms are also satisfied if, for example, the word „piece" is restricted to those limited and connected domains of the plane whose boundaries are indecomposable continua.

Notation. For each i less than 8, the notation Axiom 1_i will be used to denote the axiom obtained by retaining only that part of the statement of Axiom 1 remaining after the deletion of the phrases numbered from $(i+1)$ to (7) inclusive that occur in its formulation.

I. Consequences of Axioms 1_4, 2 and 3.

Definitions. The piece x is said to *embed* the piece y if y is embedded in x. The piece x is said to *intersect* the piece y if there exists a piece which is embedded both in x and in y. The piece x is said to be *detached* from the piece y if one of the pieces x and y is embedded in a piece that does not intersect the other one. Under these conditions x and y are also said to be *mutually detached* (and the word „mutually" may sometimes be omitted). If G and H are finite sets of pieces, G is said to be *detached* from H (and G and H are said to be *mutually detached*), if every piece of the set G is detached from every piece of the set H. A set of pieces is said to be *coherent* if it is not the sum of two sets such that no piece of one of these sets intersects any piece of the other one. The set W of pieces is said to *separate* the piece h from the piece k if every coherent set of pieces that contains h and k contains a piece that intersects some of piece of W.

Axiom 1. *There exists a sequence G_1, G_2, $G_3, \ldots$ such that*

(1) *for each n, G_n is a set of pieces,*

(2) *for each n, G_{n+1} is a subset of G_n,*

(3) *if g is a piece then, for each n, there is a piece belonging to G_n and embedded in g,*

(4) *if the piece k is embedded in the piece h there exists a number m such that every piece that belongs to G_m and intersects k is embedded in h,*

(5) *if g is a piece and n is a natural number the set of all pieces of the set G_n that are embedded in g is coherent,*

(6) *if g is a piece and n is a natural number there exist a natural number m and a finite subset H_n of G_n such that if x is a piece of the set G_m which intersects g without being embedded in it then x is embedded in some piece of the set H_n,*

(7) *the statement labelled* (6) *remains true if the phrase „without being embedded in it" is omitted.*

Axiom 2. *If the piece g is embedded in the piece h and h is embedded in the piece k then g is embedded in k.*

Axiom 3. *If the piece g is embedded in the piece h then h is not embedded in g.*

Definition. If $g_1, g_2, g_3, \ldots$ is a sequence of pieces such that, for each n, g_n belongs to G_n and g_{n+1} is embedded in g_n, then the set of all pieces x such that some piece of this sequence is embedded in x is called a *point* and the sequence $g_1, g_2, g_3, \ldots$ is said to *determine* this point.

Definition. The point 0 is said to *pertain* to the piece g if some piece of the set 0 is embedded in g. In other words, 0 is said to pertain to g if g belongs to 0.

Theorem 1. *If W is a point and α is a sequence $h_1, h_2, h_3, \ldots$ such that, for each n, h_n belongs both to W and to G_n, and h_{n+1} is embedded in h_n then α determines W.*

Proof. Let β denote a sequence of pieces $g_1, g_2, g_3, \ldots$ which determines W. Let x denote a piece of the set W. There exists a number n such that g_n is embedded in x. There exists a number δ such that every piece of G_δ that intersects g_n is embedded in x. But h_δ belongs to W and therefore intersects h_δ. Hence h_δ is embedded in x. Thus every piece of the set W embeds some piece of the sequence α. Suppose y is a piece that embeds some piece h_m of α. Since h_m belongs to W it embeds some piece g_i of β and therefore y embeds g_i and consequently belongs to W. Thus W is the set of all pieces w such that w embeds some piece of the sequence α. It follows that α determines W.

Theorem 2. *If A and B are distinct points, not every piece belonging to A intersects every one that belongs to B.*

Proof. Let α and β denote sequences $g_1, g_2, g_3, \ldots$ and $h_1, h_2, h_3, \ldots$ determining A and B respectively and therefore such that, for each n, g_n and h_n belong to G_n. Since A and B are distinct either there is a piece which embeds some piece of α but which embeds no piece of β or there is a piece which embeds some piece of β but no piece of α. Suppose there is a piece k which embeds g_m but no piece of β. There exists a number n such that k embeds every piece of the set G_n that intersects g_m. The piece h_n does not intersect g_m.

