On sequences and limiting sets.

By

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In this paper a study will be made of relations between convergent sequences of sets and their limiting sets when certain restrictions are placed on the nature of the convergence of the sequences. In the first section necessary and sufficient conditions will be established in order that the limiting set A of a sequence of sets $[A_n]$ should be (1) a γ^r-continuum, in the sense that every r-dimensional complete cycle in A is ~ 0 in A, and (2) locally γ^r-connected, in a similar sense. The former of these results may be regarded as an r-dimensional analog of a classical theorem of Zoretti which has as a consequence that the limiting set of a convergent sequence of continua (γ^0-continua) is itself a continuum (γ^0-continuum). The conditions here imposed suggest a type of convergence which we shall call *regural convergence relative to r-dimensional cycles.* This is studied in section 2, and in the remainder of the paper it is applied to sequences of arcs, simple closed curves, topological spheres, and 2-cells.

It will be shown that regular convergence relative to 0-cycles for sequences of arcs and simple closed curves gives limiting sets of the same type. For sequences of topological spheres it gives cactoids as limiting sets (see § 5), and in case the convergence is regular also relative to 1-cycles, these cactoids reduce to topological spheres. For a sequence of 2-cells whose boundaries also converge, regular convergence relative to 0-cycles gives as a limiting set a hemicactoid whose base set is bounded by a boundary curve which is the limit of the boundaries of the 2-cells (see § 6). The hemicactoid reduces to its base set in case the convergence is also

regular relative to 1-cycles, and this base set reduces to a 2-cell in case the boundaries of the 2-cells in the sequence converge regularly relative to 0-cycles.

It will be assumed throughout that all sets used lie in a compact metric space. For systematic treatments of the combinatorial notions used the reader is referred to the papers of Vietoris and Alexandroff[1]).

1. Connectivity of limiting sets.

All of our *complexes* and *cycles* will be *non-oriented.* A closed set A is called a *γ^r-continuum* provided every complete r-cycle in A is ~ 0 in A. Similarly, A is said to be *locally γ^r-connected* provided that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that every complete r-cycle in A of diameter $< \delta$ is ~ 0 in a subset of A of diameter $< \varepsilon$. The *distance* from x to y is denoted by $\varrho(x, y)$ and, for any $d > 0$, $V_d(A)$ will denote the set of all points x such that the distance from x to some point of A is $< d$. Finally, we use the arrow $\rightarrow$ to indicate *convergence* (and not to indicate bounding relationships).

(1.1) ***Lemma.*** *Let the sequence of closed sets $[A_n]$ converge to the limiting set A. If for a given $e > 0$ there exists a d, $0 < d < e$, such that for any $\varepsilon > 0$ there exist positive numbers δ and N such that if $n \geqslant N$, then any r-dimensional δ-cycle in A_n of diameter $< 3d$ is $\underset{\varepsilon}{\sim} 0$ in a subset of A_n of diameter $< e$, then any complete r-dimensional cycle in A of diameter $< d$ is ~ 0 in a subset of A of diameter $< 3e$.*

Proof. Let $\gamma^r = (C_1, C_2, \ldots)$ be any complete r-dimensional cycle in A of diameter $< d$. We have to show that for any $\varepsilon > 0$, there exists an integer K such that for $k > K$, $C_k \underset{\varepsilon}{\sim} 0$ in a subset of A of diameter $< 3e$. By hypothesis, given $\varepsilon/4$, there exist positive numbers $\delta < \varepsilon$ and N such that for any $n > N$, any r-dimensional δ-cycle in A_n of diameter $< 3d$ is $\underset{\varepsilon/4}{\sim} 0$ in a subset of A_n of diameter $< e$. Let us take K so that for $k > K$, C_k is a $\delta/3$-cycle; and with k fixed and $> K$, let $C_k = (x_0, x_1, \ldots, x_g)$, where the x_i are

[1]) See, for example, Vietoris, Math. Ann., vol. 97, pp. 545–572, and Fund. Math., vol. 19 (1932), pp. 265–273; Alexandroff, Annals of Math., vol. 30 (1928), pp. 101–187, and Math. Ann., vol. 106 (1932), pp. 161–238.

the vertices of C_k. Take an integer I such that for $i > I$, we have $A_i \subset V_{\delta'}(A)$ and $A \subset V_{\delta'}(A_i)$ where $\delta' = \min(\delta/3, d)$. Take a fixed $j > I + N$. For each $s \leqslant g$, let y_s be a point of A_j such that $\varrho(x_s, y_s) = \varrho(x_s, A_j) < \delta'$. For each simplex $(x_{i_0}, x_{i_1}, \ldots, x_{i_r})$ in C_k, let $(y_{i_0}, y_{i_1}, \ldots, y_{i_r})$ be a simplex and let C_j^* be the cycle composed of all such simplexes [1].

Now since $\varrho(y_{i_l}, y_{i_k}) \leqslant \varrho(y_{i_l}, x_{i_l}) + \varrho(x_{i_l}, x_{i_k}) + \varrho(x_{i_k}, y_{i_k}) < \delta' + \delta/3 + \delta' \leqslant \delta$, it follows that C_j^* is a δ-cycle; and since $\varrho(y_m, y_s) \leqslant \varrho(y_m, x_m) + \varrho(x_m, x_s) + \varrho(x_s, y_s) < \delta' + d + \delta' \leqslant 3d$, then C_j^* is of diameter $< 3d$. Thus by hypothesis there exists an $\varepsilon/4$-complex $Z^{r+1} = (z_0, z_1, \ldots, z_h)$ in A_j of diameter $< e$ bounded by C_j^*. For each $s \leqslant h$, let w_s be a point of A such that $\varrho(z_s, w_s) = \varrho(z_s, A) < \delta'$ and such that when $z_s = y_t$, $0 \leqslant t \leqslant g$, $w_s = x_t$. For each simplex $(z_{i_0}, z_{i_1}, \ldots, z_{i_{r+1}})$ in Z^{r+1}, let $(w_{i_0}, w_{i_1}, \ldots, w_{i_{r+1}})$ be a simplex, and let K^{r+1} be the complex of all such simplexes. (In other words, we *project* Z^{r+1} into the complex K^{r+1} in A by a δ'-projection). Then clearly K^{r+1} is an ε-complex in A of diameter $< 3e$ bounded by C_k. Thus $C_k \underset{\varepsilon}{\sim} 0$ in a subset of A of diameter $< 3e$, for any $k > K$; and our lemma is proven.

