E. Borel.

est égal à zero ou à un, on serait amené à considérer des séries telles que les suivantes, pour $x_0 = 1$, en posant $l_n = \frac{1}{n}$:

$$\sum l_n (\log l_n)^n,$$

$$\sum l_n (\log \log l_n)^n,$$

et dans le cas où $x_0 = 0$, des séries telles que les suivantes:

$$\sum \frac{1}{(\log l_n)^n},$$

$$\sum \frac{1}{(\log \log l_n)^n},$$

séries qui sont données seulement à titre d'exemple, car la variété des possibilités est considérable. Il y a un très vaste champ de recherches à explorer.

Signalons enfin, pour mémoire, l'étude des cas où la croissance des fonctions considérées ne serait pas régulière, c'est-à-dire en relation simple avec la fonction exponentielle.

Les ensembles définis par les intervalles fondamentaux normaux rattachés à des ensembles énumérables classiques sont assurément fort particuliers; ils donnent toutefois des exemples en fait fort généraux et permettent d'étudier très complètement la classification des ensembles de mesure nulle. Je souhaite que les brèves indications qui précèdent encouragent à poursuivre cette étude ceux qui s'intéressent à la théorie des ensembles.

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A set of axioms for plane analysis situs.

By

R. L. Moore (Austin).

As far as I know, Veblen was the first to conceive the idea of basing geometry on a set of axioms in terms of what he called "chunks" (of space). He had this idea as early as 1905 and discussed it with me at that time. In 1913 Huntington published a paper \(^1\) in which he founded geometry on a set of postulates in terms of "sphere" and "inclusion"; and Whitehead and Nicolas have given some thought to related questions.

Huntington makes much use of what may be termed the convexity of his undefined spatial elements. It is natural that he should do so in founding a geometry. The notion of the convexity of these elements is intimately \(^1\) related to the notion straight line which is one of the fundamental elements of geometry.

For many years I have endeavored to found point-set theoretic analysis situs on the basis of a set of axioms in terms of the notion of what I shall call "piece" and the relation "embedded in". In so far as I was endeavoring to found analysis situs, as opposed to geometry, I would naturally avoid postulating that these pieces be convex, the notions convexity and collinearity being foreign to analysis situs. In April, 1930, I delivered \(^2\) at the University of Texas, a set of five lectures in which I dealt with a set of Axioms in

\(^1\) A set of postulates for abstract geometry, expressed in terms of the simple relation of inclusion, Math. Ann., Bd. LXXIII, 522—559.

\(^2\) In order that three given points of a Euclidean space should be collinear it is necessary and sufficient that they may be lettered A, B and C in such a way that B is contained in every convex region that contains both A and C.

\(^3\) As University of Texas research lecturer for the session 1929—1930.
terms of piece and embedded in. On November 18, 1931, at a meeting of the American Mathematical Society at Pasadena, California, I delivered an address dealing with the same set of axioms except, possibly, for more or less minor modifications. This set included a set \( \mathfrak{A} \) of axioms which (except possibly for minor considerations) was identical with the set consisting of Axioms 2, 3, 4 and 5 and that portion of Axiom 1, of the present paper, which remains on the removal of the statement numbered (6). To these were added additional axioms limiting space to being topologically equivalent either to the plane or to a sphere. I was, however, not well satisfied with some of these additional axioms and it is only recently that I have obtained the formulations given below of Axioms 7 and 8 and the proposition labelled (6) in the statement of Axiom 1. In connection with Axiom 8, I am much indebted to Kuratowski’s paper “Une caractérisation topologique de la surface de la sphère” \( ^{3} \). This axiom is formulated in such a way as to enable me to prove Theorem 4 of Section III by an argument closely related to that given by Kuratowski, on pages 311 and 312 of his paper, to prove the proposition that \( {\mathcal{L}} \to {\mathcal{L}} \).

The set of Axioms 1—8 is satisfied if, speaking roughly, the word “piece” is interpreted to mean any limited piece (in the ordinary \( ^{4} \) sense) of the plane and the word “embedded” is interpreted in a natural manner. In such an interpretation, a piece of a plane is not regarded as a set of points.

Speaking accurately, if \( S \) is a plane, Axioms 1—8 are satisfied if the word “piece” is interpreted to mean any \( ^{5} \) connected and limited domain in \( S \) (regardless of what sort of boundary it has) and the piece \( x \) is said to be embedded in the piece \( y \) if, and only if, \( y \) contains \( x \) together with its boundary.

If a space satisfies Axioms 1—8 and the terms point and limit point are defined as described in this paper then the set of all

1) Every space satisfying these axioms is topologically equivalent to a connected, locally connected and complete metric space having no cut point.


3) It would seem to be accordance with ordinary usage to refrain from applying the term “piece of space” to a point or a straight line or anything else which has no interior.

4) These axioms are also satisfied if, for example, the word “piece” is restricted to those limited and connected domains of the plane whose boundaries are indecomposable continua.

A set of axioms points is topologically equivalent either to a plane or to the surface of a sphere.