Theorem 3. *If the point 0 pertains both to the piece x and to the piece y then some piece of the set 0 is embedded both in x and in y.*

Proof. Let $g_1, g_2, g_3, \ldots$ denote a sequence of pieces that determines 0. There exist numbers m and n such that g_m and g_n are embedded in x and y respectively. The piece g_{m+n} belongs to 0 and is embedded both in x and in y.

Theorem 4. *If $P_1, P_2, P_3, \ldots, P_n$ is a finite set of distinct points there exists a set of n mutually detached pieces $x_1, x_2, \ldots, x_n$ such that, for each k less than or equal to n, P_k pertains to x_k.*

Proof. If i and j are any two natural numbers less than or equal to n and such that $i < j$ then, by Theorem 2, there exist two non-intersecting pieces $x_{i,j}$ and $y_{i,j}$ belonging to P_i and P_j respectively. By Theorem 3 and Axiom 2, for every k not greater than n there exists a piece x_k belonging to P_k and embedded in all the $n-1$ pieces $y_{1,k}, y_{2,k}, \ldots, y_{k-1,k}, x_{k,k+1}, x_{k,k+2}, \ldots, x_{k,n}$. The pieces $x_1, x_2, x_3, \ldots, x_n$ are mutually detached.

Theorem 5. *If A and B are distinct points pertaining to the piece p, there exist two non-intersecting pieces both embedded in p and belonging to the sets A and B respectively.*

Proof. Let α and β denote sequences $g_1, g_2, g_3, \ldots$ and $h_1, h_2, h_3, \ldots$ that determine A and B respectively. By hypothesis there exist two numbers m_1 and n_1 such that g_{m_1} and h_{n_1} are both embedded in p. By Theorem 2 there exist two non-intersecting pieces x and y belonging to A and B respectively. There exist numbers m_2 and n_2 such that g_{m_2} and h_{m_2} are embedded in x and y respectively. The pieces $g_{m_1+m_2}$ and $h_{n_1+n_2}$ are non-intersecting and they are both embedded in p.

Theorem 6. *If r is a piece there is at least one point pertaining to r.*

Proof. By Axiom 1_3 there exist a piece g_1 belonging to G_1 and embedded in r, a piece g_2 belonging to G_2 and embedded in g_1 and so on. It follows that there exists a sequence $g_1, g_2, g_3, \dots$ such that g_1 is embedded in r and, for each n, g_n belongs to G_n, and g_{n+1} is embedded in g_n. This sequence determines a point pertaining to r.

Definition. If r is a piece, the set of all points that pertain to r is called a *region.*

Convention. If a certain letter is used to denote a piece, the same letter primed will be used to denote the region consisting of all the points that pertain to that piece and if a letter is used to denote a set of pieces then that letter primed will be used to denote the set of all regions x' such that, for some piece x of that set of pieces, x' is the set of all points that pertain to x.

Theorem 7. *If the piece q is embedded in the piece r then the region q' is a subset of the region r'.*

Theorem 8. *If A and B are distinct points there exist two regions containing A and B respectively and having no point in common.*

Theorem 8 may be proved with the aid of Theorem 2.

Theorem 9. *If q and r are pieces and the regions q' and r' have a point 0 in common there exists a piece k such that the region k' contains 0 and is a subset both of q' and of r' and such that, furthermore, the piece k is embedded both in q and in r.*

Theorem 9 may be proved with the help of Theorem 3.

Theorem 10. *If A and B are distinct points of the region R there exist two mutually exclusive regions both subsets of R and one containing A and the other containing B.*

Theorem 10 may be proved with the aid of Theorem 5.

Definition. The point P is said to be a *limit point* of the point set M if, and only if, every region that contains P contains a point of M distinct from P.

Definition. The point P is said to be a *boundary point* of the point set M if every region that contains P contains both a point of M and a point not belonging to M. The point set consisting of all the boundary points of M is called the *boundary* of M.

Notation. If M is a point set the notation $\overline{M}$ will be used to denote the point set consisting of M together with its boundary.