(1.2) ***Lemma.*** *Let $A_i \to A$. If for a given $e > 0$ there exists a d, $0 < d < e$, such that any complete γ^r in A of diameter $< 3d$ is ~ 0 in a subset of A of diameter $< e$, then for any $\varepsilon > 0$ there exist positive numbers δ and N such that if $n > N$, any r-dimensional δ-cycle in A_n of diameter $< d$ is $\underset{\varepsilon}{\sim} 0$ in a subset of A_n of diameter $< 3e$.*

Proof. From the hypothesis it follows that for $\varepsilon > 0$ there exists a δ, $0 < \delta < \varepsilon$, such that any r-dimensional δ-cycle in A of diameter $< 3d$ is $\underset{\varepsilon/3}{\sim} 0$ in a subset of A of diameter $< e$. Let us take an integer K such that for $k > K$, $A_k \subset V_{\delta'}(A)$ and $A \subset V_{\delta'}(A_k)$, where $\delta' = \min(\delta/3, d)$. Let $C = (x_0, x_1, \ldots, x_g)$ be any r-dimensional δ'-cycle in A_k of diameter $< d$, where the x_i are vertices of C and $k > K$. Now, just as in the proof of (1.1), C can be projected by a δ'-projection into a complex C^* in A. Then clearly C^* is an

[1] We shall call a complex C_j^* obtained in this way a *δ'-projection* of C_k in A_j, and we shall say that C_k *projects* into the complex C_j^* under a *δ'-projection* when each vertex in C_k is replaced by a point at a distance $< \delta'$ from it. It is to be noted that when a vertex of C_k lies in A_j, that vertex is replaced by itself.

r-dimensional δ-cycle in A of diameter $< 3d$. Thus C^* bounds an $\varepsilon/3$-complex Z^{r+1} in A of diameter $< e$. Similarly Z^{r+1} projects into a complex K^{r+1} in A_k by a δ'-projection which is so chosen that on C^* it is merely the inverse of the previous projection sending C into C^*. Then since $\delta < \varepsilon$ and $d < e$, it follows that K^{r+1} is an ε-complex in A_k of diameter $< 3e$; and since K^{r+1} is bounded by C, our lemma is established provided we take $\delta = \delta'$ and $N = K$.

(1.3) ***Theorem.*** *Let the sequence of closed sets $[A_i]$ converge to the limiting set A. In order that A be a γ^r-continuum it is necessary and sufficient that for any $\varepsilon > 0$ there exist positive numbers δ and N such that if $n > N$, then any r-dimensional δ-cycle in A_n is $\underset{\varepsilon}{\sim} 0$ in A_n.*

To prove the sufficiency, we note that if we take $e > d = \delta[\Sigma A_i]$, the conditions of (1.1) are fulfilled. Whence, any complete γ^r in A whatever is ~ 0 in A. Similarly for the necessity, taking $e > d > \delta[\Sigma A_i]$, we have the conditions of (1.2) satisfied and our result follows.

(1.4) ***Theorem.*** *Let $A_i \to A$ as before. In order that A be locally γ^r-connected it is necessary and sufficient that for any $e > 0$ there exist a $d > 0$ such that for any $\varepsilon > 0$, positive numbers δ and N exist so that if $n > N$, then any r-dimensional δ-cycle in A_n of diameter $< d$ is $\underset{\varepsilon}{\sim} 0$ in a subset of A_n of diameter $< e$.*

Proof. Sufficiency: Let $e' > 0$ be given. Taking $e = e'/3$ and applying (1.1), we see that any complete γ^r in A of diameter $< d$ is ~ 0 in a subset of A of diameter $< 3e = e'$. Accordingly, A is locally γ^r-connected.

Necessity: Let $e' = e/3$. By hypothesis there exists a $d' > 0$ such that any γ^r in A of diameter $< d'$ is ~ 0 in a subset of A of diameter $< e'$. Applying (1.2), we see that for any $\varepsilon > 0$ there exist positive numbers δ and N such that if $n > N$, any r-dimensional δ-cycle in A_n of diameter $< d = d'/3$ is $\underset{\varepsilon}{\sim} 0$ in a subset of A_n of diameter $< 3e' = e$.

2. Regular convergence.

Let the sequence of closed sets $[A_i]$ converge to the limiting set A. Then $[A_i]$ will be said to converge to A *regularly relative to r-cycles* provided that for each $\varepsilon > 0$ there exist positive numbers δ and N such that if $n > N$, any complete r-dimensional cycle in A_n of diameter $< \delta$ is ~ 0 in a subset of A_n of diameter $< \varepsilon$.

In the remainder of this paper we shall be concerned almost exclusively with the cases $r=0, 1$. It is most natural for us to interpret a 0-cycle as an even number of 0-cells (points); and since we are concerned only with questions relative to regular convergence, it is clear that no generality is lost and the treatment is considerably simplified if, as we shall do, we interpret a 0-cycle (complete or geometric) as being made up of a single pair of distinct points. Regular convergence of $[A_i]$ to A relative to 0-cycles is then equivalent to the condition that for each $\varepsilon>0$ there exist positive numbers δ and N such that if $n>N$, any pair of points $x, y \in A_n$ with $\varrho(x, y)<\delta$ lie together in a continuum in A_n of diameter $<\varepsilon$. Thus we have in this case a sort of „equi-uniform-local connectedness" for the sequence $[A_n]$ [1]).