Notation. For each \( i \) less than \( 8 \), the notation Axiom 1, will be used to denote the axiom obtained by retaining only that part of the statement of Axiom 1 remaining after the deletion of the phrases numbered from \( (i + 1) \) to \( (7) \) inclusive that occur in its formulation.

I. Consequences of Axioms 1, 2 and 3.

Definitions. The piece \( x \) is said to embed the piece \( y \) if \( y \) is embedded in \( x \). The piece \( x \) is said to intersect the piece \( y \) if there exists a piece which is embedded both in \( x \) and in \( y \). The piece \( x \) is said to be detached from the piece \( y \) if one of the pieces \( x \) and \( y \) is embedded in a piece that does not intersect the other one. Under these conditions \( x \) and \( y \) are also said to be mutually detached. (The word “mutually” may sometimes be omitted). If \( G \) and \( H \) are finite sets of pieces, \( G \) is said to be detached from \( H \) (and \( G \) and \( H \) are said to be mutually detached), if every piece of the set \( G \) is detached from every piece of the set \( H \). A set of pieces is said to be coherent if it is not the sum of two sets such that no piece of one of these sets intersects any piece of the other one. The set \( W \) of pieces is said to separate the piece \( h \) from the piece \( k \) if every coherent set of pieces that contains \( h \) and \( k \) contains a piece that intersects some piece of \( W \).

Axiom 1. There exists a sequence \( G_1, G_2, G_3, \ldots \) such that

1) for each \( n \), \( G_n \) is a set of pieces,

2) for each \( n \), \( G_{n+1} \) is a subset of \( G_n \),

3) if \( g \) is a piece then, for each \( n \), there is a piece belonging to \( G_n \) and embedded in \( g \),

4) if the piece \( k \) is embedded in the piece \( h \) there exists a number \( m \) such that every piece that belongs to \( G_m \) and intersects \( k \) is embedded in \( h \),

5) if \( g \) is a piece and \( n \) is a natural number the set of all pieces of the set \( G_n \) that are embedded in \( g \) is coherent,

6) if \( g \) is a piece and \( n \) is a natural number there exist a natural number \( m \) and a finite subset \( H \) of \( G_n \) such that if \( x \) is a piece of the set \( G_m \) which intersects \( g \) without being embedded in it then \( x \) is embedded in some piece of the set \( H \).
(7) the statement labelled (6) remains true if the phrase "without being embedded in $\mathcal{U}$" is omitted.

**Axiom 2.** If the piece $g$ is embedded in the piece $h$ and $h$ is embedded in the piece $k$ then $g$ is embedded in $k$.

**Axiom 3.** If the piece $g$ is embedded in the piece $h$ then $h$ is not embedded in $g$.

**Definition.** If $g_1, g_2, g_3, \ldots$ is a sequence of pieces such that, for each $n$, $g_n$ belongs to $G_n$ and $g_{n+1}$ is embedded in $g_n$, then the set of all pieces $x$ such that some piece of this sequence is embedded in $x$ is called a point and the sequence $g_1, g_2, g_3, \ldots$ is said to determine this point.

**Definition.** The point 0 is said to pertain to the piece $g$ if some piece of the set 0 is embedded in $g$. In other words, 0 is said to pertain to $g$ if $g$ belongs to 0.

**Theorem 1.** If $W$ is a point and $\alpha$ is a sequence $h_1, h_2, h_3, \ldots$ such that, for each $n$, $h_n$ belongs both to $W$ and to $G_n$, and $h_{n+1}$ is embedded in $h_n$, then $\alpha$ determines $W$.

**Proof.** Let $\beta$ denote a sequence of pieces $g_1, g_2, g_3, \ldots$ which determines $W$. Let $x$ denote a piece of the set $W$. There exists a number $n$ such that $g_n$ is embedded in $x$. There exists a number $\delta$ such that every piece of $G_\delta$ that intersects $g_n$ is embedded in $x$. But $h_\delta$ belongs to $W$ and therefore intersects $h_n$. Hence $h_n$ is embedded in $x$. Thus every piece of the set $W$ embeds some piece of the sequence $\alpha$. Suppose $y$ is a piece that embeds some piece $h_n$ of $\alpha$. Since $h_n$ belongs to $W$ it embeds some piece $g_\delta$ of $\beta$ and therefore $y$ embeds $g_\delta$ and consequently belongs to $W$. Thus $W$ is the set of all pieces $w$ such that $w$ embeds some piece of the sequence $\alpha$. It follows that $\alpha$ determines $W$.

**Theorem 2.** If $A$ and $B$ are distinct points, not every piece belonging to $A$ intersects every one that belongs to $B$.

**Proof.** Let $\alpha$ and $\beta$ denote sequences $g_1, g_2, g_3, \ldots$, and $h_1, h_2, h_3, \ldots$, determining $A$ and $B$ respectively and therefore such that, for each $n$, $g_n$ and $h_n$ belong to $G_n$. Since $A$ and $B$ are distinct either there is a piece which embeds some piece of $\alpha$ but which embeds no piece of $\beta$ or there is a piece which embeds some piece of $\beta$ but no piece of $\alpha$. Suppose there is a piece $k$ which embeds $g_n$ but no piece of $\beta$. There exists a number $n$ such that $k$ embeds every piece of the set $G_n$ that intersects $g_n$. The piece $h_n$ does not intersect $g_n$.