Theorem 11. *If the piece q is embedded in the piece p then $\overline{q}'$ is a subset of p'.*

P r o o f. By Theorem 7, q' is a subset of p'. Suppose now that X is a limit point of q'. Suppose r is a piece to which X pertains. Then the region r' contains a point belonging to q'. Hence, by Theorem 9, there exists a piece k which is embedded both in r and in q. Thus q intersects every piece to which X pertains. Hence if $g_1, g_2, g_3, \dots$ is a sequence of pieces which determines X, there exists a number n such that g_n is embedded in p. Hence X belongs to p'.

Definition. Two point sets are said to *intersect* if they have at least one point in common.

Definition. The *region* R will be said to be *embedded* in the region K if, and only if, there exist two pieces x and y such that x is embedded in y and such that $x' = R$ and $y' = K$.

Notation. The set of all points will be denoted by the letter S.

Theorem 12. *There exists a sequence $Q_1, Q_2, Q_3, \dots$ such that*
(1) *for each n, Q_n is a collection of regions covering S,*
(2) *for each n, Q_{n+1} is a subcollection of Q_n,*
(3) *if X is a point of a region R there exists a number m such that if x and y are intersecting regions of the collection Q_m and x contains X then y is a subset of R,*
(4) *if $M_1, M_2, M_3, \dots$ is a sequence of closed point sets such that, for each n, M_n contains M_{n+1} and, for each n, there exists a region q_n of the collection Q_n such that M_n is a subset of $\overline{q}_n$, then there is at least one point common to all the point sets of the sequence $M_1, M_2, \dots$*

P r o o f. Let Q_1 denote the set of all regions x of G_2' such that x is embedded in some region of G_1'. Let Q_2 denote the set of all regions x of G_3' such that x is embedded in some region of Q_1. This process may be continued. Thus there exists a sequence $Q_1, Q_2, Q_3, \dots$ such that Q_1 is as described and, for each n, Q_{n+1} is the set of all regions x of G_{n+2}' such that x is embedded in some region of Q_n. For each n, the collection Q_{n+1} covers S and is a subcollection of Q_n.

Suppose X is a point of a region R. There exists a piece K such that $K' = R$. Let $t_1, t_2, t_3, \dots$ denote a sequence of pieces that determines X. There exists a number k such that t_k is embedded in the piece K. The piece t_{k+1} is embedded in t_k. Hence there exist numbers n_1 and n_2 such that t_k embeds every piece of the set G_{n_1} that intersects t_{k+1} and K embeds every piece of G_{n_2} that intersects t_k. Let m denote $n_1 + n_2$. If p and q are intersecting pieces of the set G_m and X pertains to p then q is embedded in K. It follows that if p' and q' are intersecting regions of the set G'_m and p' contains X then $\bar{q}'$ is a subset of K'. But, for every n, the set Q_n is a subset of G'_n. It follows that if x and y are intersecting regions of Q_m and x contains X then $\bar{y}$ is a subset of R.

Suppose $q_1, q_2, q_3, \ldots$ is a sequence of regions such that, for each n, q_n belongs to Q_n and such that if i and j are any two natural numbers then q_i and q_j have at least one point in common. The region q_1 is embedded in some region g_1 belonging to G'_1. There exists a number n_1 such that g_1 embeds every region of G'_{n_1} that intersects q_1. The region q_{n_1} intersects q_1 and is embedded in some region g_{n_1} of G'_{n_1}. The region g_{n_1} is embedded in g_1. There exists a number n_2 such that g_{n_1} embeds every region of G'_{n_2} that intersects q_{n_1}. The region q_{n_2} intersects q_{n_1} and is embedded in some region g_{n_2} of G'_{n_2}. The region g_{n_2} is embedded in g_{n_1}. This process may be continued. Thus there exist an ascending sequence of natural numbers $n_1, n_2, n_3, \ldots$ and a sequence $g_{n_1}, g_{n_2}, g_{n_3}, \ldots$ of regions such that, for each i, (1) g_{n_i} belongs to G'_{n_i} and (2) $g_{n_{i+1}}$ and q_{n_i} are both embedded in g_{n_i}. There exists one, and only one, point P common to all the regions $g_{n_1}, g_{n_2}, g_{n_3}, \ldots$ Suppose m is a number and R is a region containing P. There exists a number k such that g_{n_k} is a subset of R. The region q_{n_k} is a subset of g_{n_k} and therefore of R. But q_m has a point in common with q_{n_k}. Hence R contains a point of q_m. Thus, for every m, P belongs to $\bar{q}_m$.