(2.1). *Let the sequence of closed sets $[M_n]$ converge to the limiting set M. Then in order that $M_n \to M$ regularly relative to 0-cycles it is necessary and sufficient that for each sequence of decompositions $M_{n_i}=A_{n_i}+B_{n_i}$ into closed sets such that $A_{n_i}\to A$, $B_{n_i}\to B$ and $A_{n_i}\cdot B_{n_i}=X_i\to X$, we have $A\cdot B=X$,* i. e.,

$$\lim (A_{n_i}\cdot B_{n_i})=(\lim A_{n_i})\cdot(\lim B_{n_i}).$$

Proof. The condition is necessary. For suppose $M_n\to M$ 0-regularly (i. e., regularly relative to 0-cycles) but that for some such sequence of decompositions, there exists a point p of $A\cdot B-X$. Then, taking $\varepsilon=\varrho(p, X)/4$ and determining δ and N for the regular convergence, it is clear that for i sufficiently large and $>N$, we will have $X_i\subset V_\varepsilon(X)$ and A_{n_i} and B_{n_i} will contain points a_i and b respectively which lie in $V_{\delta/2}(p)$. But this is impossible, since we then have $\varrho(a_i, b_i)<\delta$ and $i>N$, whereas any subcontinuum of M_{n_i} containing a_i+b_i must intersect X_i and hence be of diameter $>\varepsilon$.

The condition is also sufficient. For suppose the condition satisfied and that the convergence is not 0-regular. Then for some $\varepsilon>0$, there exist sequences of points $[a_i]$ and $[b_i]$, where $a_i+b_i\subset M_{n_i}$, such that $a_i\to p\leftarrow b_i$, where $p\in M$ and $a_i+b_i\subset V_{\varepsilon/4}(p)=V$, but such that any continuum in M_{n_i} containing a_i+b_i is of diameter $>\varepsilon$. For each i there exists a separation $M_{n_i}\overline{V}=M_{a_i}+M_{b_i}$ of $M_{n_i}\cdot V$ into disjoint closed sets containing a_i and b_i respectively as indicated. Let

$$A_{n_i}=M_{n_i}-M_{b_i}\cdot V, \qquad B_{n_i}=M_{n_i}-M_{a_i}\cdot V.$$

Then for each i, we have

$$M_{n_i}=A_{n_i}+B_{n_i} \quad \text{and} \quad A_{n_i}\cdot B_{n_i}=X_i=M-V\cdot M.$$

Thus if we select convergent subsequences $A_{n_{i_j}}\to A$ and $B_{n_{i_j}}\to B$, we will have $\lim X_{i_j}=M-V\cdot M$, whereas clearly $A\cdot B\supset p\in V$. Thus our supposition that the convergence $M_n\to M$ is not 0-regular leads to a contradiction.

(2.11). *If, in the notation of* (2.1), *the convergences $M_n\to M$, $X_n=A_n\cdot B_n\to A\cdot B=X$ are 0-regular, so also are the convergences $A_n\to A$ and $B_n\to B$.*

For let $\varepsilon>0$ be given. Using $\varepsilon/3$, select δ and N for the regular convergence $X_n\to X$. Then with this δ as a new ε, select δ' and N', $(\delta'<\delta, N'>N)$, for the regular convergence $M_n\to M$. Let $x, y\in A_n$, where $\varrho(x, y)<\delta'$ and $n>N'$. Then there exists a continuum H with $x+y\subset H\subset M$ and $\delta(H)<\delta$. Now $H\cdot A$ contains continua U and V such that $U\supset x+u$ and $V\supset y+v$, where $u+v\subset X_n$. Then since $\varrho(u, v)<\delta$, X_n contains a continuum $W\supset u+v$ with $\delta(W)<\varepsilon/3$. Then $U+V+W$ is a continuum in A_n of diameter $<\varepsilon$ containing $x+y$. Hence $A_n\to A$ 0-regularly. Similarly, $B_n\to B$ 0-regularly.

(2.2). *Let the sequence $[K_n]$ of locally connected continua converge regularly relative to 0-cycles and have limiting set K. Then K is locally connected and every simple closed curve $J\subset K$ is the limiting set of a 0-regularly convergent sequence of simple closed curves $J_1, J_2, \ldots$, where $J_n\subset K_n$.*

Proof. That K is locally connected follows from the case $r=0$ of (1.4). To prove the remainder, take a sequence $\varepsilon_1>\varepsilon_2>\ldots\to 0$. On J take a cyclicly ordered set of points $x_{11}, x_{12}, \ldots, x_{1k_1}$ $x_{1(k_1+1)}=x_{11}$. such that the arcs $x_{1i}x_{1(i+1)}$ are of diameter $<\varepsilon_1/4$. Let $\varepsilon_1'=1/4$ min $[\varrho(x_{1i}x_{1(i+1)}, x_{1j}x_{1(j+1)})]$ for $|i-j|>1$. Using ε_1', determine δ_1

[1]) Considerable similarity will be noted between this case of regular convergence and the notion of an equicontinuous set of curves as used by R. L. Moore. See his papers in Trans. Amer. Math. Soc., vol. 22 (1921), p. 42; and Fund. Math., vol. 4, p. 106; see also H. Whitney, Proc. Natl. Acad. of Sci., vol. 18 (1932), pp. 275—278 and 340—342.

and N_1', $\delta_1 < \varepsilon_1'$, for the regular convergence. For each i, take a $\delta_1/3$ subdivision $x_{1i} = x_{1i}^1, x_{1i}^2, \dots, x_{1i}^{m_1 i} = x_{1(i+1)}$ of $x_{1i}x_{1(i+1)}$, i. e., $\delta(x_{1i}^p x_{1i}^{p+1}) < \delta_1/3$. Determine $N_1 > N_1'$ such that for $n \geqslant N_1$, $K \subset V_{\delta_1/3}(K_n)$ and $K_n \subset V_{\delta_1/3}(K)$. Take a fixed $n_1 > N_1$.