**Theorem 3.** If the point 0 pertains both to the piece $x$ and to the piece $y$ then some piece of the set 0 is embedded both in $x$ and in $y$.

**Proof.** Let $g_1, g_2, g_3, \ldots$ denote a sequence of pieces that determines 0. There exist numbers $m$ and $n$ such that $g_m$ and $g_n$ are embedded in $x$ and $y$ respectively. The piece $g_{m+n}$ belongs to 0 and is embedded both in $x$ and in $y$.

**Theorem 4.** If $P_1, P_2, P_3, \ldots, P_n$ is a finite set of distinct points there exists a set of $n$ mutually detached pieces $x_1, x_2, \ldots, x_n$ such that, for each $k$ less than or equal to $n$, $P_k$ pertains to $x_k$.

**Proof.** If $i$ and $j$ are any two natural numbers less than or equal to $n$ and such that $i < j$ then, by Theorem 2, there exist two non-intersecting pieces $x_i$ and $x_j$ belonging to $P_i$ and $P_j$ respectively. By Theorem 3 and Axiom 2, for every $k$ not greater than $n$ there exists a piece $x_k$ belonging to $P_k$ and embedded in all the $n-1$ pieces $x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n$. The pieces $x_1, x_2, \ldots, x_n$ are mutually detached.

**Theorem 5.** If $A$ and $B$ are distinct points pertaining to the piece $p$, there exist two non-intersecting pieces both embedded in $p$ and belonging to the sets $A$ and $B$ respectively.

**Proof.** Let $\alpha$ and $\beta$ denote sequences $g_1, g_2, g_3, \ldots$ and $h_1, h_2, h_3, \ldots$, determining $A$ and $B$ respectively. By hypothesis there exist two numbers $m_1$ and $n_1$ such that $g_{m_1}$ and $h_{n_1}$ are both embedded in $p$. By Theorem 2 there exist two non-intersecting pieces $x$ and $y$ belonging to $A$ and $B$ respectively. There exist numbers $m_2$ and $n_2$ such that $g_{m_2}$ and $h_{n_2}$ are embedded in $x$ and $y$ respectively. The pieces $g_{m_1+m_2}$ and $g_{n_1+n_2}$ are non-intersecting and they are both embedded in $p$.

**Theorem 6.** If $r$ is a piece there is at least one point pertaining to $r$.

**Proof.** By Axiom 1, there exist a piece $g_1$ belonging to $G_1$ and embedded in $r$, a piece $g_2$ belonging to $G_2$ and embedded in $r$. Fundamenta Mathematicae, t. XXV.
A set of axioms

Theorem 11. If the piece $q$ is embedded in the piece $p$ then $q'$ is a subset of $p'$.

Proof. By Theorem 7, $q'$ is a subset of $p'$. Suppose now that $X$ is a limit point of $q'$. Suppose $r$ is a piece to which $X$ pertains. Then the region $r'$ contains a point belonging to $q'$. Hence, by Theorem 9, there exists a piece $k$ which is embedded both in $r$ and in $q$. Thus $g$ intersects every piece to which $X$ pertains. Hence if $g_1, g_2, g_3, \ldots$ is a sequence of pieces which determines $X$, there exists a number $n$ such that $g_n$ is embedded in $p$. Hence $X$ belongs to $p'$.

Definition. Two point sets are said to intersect if they have at least one point in common.

Definition. The region $R$ will be said to be embedded in the region $K$ if, and only if, there exist two pieces $x$ and $y$ such that $x$ is embedded in $y$ and such that $x' = R$ and $y' = K$.

Notation. The set of all points will be denoted by the letter $S$.

Theorem 12. There exists a sequence $Q_1, Q_2, Q_3, \ldots$ such that

1. for each $n$, $Q_n$ is a collection of regions covering $S$,
2. for each $n$, $Q_{n+1}$ is a subcollection of $Q_n$,
3. if $X$ is a point of a region $R$ there exists a number $m$ such that if $x$ and $y$ are intersecting regions of the collection $Q_m$ and $x$ contains $X$ then $y$ is a subset of $R$,
4. if $M_1, M_2, M_3, \ldots$ is a sequence of closed point sets such that, for each $n$, $M_n$ contains $M_{n+1}$ and, for each $n$, there exists a region $q_n$ of the collection $Q_n$ such that $M_n$ is a subset of $q_n$, then there is at least one point common to all the point sets of the sequence $M_1, M_2, \ldots$.

Proof. Let $Q_1$ denote the set of all regions $x$ of $G_1$ such that $x$ is embedded in some region of $G_1$. Let $Q_2$ denote the set of all regions $x$ of $G_2$ such that $x$ is embedded in some region of $Q_1$. This process may be continued. Thus there exists a sequence $Q_1, Q_2, Q_3, \ldots$ such that $Q_1$ is as described and, for each $n$, $Q_{n+1}$ is the set of all regions $x$ of $G_{n+1}$ such that $x$ is embedded in some region of $Q_n$. For each $n$, the collection $Q_{n+1}$ covers $S$ and is a subcollection of $Q_n$.