Suppose now that $M_1, M_2, M_3, \ldots$ is a sequence of closed point sets such that, for each n, M_n contains M_{n+1} and, for each n, there exists a region q_n of the set Q_n such that M_n is a subset of $\bar{q}_n$. For each n, there exist regions g_n and h_n belonging to Q_n and such that $\bar{q}_{n+2}$ is a subset of g_n and $\bar{g}_n$ is a subset of h_n. There exists a point P common to all the point sets $\bar{g}_1, \bar{g}_2, \bar{g}_3, \ldots$ and therefore to all the regions $h_1, h_2, h_3, \ldots$ Suppose there exists a number k such that P does not belong to M_k. Then there exists a region R containing P but no point of M_k. There exists a number m such that R embeds every region of Q_m that contains P. But h_m belongs to Q_m and contains P. Therefore h_m is a subset of R. But q_{m+2} is a subset of h_m. Therefore q_{m+2} contains no point of M_k. Thus the supposition that P does not belong to all the point sets of the sequence $M_1, M_2, M_3, \ldots$ has led to a contradiction.

The truth of Theorem 12 has now been established.

With the aid of Theorems 10 and 12 it easily follows that Axiom 1 of my book Foundations of Point Set Theory [1]) holds true here. Furthermore space is metric. To see this proceed exactly as in the proof of Theorem 26 of Chapter VII of this book, except for the substitution of the words „Theorem 12 of the present paper“ in place of the words „Theorem 74 of Chapter I, as being satisfied with respect to M^u“. In this proof no use is made of anything in Theorem 74 beyond the result stated under (3) in the formulation of Theorem 12 of the present paper.

II. Consequences of Axioms 1_5, 2 and 3.

Definition. If h and k are pieces, a *simple chain* from h to k is a finite sequence of pieces $h_1, h_2, h_3, \ldots, h_n$ such that $h_1 = h$ and $h_n = k$ and such that if $1 \leqq i \leqq n$, $1 \leqq j \leqq n$ and $i < j$ then h_i intersects h_j if, and only if, $j = i + 1$. The piece h_k $(1 \leqq k \leqq n)$ is said to be the *k th link* of the chain $h_1, h_2, h_3, \ldots, h_n$.

Theorem 1. *If h and k are two pieces belonging to the coherent collection W there is a simple chain from h to k such that every link of this chain is a piece belonging to W.*

Proof. Suppose there is no such simple chain. Let W_h denote the set of all pieces x of the set W such that there is a simple chain, from h to x, whose links all belong to W; and let W_k denote the set $W - W_h$. Since W is coherent there exists a piece y_h belonging to W_h and intersecting some piece y_k of the set W_k. There exists a simple chain $h_1, h_2, h_3, \ldots, h_n$ whose links all belong to W and whose first and last links are h and y_k respectively. If h_m is the first link of this chain which intersects y_k then $h_1, h_2, h_3, \ldots, h_m, y_k$ is a simple chain from h to y_k. But this involves a contradiction.

[1]) American Mathematical Society Colloquium Publications, volume XIII, New York, 1932. Hereafter, in this paper, this book will be referred to as *Point Set Theory*.

Theorem 2. *If, for each n, H_n is a finite set of pieces and, for each n, each piece of the set H_{n+1} is embedded in some piece of the set H_n, then there exists a sequence of pieces $h_1, h_2, h_3, \ldots$ such that, for each n, h_n belongs to H_n and h_{n+1} is embedded in h_n.*

Theorem 2 may be proved by an argument closely analogous to that employed to establish Theorem 78 of Chapter I of Point Set Theory.

Notation. If a letter is used to denote a collection of point sets, then that letter with an asterisk as a suffix will be used to denote the sum of all the point sets of this collection.

Theorem 3. *If $H_1, H_2, H_3, \ldots$ is a sequence such that, for each n,*
(1) *H_n is a finite subset of G_n and*
(2) *each piece that belongs to H_{n+1} is embedded in some piece that belongs to H_n;*
then there exists at least one point common to all the point sets $H_1'^*, H_2'^*, H_3'^*, \ldots$ and K, the set of all their common points, is a compact closed point set and if, for each n, P_n is a point of $H_n'^*$, some subsequence of the sequence $P_1, P_2, P_3, \ldots$ converges to a point of K.