Then the points x_{1i}^m project into points y_{1i}^m on K_{n_1} under a $\delta_1/3$-projection so that $\varrho(y_{1i}^m, y_{1i}^{m+1}) < \delta_1$. Hence by regular convergence, K_{n_1} contains ε_1' — arcs $y_{1i}^m y_{1i}^{m+1}$. Now for each fixed i, $\sum_{m=1}^{m_1 i} y_{1i}^m y_{1i}^{m+1}$ contains an arc $y_{1i}y_{1(i+1)}$, where $y_{1i} = y_{1i}^1$, $y_{1(i+1)} = y_{1(i+1)}^{m_1 i}$; and since $y_{1i}y_{1(i+1)} \subset V_{2\varepsilon_1'}(x_{1i}x_{1(i+1)})$, we have

(i) $\qquad \delta(y_{1i}y_{1(i+1)}) < \varepsilon_1/2 \quad$ and $\quad \varrho(y_{1i}y_{1(i+1)}, y_{1j}y_{1(j+1)}) > 2\varepsilon_1'$,

for $|i-j| > 1$. Thus if we let z_{12} be the first point of $y_{11}y_{12}$ which is on $y_{12}y_{13}$, let z_{13} be the first point of $z_{12}y_{13}$ which is on $y_{13}y_{14}$, and so on to z_{1k_1} as the first point of $z_{1(k_1-1)}y_{1k_1}$ which is on $y_{1k_1}y_{11}$, and let $z_{1(k_1+1)} = z_{11} =$ the first point of $z_{1k_1}y_{11}$ which is on $y_{11}z_{12}$, we obtain a set of arcs $z_{11}z_{12}, z_{12}z_{13}, \dots, z_{1k_1}z_{11}$ each contained in the corresponding arc $y_{1i}y_{1(i+1)}$ and whose sum is a simple closed curve $J_{n_1} \subset K_{n_1}$. By (i) we have

(ii) $\qquad \delta(z_{1i}z_{1(i+1)}) < \varepsilon_1/2 \quad$ and $\quad \varrho(z_{1i}z_{1(i+1)}, z_{1j}z_{1(j+1)}) > 2\varepsilon_1'$

for $|i-j| > 1$.

Now, using ε_2, let us make a similar construction, choosing the points x_{2i} so that they include all of the points x_{1i} (i. e., the subdivision of J is monotonic), finding ε_2' and determining $N_2 > N_1$. Choosing $n_2 > N_2$, we construct similarly a simple closed curve $J_{n_2} \subset K_{n_2}$ consisting of arcs $z_{21}z_{22}, z_{22}z_{23}, \dots, z_{2k_2}z_{21}$ such that

(iii) $\qquad \delta(z_{2i}z_{2(i+1)}) < \varepsilon_2/2 \quad$ and $\quad \varrho(z_{2i}z_{2(i+1)}, z_{2j}z_{2(j+1)}) > 2\varepsilon_2'$

for $|i-j| > 1$, and

(iv) $\qquad J_{n_2} \subset V_{2\varepsilon_2'}(J) \quad$ and $\quad J \subset V_{2\varepsilon_2'}(J_{n_2})$.

Similarly for each i we make such a construction for ε_i. This gives a sequence of simple closed curves $J_{n_i} \subset K_{n_i}$ consisting of arcs $z_{i1}z_{i2}, z_{i2}z_{i3}, \dots, z_{ik_i}z_{i1}$ such that

(v) $\qquad \delta(z_{ip}z_{i(p+1)}) < \varepsilon_i/2 \quad$ and $\quad \varrho(z_{ip}z_{i(p+1)}, z_{iq}z_{i(q+1)}) > 2\varepsilon_i'$

for $|p-q| > 1$, and

(vi) $\qquad J_{n_i} \subset V_{2\varepsilon_i'}(J) \quad$ and $\quad J \subset V_{2\varepsilon_i'}(J_{n_i})$.

Now from (vi) we have that $\lim J_{n_i} = J$, since $\varepsilon_i > \varepsilon_i' \to 0$. Furthermore, the convergence is regular. For let $\varepsilon > 0$ be given. For some k we have $\varepsilon_k < \varepsilon$. Then if we set $\delta = \varepsilon_k'$ and $N = n_k$, the conditions for 0-regular convergence must be satisfied. For by (v), we have that if $x, y \in J_{n_k}$ and $\varrho(x, y) < \delta = \varepsilon_k'$, then x and y are on consecutive arcs $z_{ki}z_{k(i+1)}, z_{k(i+1)}z_{k(i+2)}$ of J_{n_k}; and since by (v) the diameter of the sum of these arcs is $< \varepsilon_k < \varepsilon$, the conditions hold for J_{n_k}. It is readily seen that, since monotonic subdivisions are used in J, they likewise hold for any index $> n_k$. Thus $J_{n_k} \to J$ regularly relative to 0-cycles.

Finally, if for each n with $n_k < n < n_{k+1}$, we construct on K_n a curve J_n just as J_{n_k} was constructed on K_{n_k}, i. e., using ε_k, and if we neglect all indices $< n_1$, it is clear that $J_k \to J$ 0-regularly.

By exactly similar reasoning, we have

(2.21) *Under the same hypothesis as in* (2.2), *every simple arc* ab *in* K *is the limiting set of a 0-regularly convergent sequence of arcs* $[a_n b_n]$, *where* $a_n b_n \subset K_n$.

(2.3). *Let the sequence of* γ^1*-continua* $A_1, A_2, \dots$, *converge regularly relative to 0-cycles and have limiting set* A. *Then* A *is a* γ^1*-continuum.*

To prove this it suffices to show that the condition in (1.3) is satisfied. Let $\varepsilon > 0$ be given. Determine δ and N for the 0-regular convergence. Now let $Z^1 = (x_1, x_2, \dots, x_n)$ be any 1-dimensional δ-cycle on A_n, where the x_i are vertices of Z^1 and where $n > N$. Then for each 1-simplex (x_i, x_j) in Z^1, by virtue of 0-regular convergence, we have in A_n a continuum $x_i x_j$ of diameter $< \varepsilon$. Each such continuum $x_i x_j$ carries a 1-dimensional semi-cycle Z_{ij}[1]) and the sum of all the semi-cycles Z_{ij} is a complete 1-dimensional cycle Z. Let us write $Z = (z_1, z_2, \dots)$, where z_l is a δ_l-cycle and $\delta_l \to 0$.