Suppose $X$ is a point of a region $R$. There exists a piece $K$ such that $K' = R$. Let $t_1, t_2, t_3, \ldots$ denote a sequence of pieces that
determines $X$. There exists a number $k$ such that $t_k$ is embedded in the piece $K$. The piece $t_{k+1}$ is embedded in $t_k$. Hence there exist numbers $n_1$ and $n_2$ such that $t_k$ embeds every piece of the set $G_n$ that intersects $t_{k+1}$ and $K$ embeds every piece of $G_n$ that intersects $t_k$. Let $m$ denote $n_1 + n_2$. If $p$ and $q$ are intersecting pieces of the set $G_n$ and $X$ pertains to $p$ then $q$ is embedded in $K$. It follows that if $p'$ and $q'$ are intersecting regions of the set $G_n$ and $p'$ contains $X$ then $q'$ is a subset of $K$. But, for every $n$, the set $Q_n$ is a subset of $G_n$. It follows that if $x$ and $y$ are intersecting regions of $Q_n$ and $x$ contains $X$ then $y$ is a subset of $B$.

Suppose $q_1, q_2, q_3, \ldots$ is a sequence of regions such that, for each $n$, $q_n$ belongs to $Q_n$, and such that if $i$ and $j$ are any two natural numbers then $q_i$ and $q_j$ have at least one point in common. The region $q_1$ is embedded in some region $g_1$ belonging to $G_1$. There exists a number $n_1$ such that $g_1$ embeds every region of $G_n$ that intersects $q_1$. The region $q_2$ intersects $q_1$, and is embedded in some region $g_2$, of $G_n$. The region $q_2$ is embedded in $g_1$. There exists a number $n_2$ such that $g_2$ embeds every region of $G_n$ that intersects $q_2$. The region $g_3$ intersects $g_2$ and is embedded in some region $g_3$, of $G_n$. The region $g_3$ is embedded in $g_1$. This process may be continued. Thus there exist an ascending sequence of natural numbers $n_1, n_2, n_3, \ldots$ and a sequence $g_1, g_2, g_3, \ldots$ of regions such that, for each $i$, (1) $g_i$ belongs to $G_i$ and (2) $g_{i+1}$ and $g_i$ are both embedded in $g_i$. There exists one and only one point $P$ common to all the regions $g_1, g_2, g_3, \ldots$ Suppose $m$ is a number and $B$ is a region containing $P$. There exists a number $k$ such that $g_k$ is a subset of $B$. The region $g_k$ is a subset of $g_{k+1}$ and therefore of $R$. But $g_k$ has a point in common with $g_{k+1}$. Hence $R$ contains a point of $g_{k+1}$. Thus, for every $m$, $P$ belongs to $g_{k+1}$.

Suppose now that $M_1, M_2, M_3, \ldots$ is a sequence of closed point sets such that, for each $n$, $M_n$ contains $M_{n-1}$ and, for each $n$, there exists a region $g_n$ of the set $Q_n$ such that $M_n$ is a subset of $g_n$. For each $n$, there exist regions $g_n$ and $h_n$ belonging to $Q_n$ and such that $g_{n+1}$ is a subset of $g_n$ and $g_n$ is a subset of $h_n$. There exists a point $P$ common to all the point sets $g_1, g_2, g_3, \ldots$ and therefore to all the regions $h_1, h_2, h_3, \ldots$. Suppose there exists a number $k$ such that $P$ does not belong to $M_k$. Then there exists a region $R$ containing $P$ but no point of $M_k$. There exists a number $m$ such that $R$ embeds every region of $Q_m$ that contains $P$. But $h_n$ belongs to $Q_m$ and contains $P$. Therefore $h_n$ is a subset of $R$. But $g_{n+1}$ is a subset of $h_n$. Therefore $g_{n+1}$ contains no point of $M_n$. Thus the supposition that $P$ does not belong to all the point sets of the sequence $M_1, M_2, M_3, \ldots$ has led to a contradiction.

The truth of Theorem 12 has now been established.

With the aid of Theorems 10 and 12 it easily follows that Axiom 1 of my book Foundations of Point Set Theory holds true here. Furthermore space is metric. To see this proceed exactly as in the proof of Theorem 26 of Chapter VII of this book, except for the substitution of the words "Theorem 12 of the present paper" in place of the words "Theorem 74 of Chapter I", as being satisfied with respect to $M$. In this proof no use is made of anything in Theorem 74 beyond the result stated under (3) in the formulation of Theorem 12 of the present paper.

II. Consequences of Axioms 15, 2 and 3.

Definition. If $h$ and $k$ are pieces, a simple chain from $h$ to $k$ is a finite sequence of pieces $h_1, h_2, \ldots, h_n$ such that $h_1 = h$ and $h_n = k$ and such that if $1 \leq i \leq n$, $1 \leq j \leq n$ and $i < j$ then $h_i$ intersects $h_j$ if, and only if, $j = i + 1$. The piece $h_i$ ($1 \leq i \leq n$) is said to be the $i$th link of the chain $h_1, h_2, \ldots, h_n$.