Theorem 3 may be proved by an argument involving reasoning closely related to that employed in portions of the proof of Theorem 79 in Chapter I of Point Set Theory.

Theorem 4. *If D is a point set and H is a coherent set of regions and some region of the set H contains a point of D and some region of H contains a point of $S-D$, then there exists a region of D which contains both a point of D and a point of $S-D$.*

Proof. Suppose, on the contrary, that H is the sum of two collections H_1 and H_2 such that no region of H_1 contains a point of $S-D$ and no region of H_2 contains a point of D. Then no region of H_1 intersects any region of H_2 contrary to the hypothesis that H is coherent.

Theorem 5. *If, under the same hypothesis as in Theorem 3, each set of the sequence $H_1, H_2, H_3, \ldots$ is coherent then the common part of the point sets $H_1'^*, H_2'^*, H_3'^*, \ldots$ is connected.*

Proof. Let K denote the common part of these point sets. Suppose K is not a continuum. Then, since by Theorem 3, K is closed and compact, it is the sum of two mutually exclusive closed and compact point sets T and L. There exists a domain D containing T and such that $\bar{D}$ contains no point of L. For each n, H_n' is a coherent set of regions and some region of H_n' contains a point of D and some region of H_n' contains a point of $S-D$. Therefore, by Theorem 4, for each n, some region of H_n' contains both a point of D and a point not belonging to D. For each n, let W_n denote the set of all those that do so. Then, for each n, W_n is a finite subset of G_n' and, for each n, each region of W_{n+1} is embedded in some region of W_n. It follows, with the help of Theorem 2, that there is a point P common to all the point sets of the sequence $W_1'^*, W_2'^*, W_3'^*, \ldots$ The point P belongs to $\bar{D}-D$ and therefore neither to T nor to L. But it belongs to K. Thus the supposition that K is not a continuum leads to a contradiction.

Theorem 6. *Every region is arcwise connected.*

With the help of Theorems 1, 2, 3 and 5, Theorem 6 may be proved by an argument closely analogous to that used to prove Theorem 1 in Chapter II, with the aid of Theorems 77, 78, 79 and 80 of Chapter II, of Point Set Theory.

It is clear that Axioms 1 and 2 of Point Set Theory are consequences of Theorems 10 and 12 of Section I, and Theorem 6 of this Section, of the present paper. Therefore from Axioms 1_5, 2 and 3 of this paper follow all the numbered theorems of Chapters I and II of Point Set Theory except, of course, in those cases where such theorems are not [1]) properly stated.

[1]) In Chapter I write „completely" before „separable" in the statement of Theorem 72.

In Chapter II, in the statement of Theorem 25, in line 4 from the bottom of the page, replace „every" by „if T is a"and, in line 3 from the bottom, write „then $\bar{T}$" after „$(M-L\cdot M)$". In Theorem 36, after „If" write „S is connected,". In Theorem 60 substitute „point" for „dendron". In the statement of „Theorem 73", between „connected" and „there", interpolate „and no point of $K-\boldsymbol{B}$ is a limit point of the component of $S-K$ that contains A".

In Chapter IV after the last „G" in Theorem 113, write „or by 0 plus any one of these sets that contains a point of 0".

In Chapter V write „compact" before „element" in Theorem 9 and, in Theorems 9′ and 10, after „If" write „the elements of the collection G are all compact and". In Theorem 13, replace „continuum" by „closed point set".

In Chapter VII, in Theorem 8, write „and compact" before „point sets" in line 2 and „compact" before continua in line 4. In numerous places, in this chapter, write „reversibly" before „continuous".

III. Consequences of Axioms 1_5, 2, 3, 4, 5, 6 and 7.

Axiom 4. *There exists three mutually non-intersecting pieces.*

Axiom 5. *If a and b are detached pieces and $p_1, p_2, p_3, \ldots$ is a sequence of pieces such that*

(1) *for each n, p_{n+1} is embedded in p_n and*

(2) *for each n, p_n separates a from b,*

then there exist two pieces x and y such that x is embedded in y and such that every piece of the sequence $p_1, p_2, p_3, \ldots$ intersects x but no one of them is embedded in y.