Now since by hypothesis $Z \sim 0$ in A_n, it follows that for some k, $z_k \underset{\varepsilon}{\sim} 0$ in A_n. But clearly we have $Z^1 \underset{\varepsilon}{\sim} z_k$, since z_k is merely a subdivision of Z^1. In fact z_k breaks up into δ_k-semi-cycles $z_{ij}^k \subset x_i x_j$; and if we choose a point c_{ij} in $x_i x_j$ and form the 2-simplexes (c_{ij}, x_i, x_j) and $(c_{ij}, y_{ij}^m, y_{ij}^s)$ for each 1-simplex (y_{ij}^m, y_{ij}^s) in z_{ij}^k, clearly

[1]) That is, Z_{ij} is a sequence of 1-dimensional complexes $(x_i, x_j) = C_1, C_2, \dots$ in $x_i x_j$ each bounded by the 0-cycle $x_i + x_j$, where C_l is a δ_l-complex which is merely a finer subdivision of C_{l-1} and $\delta_l \to 0$.

the sum K_{ij} of all such 2-simplexes is an ε-complex of dimension 2 bounded by $(x_i, x_j) + z_{ij}^k$. Accordingly, ΣK_{ij} is an ε-complex bounded by

$$\sum (x_i, x_j) + \sum z_{ij}^k = Z^1 + z_k.$$

Whence, $Z^1 \,_\varepsilon\, z_k \underset{\varepsilon}{\sim} 0$, so that $Z^1 \underset{\varepsilon}{\sim} 0$ in A_n, as was to be shown.

(2.31). **Corollary.** *The same conclusion holds, if, instead of supposing each A_i to be a γ^1-continuum, we suppose merely that every 1-dimensional complete cycle in A_i is $\underset{\varepsilon_i}{\sim} 0$ in A_i and that $\varepsilon_i \to 0$.*

3. Sequences of arcs and simple closed curves.

(3.1). *Let the sequence of simple arcs $[a_n b_n]$ converge 0-regularly to the limiting set C. Then C is a simple arc ab (or a single point), and if the notation is suitably chosen, we have $a_n \to a$ and $b_n \to b$.*

For let us suppose C contains more than one point. Let $L = \limsup (a_n + b_n)$. Then L contains more than one point. For otherwise we would have $\varrho(a_n, b_n) \to 0$; and this, by virtue of regular convergence, would give $\delta(a_n b_n) \to 0$, which is impossible since $\delta(C) > 0$. Let a and b be any two distinct points of L, and let x be any point of C distinct from a and from b. Let $x_n \in a_n b_n$ be chosen so that $x_n \to x$. Then for a suitably chosen sequence of integers (n_i) together with a suitable interchange of labels on the endpoints of $a_n b_n$ if necessary, we have

$$a_{n_i} x_{n_i} \to A, \quad x_{n_i} b_{n_i} \to B, \quad a \in A, \quad b \in B, \quad \text{and} \quad A + B = C.$$

Now by virtue of the regular convergence and (2.1), we have $A \cdot B = x$. Thus any point x of $C - (a + b)$ separates C between a and b. Accordingly, C is a simple arc ab.

Now since a and b were any two points of L whatever, it follows that L must reduce to $a + b$. Consequently, if the end points of the arcs $a_n b_n$ are suitably labelled, we will have $a_n \to a$ and $b_n \to b$.

(3.2). *Let the sequence of simple closed curves $[J_n]$ converge regularly relative to 0-cycles and have limiting set J. Then J is a simple closed curve (or a single point).*

Proof. Supposing J contains more than one point, let us take any two distinct points a and b of J. Let us select points $a_n, b_n \in J_n$ so that $a_n \to a$ and $b_n \to b$ and decompose J_n into simple arcs $a_n x_n b_n$ and $a_n y_n b_n$. Then for a suitably chosen sequence (n_i), the sequences $[a_{n_i} x_{n_i} b_{n_i}]$ and $[a_{n_i} y_{n_i} b_{n_i}]$ will converge to limiting sets J_x and J_y respectively. Now $J = J_x + J_y$; and since $a_n + b_n \to a + b$, by regular convergence and (2.1) we have $J_x \cdot J_y = a + b$. Thus every pair of points a, b of J separates J; and hence, by a well known result due to R. L. Moore, it follows that J is a simple closed curve.

4. Characterizations of boundary curves and cactoids.

A locally connected continuum K is said to be a *boundary curve* [1]) provided every true cyclic element of K is a simple closed curve; and such a continuum K is said to be a *cactoid* [2]) provided every true cyclic element of K is a simple closed surface (topological sphere). In this section we establish characterizations of these types of sets which will be needed in the sequel.

(4.1). *In order that a compact continuum K be a boundary curve it is necessary and sufficient that every conjugate* [3]) *pair of points of K should disconnect K.*

Proof. In the light of the cyclic element theory of the structure of a locally connected continuum, the necessity of the condition is obvious. To prove the sufficiency of the condition, we first show that K is locally connected. If this were not so, K would have a continuum of convergence H. But every pair of points of H would be conjugate; and since H would contain uncountably many such pairs which are disjoint and since each such pair would contain a local separating point [4]) of K, H would necessarily contain [4]) a point of Menger-Urysohn order 2 of K. Clearly this is impossible, and hence K is locally connected.

Now let C be any true cyclic element of K. Then C contains a simple closed curve J. Furthermore, we must have $C = J$. For if not, $C - J$ would contain a non-cut point x of K, since the cut points of K on C are countable. But then from our hypothesis it

[1]) See my paper in Amer. Jour. Math., vol. 56 (1934), p. 301.

[2]) See R. L. Moore, Monatsh. f. Math. u. Phys., vol. 36 (1929), p. 81.

[3]) Two points $a, b \in K$ are said to be *conjugate* provided no point separates them in K.

[4]) See my paper in Monatsh. f. Math. u. Phys., vol. 36 (1929), pp. 305—314.

follows that every point of J is a cut point of the connected set $K-x$, which clearly is impossible [1]) since J itself has no cut point.

(4.2). *In order that a locally connected compact continuum K be a cactoid it is necessary and sufficient that for every simple closed curve J in K, $K-J$ have at least two components each for which is bounded by J.*

Proof. Again the necessity of the condition is obvious in view of the structure of K relative to its cyclic elements. In fact we can make the following much stronger statement: If K is a cactoid, then every simple closed curve J in K is contained in a uniquely determined cyclic element C of K, J decomposes C into two open 2-cells C_1 and C_2 bounded by J which lie in distinct components of $K-J$, and every other component of $K-J$ is a component of $K-C$ and hence is bounded by a single point of J.