Theorem 1. If $h$ and $k$ are two pieces belonging to the coherent collection $W$ there is a simple chain from $h$ to $k$ such that every link of this chain is a piece belonging to $W$.

Proof. Suppose there is no such simple chain. Let $W_x$ denote the set of all pieces $x$ of the set $W$ such that there is a simple chain, from $h$ to $x$, whose links all belong to $W_x$ and let $W_k$ denote the set $W - W_x$. Since $W$ is coherent there exists a piece $y_x$ belonging to $W_x$ and intersecting some piece $y_k$ of the set $W_k$. There exists a simple chain $h_1, h_2, \ldots, h_n$ whose links all belong to $W$ and whose first and last links are $h$ and $k$ respectively. If $h_m$ is the first link of this chain which intersects $y_k$ then $h_1, h_2, \ldots, h_m, y_k$ is a simple chain from $h$ to $y_k$. But this involves a contradiction.

1) American Mathematical Society Colloquium Publications, volume XIII, New York, 1933. Hereafter, in this paper, this book will be referred to as Point Set Theory.
Theorem 2. If, for each $n$, $H_n$ is a finite set of pieces and, for each $n$, each piece of the set $H_{n+1}$ is embedded in some piece of the set $H_n$, then there exists a sequence of pieces $h_1, h_2, h_3, \ldots$ such that, for each $n$, $h_n$ belongs to $H_n$ and $h_{n+1}$ is embedded in $h_n$.

Theorem 2 may be proved by an argument closely analogous to that employed to establish Theorem 1 of Chapter I of Point Set Theory.

Notation. If a letter is used to denote a collection of point sets, then that letter with an asterisk as a suffix will be used to denote the sum of all the point sets of this collection.

Theorem 3. If $H_1, H_2, H_3, \ldots$ is a sequence such that, for each $n$,

1. $H_n$ is a finite subset of $G_x$ and

2. each piece that belongs to $H_{n+1}$ is embedded in some piece that belongs to $H_n$;

then there exists at least one point common to all the point sets $H_1, H_2, H_3, \ldots$ and $K$, the set of all their common points, is a compact closed point set and if, for each $n$, $P_n$ is a point of $H_n$, some subsequence of the sequence $P_1, P_2, P_3, \ldots$ converges to a point of $K$.

Theorem 3 may be proved by an argument involving reasoning closely related to that employed in portions of the proof of Theorem 1 in Chapter I of Point Set Theory.

Theorem 4. If $D$ is a point set and $H$ is a coherent set of regions and some region of the set $H$ contains a point of $D$ and some region of $H$ contains a point of $S - D$, then there exists a region of $D$ which contains both a point of $D$ and a point of $S - D$.

Proof. Suppose, on the contrary, that $H$ is the sum of two collections $H_1$ and $H_2$ such that no region of $H_1$ contains a point of $S - D$ and no region of $H_2$ contains a point of $D$. Then no region of $H_1$ intersects any region of $H_2$ contrary to the hypothesis that $H$ is coherent.

Theorem 5. If, under the same hypothesis as in Theorem 3, each set of the sequence $H_1, H_2, H_3, \ldots$ is coherent then the common part of the point sets $H_1, H_2, H_3, \ldots$ is connected.

Proof. Let $K$ denote the common part of these point sets. Suppose $K$ is not a continuum. Then, since by Theorem 3, $K$ is closed and compact, it is the sum of two mutually exclusive closed and compact point sets $T$ and $L$. There exists a domain $D$ containing $T$ and such that $D$ contains no point of $L$. For each $n$, $H_n$ is a coherent set of regions and some region of $H_n$ contains a point of $D$ and some region of $H_n$ contains a point of $S - D$. Therefore, by Theorem 4, for each $n$, some region of $H_n$ contains both a point of $D$ and a point not belonging to $D$. For each $n$, let $W_n$ denote the set of all those that do so. Then, for each $n$, $W_n$ is a finite subset of $G_x$ and, for each $n$, each region of $W_{n+1}$ is embedded in some region of $W_n$. It follows, with the help of Theorem 2, that there is a point $P$ common to all the point sets of the sequence $W_1, W_2, W_3, \ldots$ The point $P$ belongs to $D - D$ and therefore neither to $T$ nor to $L$. But it belongs to $K$. Thus the supposition that $K$ is not a continuum leads to a contradiction.

Theorem 6. Every region is arcwise connected.

With the help of Theorems 1, 2, 3 and 5, Theorem 6 may be proved by an argument closely analogous to that used to prove Theorem 1 in Chapter II, with the aid of Theorems 77, 78, 79 and 80 of Chapter II, of Point Set Theory.

It is clear that Axioms 1 and 2 of Point Set Theory are consequences of Theorems 10 and 12 of Section I, and Theorem 6 of this Section, of the present paper. Therefore from Axioms 1, 2 and 3 of this paper follow all the numbered theorems of Chapters I and II of Point Set Theory except, of course, in those cases where such theorems are not properly stated.  