Theorem 1. *No point separates*[1]) *S.*

Proof. Suppose A, B and P are three distinct points. By Theorem 4 of Section I, there exist three mutually detached pieces a, b and p belonging to A, B and P respectively. The point P is determined by a sequence of pieces $p_1, p_2, p_3, \ldots$ all embedded in p. If x and y are two pieces such that x is embedded in y and intersects every piece of this sequence then there are not more than a finite number of the pieces of this sequence that are not embedded in y. Hence, by Axiom 5, there exists a number m such that p_m does not separate a from b. Hence there exists a coherent set H of pieces such that both a and b belong to H but p does not intersect any piece belonging to H. With the help of Theorem 6 of Section II, it is easy to se that the point set H'^* is a connected subset of $S-P$. But it con-tains both A and B.

Theorem 2. *The set of all points is connected.*

Proof. Suppose A and B are two distinct points. By Axiom 4 there exist three mutually non-intersecting pieces. Hence there exists a piece γ belonging neither to A nor to B. By Theorem 6 of Section I, there exists a point P pertaining to γ. By Theorem 1, $S-P$ is connected. Thus if A and B are any two points there is a connected point set containing both A and B. Therefore S is connected.

Axiom 6. *If U and V are finite collections of pieces and $U\cdot V$ is a coherent collection and the collections $U-U\cdot V$ and $V-U\cdot V$*

[1]) The subset K of the point set M is said to *separate* M if $M-K$ is not connected.

are mutually detached and neither U nor V separates the piece α from the piece β then the collection $U+V$ does not separate α from β.

Theorem 3. *If H and K are two closed and compact point sets and $H\cdot K$ is connected and neither H nor K separates the point A from the point B then $H+K$ does not separate A from B.*

Proof. If A or B belongs to H or to K then clearly $H+K$ does not separate A from B. Suppose neither A nor B belongs to $H+K$. There exist arcs AXB and AYB such that AXB contains no point of H and AYB contains no point of K. There exist pieces α and β belonging to A and B respectively but such that neither $\bar\alpha'$ non $\bar\beta'$ contains any point of $H+K$. There exists a finite collection T of pieces such that T' properly[1]) covers $H\cdot K$ and such that $\bar T'$ has no point in common with $AXB+AYB$, and T is detached from α and from β. Let H_1 and K_1 respectively denote the closed point sets $H-H\cdot T'^*$ and $K-K\cdot T'^*$. There exist two mutually detached finite collections Q_H and Q_K of pieces such that (1) Q'_H and Q'_K properly cover $\bar H_1$ and $\bar K_1$ respectively and (2) $(K+AXB)\cdot \bar Q'_H$ and $(H+AYB)\cdot \bar Q'_K$ are vacuous. There exist finite collections L_x and L_y of pieces such that (1) L'_x and L'_y properly cover AXB and AYB respectively, (2) α and β belong both to L_x and to L_y, (3) L_x and Q_H+T are mutually detached and so are L_y and Q_K+T. The common part of the collections Q_H+T and Q_K+T is the coherent collection T and $(Q_H+T)-T$ and $(Q_K+T)-T$ are the mutually detached collections Q_H and Q_K. Furthermore, the coherent collection L_x contains α and β but is detached from Q_H+T, and L_y is a coherent collection containing α and β but detached from Q_K+T. Therefore neither Q_H+T nor Q_K+T separates α from β. It follows, by Axiom 6, that $(Q_H+T)+(Q_K+T)$ does not separate α from β. Hence $(Q'_H+T')+(Q'_K+T')$ does not separate α' from β'. But $H+K$ is a subset of $(Q'_H+T')+(Q'_K+T')$ and α' and β' are connected point sets containing A and B respectively. Therefore $H+K$ does not separate A from B.