To prove the sufficiency of the condition, let S be any true cyclic element of K. We have to show that S is a topological sphere, and for this purpose it will suffice to prove that Zippin's conditions [2]) (A, B, C) are satisfied. Now condition (A) states that S contains at least one simple closed curve, and this is obvious. Condition (C) states that S is disconnected by any simple closed curve $J \subset S$. To prove this, let J be any simple closed curve in S. By hypothesis, $K-J$ has at least two components K_1 and K_2 each bounded by J. Since each component of $K-S$ is bounded by a single point of S, it follows that $S \cdot K_1 \neq 0 \neq S \cdot K_2$. Accordingly, J disconnects S, and (C) is proven. We note also that each of the sets $S \cdot K_1$ and $S \cdot K_2$ is connected and, in fact each of these sets is a component of $S-J$ bounded by J.

Now finally, condition (B) states that no arc axb of a simple closed curve J in S can disconnect S. Let us suppose, on the contrary, that some such arc axb disconnects S. Then if we write $J = axb + ayb$, there will exist a component S_1 of $S - axb$ which contains no point of ayb. Now $S_1 + axb$ contains an arc uzv such that $uzv \cdot axb = u + v$ and, by suitably choosing x, we have some such order as a, u, x, v, b on axb. Let C be the simple closed curve $uzv +$ arc $uaybv$ of J. By what was just shown above, $S-C$ has at least two components H_1 and H_2 each bounded by C. Now both H_1 and H_2 must contain points of S_1 as well as points of $S-\overline{S}_1$, since $y + z \subset \overline{H}_1 \cdot \overline{H}_2$. Accordingly, both H_1 and H_2 must intersect the open arc uxv of J. This is impossible, since $uxv - (u+v)$ is connected and contains no point of C. Thus our supposition that axb disconnects S leads to a contradiction, and hence (B) is satisfied. This completes the proof.

[1]) See R. L. Moore, Proc. Ntl. Acad. Sci., vol. 9 (1923), p. 102.

[2]) See Amer. Jour. Math., vol. 52 (1930), p. 333. These conditions of Zippin are an improvement on, and yet they are based on, the original characterization of the topological sphere due to R. L. Moore. See R. L. Moore, Trans. Amer. Math. Soc., vol. 17 (1916) pp. 131—164.

5. Sequences of topological spheres.

(5.1). *Let the sequence $S_1, S_2, \ldots$, of topological spheres converge regularly relative to 1-cycles and have limit C. Then C has no cut point.*

Proof. Suppose on the contrary that for some $x \in C$ we have a separation $C - x = C_a + C_b$, where $a \in C_a$, $b \in C_b$. Let $\varepsilon = 1/8 \min [\varrho(a, x), \varrho(b, x)]$, and determine $\delta' = 2\delta$ and $N(\delta' < \varepsilon)$, for the regular convergence. Let $C'_a = C_a - C_a \cdot V_\delta(x)$ and $C'_b = C_b - C_b \cdot V_\delta(x)$. For $d < \delta$ sufficiently small, we have $V_d(C'_a) \cdot V_d(C'_b) = 0$. Let $G = V_d(C'_a) + V_d(C'_b) + V_\delta(x)$. Select points $a_i, b_i \in S_i$ so that $a_i \to a$, $b_i \to b$. Then for $i >$ some I, we have $a_i \subset V_d(a) \subset G$, $b_i \subset V_d(b) \subset G$ and $S_i \subset G$.

Let n be any integer $> N + I$. Then $V_\delta(x) \cdot S_n$ separates a_n and b_n in S_n, since $V_d(C'_a) \cdot V_d(C'_b) = 0$ and a, b non-$\in V_\delta(x)$. Hence $V_\delta(x) \cdot S_n$ contains a simple closed curve J_n separating a_n and b_n on S_n. But this is impossible in view of the 1-regular convergence, since $\delta(J_n) < \delta'$ whereas each of the 2-cells on S_n determined by J_n must be of diameter $> \varepsilon$, since one of these contains a_n and the other contains b_n.

(5.2). ***Theorem.*** *Let the sequence $S_1, S_2, \ldots$, of topological spheres converge regularly relative to 0-cycles and have limit K. Then K is a cactoid.*

Proof. By (1.4), K is locally connected. Hence to prove K a cactoid, by (4.2) we have only to show that every simple closed curve J in K disconnects K into components at least two of which are bounded by J. To this end, let J be any simple closed curve in K.

By (2.2) there exists a 0-regularly convergent sequence of simple closed curves $J_1, J_2, \dots$, with limit J such that $J_n \subset S_n$. For each n, we have $J_n = A_n + B_n$, where A_n and B_n are 2-cells with $A_n \cdot B_n = J_n$. Furthermore for a suitably chosen sequence of integers (n_i), the sequences A_{n_i} and B_{n_i} will converge to limits A and B respectively. This gives $K = A + B$, $A \cdot B \supset J$; and by virtue of the 0-regular convergence and (2.1), we have $A \cdot B = J$. Furthermore, by (2.11), the convergences $A_{n_i} \to A$ and $B_{n_i} \to B$ are 0-regular.

Now by (2.3), A and B are γ^1-continua. Accordingly, $J \sim 0$ in A and also in B, i. e., the essential complete 1-cycle carried by J is ~ 0 in A and in B. Thus A contains an irreducible membrane M carrying the homology $J \sim 0$. Let $a \,\epsilon (M - J)$ and let K_a be the component of $K - J$ containing a. Then we must have $F(K_a) = J$, where $F(K_a)$ denotes the boundary of K_a relative to K. For if not, then $F(K_a)$ is a T_1-set[1]) disconnecting M into two sets $M \cdot \overline{K}_a$ and $M \cdot (K - K_a)$ intersecting in $F(K_a)$ such that $J \subset M \cdot (K - K_a)$. This is impossible by a result of the author[2]). Hence $F(K_a) = J$. Similarly, using B, we find another component K_b of $K - J$ bounded by J. This completes our proof.

(5.21). ***Corollary.*** *For each n, let S_n be a 2-cell with boundary C_n. Let $S_n \to K$ regularly relative to 0-cycles and let $\delta(C_n) \to 0$. Then K is a cactoid.*

Essentially the same argument applies here as in the proof of (5.2). The only essential difference is that now either A_n or B_n, say B_n, instead of being a 2-cell is a cylinder bounded by J_n and C_n. Results (2.2) and (2.21) now apply just as (2.2) alone applied before, since $\delta(C_n) \to 0$, and the corollary follows.