1) In Chapter I write "completely" before "separable" in the statement of Theorem 78.

In Chapter II, in the statement of Theorem 25, in line 4 from the bottom of the page, replace "every" by "all" and, in line 3 from the bottom, write $S$ then $T^m$ after $\sigma(M - L - M)^s$. In Theorem 86, after $\chi^m$ write $\sigma$ is connected. In Theorem 60 substitute "point" for "derivative". In the statement of Theorem 78, between "connected" and "there", interpolate "and no point of $K - B$ is a limit point of the component of $S - K$ that contains $A'".

In Chapter IV after the last $\sigma$ in Theorem 118, write 0 or by 0 plus any one of these sets that contains a point of 0'.

In Chapter V write "compact" before "element" in Theorem 9 and, in Theorems 9' and 10, after $\chi^m$ write "the elements of the collection $G$ are all compact and". In Theorem 13 replace "continuum" by "closed point set".

In Chapter VII, in Theorem 8, write "and compact" before "point sets" in line 5 and "compact" before contains in line 4. In numerous places, in this chapter, write "reversibly" before "continuously".
III. Consequences of Axioms 15, 2, 3, 4, 5, 6 and 7.

Axiom 4. There exists three mutually non-intersecting pieces.

Axiom 5. If a and b are detached pieces and \( p_1, p_2, p_3, \ldots \) is a sequence of pieces such that

1. for each \( n \), \( p_{n+1} \) is embedded in \( p_n \) and
2. for each \( n \), \( p_n \) separates a from b,

then there exist two pieces \( x \) and \( y \) such that \( x \) is embedded in \( y \) and such that every piece of the sequence \( p_1, p_2, p_3, \ldots \) intersects \( x \) but no one of them is embedded in \( y \).

Theorem 1. No point separates \(^1\) S.

Proof. Suppose A, B and P are three distinct points. By Theorem 4 of Section I, there exist three mutually detached pieces \( a, b \) and \( p \) belonging to A, B and P respectively. The point \( P \) is determined by a sequence of pieces \( p_1, p_2, p_3, \ldots \) all embedded in \( p \). If \( x \) and \( y \) are two pieces such that \( x \) is embedded in \( y \) and intersects every piece of this sequence then there are not more than a finite number of the pieces of this sequence that are not embedded in \( y \). Hence, by Axiom 5, there exists a number \( n \) such that \( p_n \) does not separate \( a \) from \( b \). Hence there exists a coherent set \( H \) of pieces such that both \( a \) and \( b \) belong to \( H \) but \( p \) does not intersect any piece belonging to \( H \). With the help of Theorem 6 of Section II, it is easy to see that the point set \( H^* \) is a connected subset of \( S - P \). But it contains both A and B.

Theorem 2. The set of all points is connected.

Proof. Suppose A and B are two distinct points. By Axiom 4 there exist three mutually non-intersecting pieces. Hence there exists a piece \( y \) belonging neither to A nor to B. By Theorem 6 of Section I, there exists a point \( P \) pertaining to \( y \). By Theorem 1, \( S - P \) is connected. Thus if A and B are any two points there is a connected point set containing both A and B. Therefore S is connected.

Axiom 6. If \( U \) and \( V \) are finite collections of pieces and \( U \cdot V \) is a coherent collection and the collections \( U - U \cdot V \) and \( V - U \cdot V \)

are mutually detached and neither \( U \) nor \( V \) separates the piece \( a \) from the piece \( \beta \) then the collection \( U + V \) does not separate \( a \) from \( \beta \).

Theorem 3. If H and K are two closed and compact point sets and \( H \cdot K \) is connected and neither \( H \) nor \( K \) separates the point \( a \) from the point \( B \) then \( H + K \) does not separate \( a \) from \( B \).