[1]) The set W of point sets is said to properly cover the point set M if every point of M belongs to some point set of W and *every point set of W contains some point of M.*

Axiom 7. *If G and H are finite and mutually detached collections of pieces and g and h are pieces belonging to G and H respectively then there exists a finite collection V of pieces such that*

(1) *V separates g from h,*

(2) *V and $G+H$ are mutually detached,*

(3) *if U and W are finite and mutually detached collections of pieces there exists a finite and coherent collection Z of pieces such that*

(a) *each piece of the collection Z is embedded in some piece of V*

(b) *Z separates g from h and*

(c) *every piece of the collection Z which is not detached from U is detached from every piece of Z which is not detached from W.*

Theorem 4. *If the common part of two compact continua exists and is not connected then their sum separates S.*

Proof. Suppose α and β are two compact continua such that $\alpha\cdot\beta$ is the sum of two mutually exclusive closed point sets γ and δ. There exist finite collections H_α, H_β, H_γ and H_δ of pieces such that (1) H'_α, H'_β, H'_γ and H'_δ cover α, β, γ and δ respectively, (2) H_γ and H_δ are mutually detached, (3) $H_\alpha\cdot H_\beta=H_\gamma+H_\delta$ and (4) $H_\alpha-(H_\gamma+H_\delta)$ and $H_\beta-(H_\gamma+H_\delta)$ are mutually detached. Let P_γ and P_δ denote points belonging to γ and δ respectively and let h_γ and h_δ denote definite pieces belonging respectively to H_γ and H_δ and also belonging respectively to P_γ and P_δ. By Axiom 7 there exists a finite set V of pieces such that (1) V separates h_γ from h_δ, (2) V and $H_\gamma+H_\delta$ are mutually detached and (3) if U denotes the set of all pieces x of H_α such that x intersects some piece of V and W denotes the set of all pieces y of H_β such that y intersects some piece of V then there exists a finite collection Z of pieces such that (a) each piece of the collection Z is embedded in some piece of V, (b) Z separates h_γ from h_δ and (c) every piece of Z which is not detached from U is detached from every piece of Z which is not detached from W. Let Z_α denote the collection of all pieces of Z which are not detached from H_α and let Z_β denote the collection of all pieces of Z which are not detached from H_β. Let $z_1, z_2, z_3,\ldots$ denote the pieces of the collection Z which belong neither to Z_α nor to Z_β

Suppose now that $\alpha+\beta$ does not separate S. For each i less than n, there exists an arc t_i joining some point of z'_i to some point of z'_{i+1} and containing no point of $\alpha+\beta$. For each i less than n there exists a finite collection T_n of pieces such that (1) T'_n covers t_n properly, (2) no point of $\alpha+\beta$ belongs to, or is on the boundary of, any region of the set T'_n. Let E denote the collection $z_1+z_2+\ldots+z_n+T_1+T_2+\ldots+T_n$. There exist pieces x_γ and x_δ belonging to P_γ and P_δ respectively, embedded in h_γ and h_δ respectively and detached from E. There exist finite and coherent collections I_α and I_β of pieces such that (1) I_α and I'_β cover α and β respectively, (2) every piece of I_α is embedded in some piece of H_α and every piece of I_β is embedded in some piece of H_β, (3) both I_α and I_β are detached from E, (4) x_γ and x_δ belong to I_α and also to I_β. The finite collection I_α is coherent, contains both x_γ and x_δ and is detached from $E+Z_\beta$; and the finite and coherent collection I_β contains both x_γ and x_δ and is detached from $E+Z_\alpha$. Therefore neither $E+Z_\alpha$ nor $E+Z_\beta$ separates x_γ from x_δ. But the collection E is coherent and Z_α is detached from Z_β. Therefore, by Axiom 6, the collection $E+Z_\alpha+Z_\beta$ does not separate x_γ from x_δ. But x_γ and x_δ are detached from $E+Z_\alpha+Z_\beta$, Z is a subcollection of $E+Z_\alpha+Z_\beta$ and Z separates x_γ from x_δ. Thus the supposition that Theorem 4 is false has led to a contradiction.

Theorem 5. *If J is a simple closed curve, $S-J$ is the sum of two mutually separated connected point sets such that J is the boundary of each of them*

Theorem 5 may be established by an argument identical, except for obvious modifications, with that employed by Kuratowski, on pages 313 and 314 of the article cited above, to prove the proposition 1⁰.