(5.3). ***Theorem.*** *Let the sequence $S_1, S_2, S_3, \dots$, of topological spheres converge to the limit S regularly relative to 0- and 1-cycles. Then S is a topological sphere.*

[1]) That is, a set carrying no essential complete 1-dimensional cycle. See my paper in American Journal of Mathematics, Vol. 56 (1934) pp. 133—146.

[2]) See my paper loc. cit., p. 134.

For by (5.2) S is a cactoid; and by (5.1) S has no cut point. Hence S reduces to a single cyclic element and thus is a topological sphere.

6. Sequences of 2-cells.

The study of regularly convergent sequences of 2-cells leads us into the types of sets which have been called *hemicactoids* and *base sets* by C. B. Morrey[1]). A base set is a locally connected continuum which is homeomorphic with a plane bounded continuum not separating the plane. Clearly a locally connected continuum B is a base set if and only if every true cyclic element of B is a 2-cell whose interior is open in B. If we delete the interiors of all such 2-cells from a base set B, there will remain a boundary curve J; and we shall say that B is „bounded by" the boundary curve J obtained in this way.

A locally connected continuum H which is the sum of a base set B and a null sequence of disjoint cactoids each of which has exactly one point in common with B is called (with, Morrey) a *hemicactoid.*

(6.1). *If the sequence $[Z_n]$ of 2-cells converges regularly relative to 0-cycles to the limiting set Z and if their boundaries J_n converge to the limit J, then J is a boundary curve. In order that $J_n \to J$ regularly relative to 0-cycles it is necessary and sufficient that J be a simple closed curve*[2]).

To prove that J is a boundary curve it suffices, by virtue of (4.1), to show that every conjugate pair of points of J disconnects J. To this end let $a, b \,\epsilon\, J$ be conjugate. Select points a_n and b_n on J_n in such a way that $a_n \to a$ and $b_n \to b$. Denote the two arcs of J_n from a_n to b_n by $a_n x_n b_n$ and $a_n y_n b_n$ respectively. Then for a suitably chosen sequence of integers (n_i), the sequences $[a_{n_i} x_{n_i} b_{n_i}]$ and $[a_{n_i} y_{n_i} b_{n_i}]$ will converge to limits X and Y respectively. We then have $J = X + Y$.

Furthermore, we must have $X \cdot Y = a + b$. For otherwise there would exist a point x belonging to $X \cdot Y - (a + b)$; and we may

[1]) See Amer. Jour. Math., vol. 57 (1935), pp. 27—29.

[2]) We regard a set consisting of a single point as a degenerate simple closed curve, boundary curve, hemicactoid, etc., as the case may be.

suppose $x_{n_i} \to x \leftarrow y_{n_i}$. Now by the 0-regular convergence of Z_n to Z, it follows that there exist arcs $x_{n_i} y_{n_i} \subset Z_{n_i}$ such that $x_{n_i} y_{n_i} \to x$. For each n_i we have a decomposition

$$Z_{n_i} = A_{n_i} + B_{n_i},$$

where A_{n_i} and B_{n_i} are closed, $A_{n_i} \cdot B_{n_i} = x_{n_i} y_{n_i}$, $A_{n_i} \supset a_{n_i}$ and $B_{n_i} \supset b_{n_i}$. Furthermore, suitably chosen subsequences $[A_{n_{i_j}}]$ and $[B_{n_{i_j}}]$ will converge to limits A and B respectively, so that

$$Z = A + B, \quad \text{where } A \supset a \text{ and } B \supset b.$$

But by (2.1) we must have $A \cdot B = x$, since $Z_{n_{i_j}} \to Z$ 0-regularly. Hence x separates a and b in Z and therefore also in J, which is impossible since a and b are conjugate in J.

Therefore $X \cdot Y = a + b$, and thus $a + b$ separates J. Accordingly, J is a boundary curve.

Now to prove the second part of the theorem, we first note that the necessity of the condition results from (3.2). To prove the sufficiency of the condition, let us suppose on the contrary that J is a simple closed curve, but that the convergence of J_n to J is not regular relative to 0-cycles. It follows that there exists a sequence of integers (n_i) and a sequence of pairs of points $x_i y_i \in J_{n_i}$ such that $x_i \to x \leftarrow y_i$, where $x \in J$, and so that if $x_i a_i y_i$ and $x_i b_i y_i$ denote the two arcs of J_{n_i} from x_i to y_i, then the sequences $[x_i a_i y_i]$ and $[x_i b_i y_i]$ converge to limits J_a and J_b respectively each of which contains more than one point. We may suppose, furthermore, that the points a_i and b_i are so chosen that $a_i \to a \subset J_a - x$ and $b_i \to b \subset J_b - x$. Now by virtue of the regular convergence $Z_{n_i} \to Z$, we can find arcs $x_i y_i \subset Z_{n_i}$ such that $x_i y_i \to x$. And just as before, $x_i y_i$ separates Z_{n_i} into closed sets A_i and B_i such that $A_i \cdot B_i = x_i y_i$; and subsequences $[A_{i_j}]$ and $[B_{i_j}]$ of these converge to limits A and B respectively, where $a \in A$ and $b \in B$. Likewise, by (2.1), we have $A \cdot B = x$; and x separates a and b in Z and thus in J, which is impossible. Therefore $J_n \to J$ regularly relative to 0-cycles.

(6.2). *Let $[Z_n]$ be a sequence of 2-cells with boundaries $[J_n]$. Then if $Z_n \to Z$ regularly relative to 0-cycles and $J_n \to J$, then Z is a hemicactoid whose base set B is bounded by J.*

Proof. By virtue of (2.2) we have that Z is a locally connected continuum. Also we may suppose that J contains more than one point, since otherwise it follows from (5.21) that Z is a cactoid. The rest of the proof we give in the form of three lemmas of which the first follows.