Proof. If \( A \) or \( B \) belongs to \( H \) or to \( K \) then clearly \( H + K \) does not separate \( A \) from \( B \). Suppose neither \( A \) nor \( B \) belongs to \( H + K \). There exist areas \( AXB \) and \( AYB \) such that \( AXB \) contains no point of \( H \) and \( AYB \) contains no point of \( K \). There exist pieces \( \alpha \) and \( \beta \) belonging to \( A \) and \( B \) respectively but such that neither \( \alpha' \) nor \( \beta' \) contains any point of \( H + K \). There exists a finite collection \( T \) of pieces such that \( T' \) properly \(^1\) covers \( H \cdot K \) such that \( T' \) has no point in common with \( AXB \cdot AYB \), and \( T' \) is detached from \( \alpha \) and from \( \beta \). Let \( H_i \) and \( K_i \) respectively denote the closed point sets \( H - H \cdot T'^* \) and \( K - K \cdot T'^* \). There exist two mutually detached finite collections \( Q_H \) and \( Q_K \) of pieces such that \( 1 \) \( Q_H \) and \( Q_K \) properly cover \( H_i \) and \( K_i \) respectively and \( 2 \) \( (K + AXB) \cdot \overline{Q_H} \) and \( (H + AYB) \cdot \overline{Q_K} \) are vacuous. There exist finite collections \( L_H \) and \( L_K \) of pieces such that \( 1 \) \( L_H \) and \( L_K \) properly cover \( AXB \cdot AYB \) respectively, \( 2 \) \( \alpha \) and \( \beta \) belong both to \( L_H \) and to \( L_K \), \( 3 \) \( L_H \) and \( Q_K + T \) are mutually detached and so are \( L_K \) and \( Q_H + T \). The common part of the collections \( Q_H + T \) and \( Q_K + T \) is the coherent collection \( T \) and \( (Q_H + T) - T \) and \( (Q_K + T) - T \) are mutually detached \( Q_H \) and \( Q_K \). Furthermore, the coherent collection \( L_H \) contains \( \alpha \) and \( \beta \) but is detached from \( Q_H + T \), and \( L_K \) is a coherent collection containing \( \alpha \) and \( \beta \) but detached from \( Q_K + T \). Therefore neither \( Q_H + T \) nor \( Q_K + T \) separates \( a \) from \( \beta \). It follows, by Axiom 6, that \( (Q_H + T) + (Q_K + T) \) does not separate \( a \) from \( \beta \). Hence \( (Q_H + T') + (Q_K + T') \) does not separate \( \alpha' \) from \( \beta' \). But \( H + K \) is a subset of \( (Q_H + T') + (Q_K + T') \) and \( \alpha' \) and \( \beta' \) are connected point sets containing \( A \) and \( B \) respectively. Therefore \( H + K \) does not separate \( A \) from \( B \).

\(^1\) The set \( W \) of point sets is said to properly cover the point set \( M \) if every point of \( M \) belongs to some point set of \( W \) and every point set of \( W \) contains some point of \( M \).
Axiom 7. If \( G \) and \( H \) are finite and mutually detached collections of pieces and \( g \) and \( h \) are pieces belonging to \( G \) and \( H \) respectively then there exists a finite collection \( V \) of pieces such that

1. \( V \) separates \( g \) from \( h \),
2. \( V \) and \( G + H \) are mutually detached,
3. if \( U \) and \( W \) are finite and mutually detached collections of pieces there exists a finite and coherent collection \( Z \) of pieces such that
   a. each piece of the collection \( Z \) is embedded in some piece of \( V \)
   b. \( Z \) separates \( g \) from \( h \) and
   c. every piece of the collection \( Z \) which is not detached from \( U \) is detached from every piece of \( Z \) which is not detached from \( W \).

Theorem 4. If the common part of two compact continua exists and is not connected then their sum separates \( S \).

Proof. Suppose \( \alpha \) and \( \beta \) are two compact continua such that \( \alpha \cdot \beta \) is the sum of two mutually exclusive closed point sets \( \gamma \) and \( \delta \). There exist finite collections \( H_\alpha, H_\beta, H_\gamma, \) and \( H_\delta \) of pieces such that

1. \( H_\alpha \cdot H_\beta \cdot H_\gamma \cdot H_\delta \) cover \( \alpha, \beta, \gamma, \delta \) respectively,
2. \( H_\beta \) and \( H_\delta \) are mutually detached, and
3. \( H_\gamma \cdot H_\alpha \cdot H_\gamma \cdot H_\delta \) are mutually detached.

Let \( P_\gamma \) and \( P_\delta \) denote points belonging to \( \gamma \) and \( \delta \) respectively and let \( h_\gamma \) and \( h_\delta \) denote definite pieces belonging respectively to \( H_\gamma \) and \( H_\delta \) and also belonging respectively to \( P_\gamma \) and \( P_\delta \). By Axiom 7 there exists a finite set \( V \) of pieces such that

1. \( V \) separates \( h_\gamma \) from \( h_\delta \),
2. \( V \) and \( H_\gamma + H_\delta \) are mutually detached, and
3. if \( U \) denotes the set of all pieces \( x \) of \( H_\gamma \) such that \( x \) intersects some piece of \( V \) then there exists a finite collection \( Z \) of pieces such that
   a. each piece of the collection \( Z \) is embedded in some piece of \( V \),
   b. \( Z \) separates \( h_\gamma \) from \( h_\delta \), and
   c. every piece of \( Z \) which is not detached from \( U \) is detached from every piece of \( Z \) which is not detached from \( W \).

Let \( Z_\alpha, Z_\beta, Z_\gamma, \) and \( Z_\delta \) denote the collection of all pieces of \( Z \) which are not detached from \( H_\alpha \) and let \( Z_{\gamma+\beta} \) denote the collection of all pieces of \( Z \) which are not detached from \( H_\gamma \). Let \( x_1, x_2, x_3, \ldots \) denote the pieces of the collection \( Z \) which belong neither to \( Z_\alpha \) nor to \( Z_\beta \).