IV. Consequences of Axioms 1_6, 2, 3, 4, 5, 6, and 7.

Theorem 1. *If a region has a boundary its boundary is compact.*

Proof. Suppose g is a piece such that g' has a non-vacuous boundary T. By Axiom 1_6 there exist a natural number m_1 (greater than 1) and a finite subset H_1 of G_1 such that if x is a piece of the set G_{m_1} which intersects g, without being embedded in it, then x is embedded in some piece of H_1. Similarly there exist a natural number m_2 (greater than m_1) and a finite subset H_2 of G_{m_1} such that if x is a piece of the set G_{m_2} which intersects g, without being embedded in it, then x is embedded in some piece of the set H_2. This process may be continued indefinitely. Thus there exist an ascending sequence $m_1, m_2, m_3, \dots$ of natural numbers and a sequence $H_1, H_2, H_3, \dots$ such that (1) H_1 is a finite subset of G_1 and, for each i greater than 1, H_i is a finite subset of $G_{m_{i-1}}$, (2) for each i, every piece of the set G_{m_i} which intersects g, without being embedded in it, is embedded in some piece of the set H_i. For each i, let W_i denote the set of all pieces of the set H_i that intersect g without being embedded in it. For each i, each piece of the set W_{i+1} is embedded in some piece of W_i. It follows, by Theorem 3 of Section II, that if K denotes the set of all points X such that X is common to all the point sets $W_1'^*$, $W_2'^*$, $W_3'^*, \dots$ then K is compact. But K is identical with T.

It has been shown that a space satisfying Axioms 1_4, 2 and 3 is metric. Now Mr. F. B. Jones of the University of Texas has shown that every locally connected, connected and locally peripherally separable metric space is completely separable. He has also shown that in a space in which Axioms 1—4 of Point Set Theory and Theorem 1 of the present Section hold true, Theorem 2 of Chapter VI of Point Set Theory holds true. If a space satisfies Theorem 1 of this section then it is locally peripherally separable. From these results and results established in preceding sections of this paper it follows that if a space satisfies Axioms 1_6, 2, 3, 4, 5, 6 and 7 then it satisfies Axioms 1—5 of Point Set Theory.

V. Consequences of Axioms 1—7.

With the use of Axiom 1_7, by an argument similar to that employed to prove Theorem 1 of Section IV, it may be shown that every region is compact. With the help of results established in Chapter VII of Point Set Theory it follows that if a space satisfies Axioms 1—7 then the set of all points is topologically equivalent either to a plane or to the surface of a sphere.

Le théorème de Souslin dans la théorie générale des ensembles.

Par

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Soient $S\{E_{n_1,n_2,\dots,n_k}\}$ et $T\{H_{n_1,n_2,\dots,n_k}\}$ deux systèmes déterminants [1]) formés d'ensembles quelconques. Soient $N(S)$ et $N(T)$ respectivement leurs noyaux [1]). Nous dirons que *les systèmes S et T sont en relation R*, si pour toutes deux suites infinies d'indices naturels $p_1, p_2, p_3, \dots$ et $q_1, q_2, q_3, \dots$ il existe un nombre naturel s tel que

$$E_{p_1,p_2,\dots,p_s} H_{q_1,q_2,\dots,q_s} = 0.$$

Désignons respectivement par $\mathfrak{B}(S)$ par $\mathfrak{B}(T)$ la famille de tous les ensembles qu'on obtient en partant des ensembles qui forment respectivement les systèmes S et T et en effectuant avec eux un nombre fini ou une infinité dénombrable d'additions et de multiplications d'ensembles.

Théorème I: *Si deux systèmes déterminants S et T sont en relation R, il existe deux ensembles P et Q tels que $P \in \mathfrak{B}(S)$, $Q \in \mathfrak{B}(T)$, $PQ = 0$, $N(S) \subset P$ et $N(T) \subset Q$.*

Pour démontrer le théorème I (par voie apagogique, mais sans utiliser les nombres transfinis) il suffit de modifier légèrement la démonstration (basée sur une idée de M. Lusin) du théorème de Souslin publiée p. 265—267 du vol. XXI des *Fund. Math.*

Une autre démonstration (constructive, mais utilisant les nombres transfinis) peut être obtenue comme il suit.

[1]) Pour les définitions de ces notions voir p. ex. *Fund. Math.* XXI, p. 250 ou bien *Fund. Math.* VIII, p. 362.