Suppose now that \( \alpha + \beta \) does not separate \( S \). For each \( i \) less than \( n \), there exists an arc \( t_i \) joining some point of \( x_i \) to some point of \( x_{i+1} \) and containing no point of \( \alpha + \beta \). For each \( i \) less than \( n \) there exists a finite collection \( T_i \) of pieces such that

1. \( T_i \) covers \( t_i \) properly, and
2. no point of \( \alpha + \beta \) belongs to, or is on the boundary of, any region of the set \( T_i \).

Let \( E \) denote the collection \( x_1 + x_2 + \cdots + x_n + T_1 + T_2 + \cdots + T_n \). There exist pieces \( x_\gamma \) and \( x_\delta \) belonging to \( P_\gamma \) and \( P_\delta \) respectively, embedded in \( h_\gamma \) and \( h_\delta \) respectively and detached from \( E \). There exist finite and coherent collections \( I_\alpha \) and \( I_\beta \) of pieces such that

1. \( I_\alpha \) and \( I_\beta \) cover \( \alpha \) and \( \beta \) respectively,
2. every piece of \( I_\alpha \) is embedded in some piece of \( H_\gamma \) and every piece of \( I_\beta \) is embedded in some piece of \( H_\delta \), and
3. both \( I_\alpha \) and \( I_\beta \) are detached from \( E \),

(4) \( x_\gamma \) and \( x_\delta \) belong to \( I_\alpha \) and also to \( I_\beta \). The finite collection \( I_\alpha \) is coherent, contains both \( x_\gamma \) and \( x_\delta \) and is detached from \( E + Z_\gamma \); and the finite and coherent collection \( I_\beta \) contains both \( x_\gamma \) and \( x_\delta \) and is detached from \( E + Z_\delta \). Therefore neither \( E + Z_\alpha \) nor \( E + Z_\beta \) separates \( x_\gamma \) from \( x_\delta \). But the collection \( E \) is coherent and \( Z_\alpha \) is detached from \( Z_\gamma \). Therefore, by Axiom 6, the collection \( E + Z_\alpha + Z_\gamma \) does not separate \( x_\gamma \) from \( x_\delta \). But \( x_\gamma \) and \( x_\delta \) are detached from \( E + Z_\alpha + Z_\gamma \), \( Z \) is a subcollection of \( E + Z_\alpha + Z_\gamma \), and \( Z \) separates \( x_\gamma \) from \( x_\delta \). Thus the supposition that Theorem 4 is false has led to a contradiction.

Theorem 5. If \( J \) is a simple closed curve, \( S - J \) is the sum of two mutually separated connected point sets such that \( J \) is the boundary of both of them.

Theorem 5 may be established by an argument identical, except for obvious modifications, with that employed by Kuratowski on pages 313 and 314 of the article cited above, to prove the proposition 12.

IV. Consequences of Axioms 1, 2, 3, 4, 5, 6, and 7.

Theorem 1. If a region has a boundary its boundary is compact.

Proof. Suppose \( g \) is a piece such that \( g' \) has a non-vacuous boundary. By Axiom 4, there exist a natural number \( m_1 \) (greater than 1) and a finite subset \( H_1 \) of \( G_1 \) such that if \( x \) is a piece of the set \( G_{m_1} \) which intersects \( g \), without being embedded in it, then \( x \) is embedded in some piece of \( H_1 \). Similarly there exist a natural number \( m_2 \) (greater than \( m_1 \)) and a finite subset \( H_2 \) of \( G_{m_1} \) such that if \( x \) is a piece of the set \( G_{m_2} \) which intersects \( g \), without being embedded in it, then \( x \) is embedded in some piece of the set \( H_2 \). This process
Le théorème de Souslin dans la théorie générale des ensembles.

Par

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Soient $S\{k_{n,m,...,q}\}$ et $T\{h_{n,m,...,q}\}$ deux systèmes déterminants formés d'ensembles quelconques. Soient $N(S)$ et $N(T)$ respectivement leurs noyaux. Nous dirons que les systèmes $S$ et $T$ sont en relation $R$, si pour toutes deux suites infinies d'indices naturels $p_1, p_2, p_3,...$ et $q_1, q_2, q_3,...$ il existe un nombre naturel $s$ tel que

$$E_{p_n,...,s} H_{n,m,...,q_s} = 0.$$ 

Désignons respectivement par $S(T)$ par $B(T)$ la famille de tous les ensembles qu'on obtient en partant des ensembles qui forment respectivement les systèmes $S$ et $T$ et en effectuant avec eux un nombre fini ou une infinité dénombrable d'additions et de multiplications d'ensembles.

**Théorème I:** Si deux systèmes déterminants $S$ et $T$ sont en relation $R$, il existe deux ensembles $P$ et $Q$ tels que $P \in B(S)$, $Q \in B(T)$, $PQ = 0$, $N(S) \subset P$ et $N(T) \subset Q$.

Pour démontrer le théorème I (par voie apagogique, mais sans utiliser les nombres transfinis) il suffit de modifier légèrement la démonstration (basée sur une idée de M. Luzin) du théorème de Souslin publiée p. 265—267 du vol. XXI des *Fund. Math.*

Une autre démonstration (constructive, mais utilisant les nombres transfinis) peut être obtenue comme il suit.