(i). *If x is any cut point of Z and S is a component of $Z - x$ such that $S \cdot J = 0$, then $S + x$ a cactoid.*

To prove this, let $a \in S$ and select points $a_n \in Z_n$ so that $a_n \to a$. Let ε be any positive number less than $\varrho(a, x)$ and $\delta(J)$. Let $V = V_\varepsilon(x)$. Since the sets $S - S \cdot V$ and $Z - S - (Z - S) \cdot V$ are closed and disjoint, there exist disjoint ε-neighborhoods G and D respectively of these sets. There exists an integer n_1 such that

$$Z_{n_1} \subset G + V + D,\ a_{n_1} \subset G - G \cdot \bar{V} \text{ and } J_{n_1} \subset D + V - V \cdot G.$$

Consequently, the set $Z_{n_1} \cdot G \cdot V$ separates a_{n_1} and J_{n_1} in Z_{n_1}; and since Z_{n_1} is a 2-cell, this set contains a simple closed curve C_1 which also separates a_{n_1} and J_{n_1} in Z_{n_1}. Then C_1 bounds a 2-cell $S_1 \subset Z_{n_1}$ which contains a_{n_1}, and C_1 and J_{n_1} together bound a cylinder $H_1 \subset Z_{n_1}$ such that $S_1 + H_1 = Z_{n_1}$ and $S_1 \cdot H_1 = C_1$.

Now since ε was arbitrarily small, it is clear from what has just been shown that we can select a sequence of integers (n_i) and a sequence of simple closed curves (C_i), $C_i \subset Z_{n_i}$ such that $C_i \to x$, C_i separated Z_{n_i} into a 2-cell S_i containing a_{n_i} and a cylinder H_i containing J_{n_i}, and finally so that the sequences $[S_i]$ and $[H_i]$ converge to limiting sets K and H respectively. Since $\delta(C_i) \to 0$, it follows from (2.11) that the convergence $S_i \to K$ is 0-regular. Therefore, by (5.21), K is a cactoid. Furthermore, K must contain $S + x$. For $K \supset a \in S$, $H + K = Z$ and $H \cdot K = x$. Thus since S is a component of $Z - x$, S is also a component of $K - x$; and since K is a cactoid, it follows that S is a cactoid.

We next demonstrate

(ii). *If a, b and x are points of J such that x separates a and b in J, then x also separates a and b in Z.*

The proof for this is very similar to the first part of the proof of (6.1). Just as in that proof, we select sequences $a_{n_i} \to a$, $b_{n_i} \to b (a_{n_i} + b_{n_i} \subset J_{n_i})$ such that J_{n_i} breaks into two arcs $a_{n_i} x_{n_i} b_{n_i}$ and

$a_{n_i} y_{n_i} b_{n_i}$, where $a_{n_i} x_{n_i} b_{n_i} \to X$, $a_{n_i} y_{n_i} b_{n_i} \to Y$, and $x_{n_i} \to x \leftarrow y_{n_i}$ so that $x \subset X \cdot Y$. Then, choosing the arcs $x_{n_i} y_{n_i} \to x$ just as in the former proof and proceeding in the same way, we reach the conclusion that x separates a and b in Z.

As a final lemma, we have

(iii). *If C is any cyclic element of Z intersecting J in more than one point, then $C \cdot J$ is a simple closed curve W, and C is a 2-cell bounded by W.*

Let $a, b \in C \cdot J$. By (ii) it follows that a and b are conjugate in J. Hence there exists a cyclic element W of J which contains $C \cdot J$. Since J is a boundary curve, W is a simple closed curve. Moreover, since $W \cdot C \supset a + b$, we have $W \subset C$. This gives $W \subset C \cdot J$, and therefore $W = C \cdot J$.

Now to prove that C is a 2-cell bounded by W, by virtue of a theorem of Zippin's[1]), we have only to show that every arc in C spanning[2]) W separates C irreducibly. To this end we first prove that any arc $a z b$ spanning W separates C irreducibly between any pair of points x and y, where x lies on one arc of W from a to b and y on the other.

Now from (2.21) it follows that there exist arcs $a_n z_n b_n \subset Z_n$ such that $a_n z_n b_n \cdot J_n = a_n + b_n$, $a_n \to a$, $b_n \to b$ and $a_n z_n b_n \to a z b$ regularly relative to 0-cycles. For each n, we have $Z_n = X_n + Y_n$ where X_n and Y_n are 2-cells bounded, respectively, by $a_n z_n b_n +$ an arc $a_n x_n b_n$ of J_n and $a_n z_n b_n +$ the other arc $a_n y_n b_n$ of J_n. Furthermore, we can select a sequence of integers (n_i) such that $[X_{n_i}]$ and $[Y_{n_i}]$ will converge to limits X and Y respectively, where $X \supset x$ and $Y \supset y$. Since $Z_n \to Z$ 0-regularly, it follows by (2.1) that we have $Z = X + Y$, where $X \cdot Y = \lim a_n z_n b_n = a z b$. Since also $a_n z_n b_n \to a z b$ 0-regularly, it follows by (2.11) that the convergences $X_{n_i} \to X$ and $Y_{n_i} \to Y$ are 0-regular. Accordingly, by (2.3), X and Y are γ^1-continua.

Whence $(a x b + a z b)$ is homologous to 0 in X and thus also in an irreducible membrane $M_x \subset X$ (see Alexandroff, loc. cit.). Similarly $(a y b + a z b) \sim 0$ in an irreducible membrane $M_y \subset Y$.

[1]) See Amer. Jour. Math., vol. 55 (1933), pp. 201—217.

[2]) An arc „spans" W provided it has in common with W exactly its two end points.

Clearly we have $M_x + M_y \subset C$ (see my paper cited above in § 5). Let C_x and C_y be the components of $C - a z b$ containing x and y respectively. Then we must have $M_x - a z b \subset C_x$ and $M_y - a z b \subset C_y$ so that $a z b$ is the boundary relative to C of both C_x and C_y. For otherwise $a z b$ would separate M_x (or M_y) into two sets one of which contains $a z b + a x b$; and since $a z b$ is a T_1-set, this is impossible (see my paper just cited above). Thus $a z b$ separates C irreducibly between x and y.

Now we must have $C - a z b = C_x + C_y$. For if not, there would exist an arc rus in C, where r and s lie on $a z b$ in some such order as a, r, z, s, b and $rus \cdot (C_x + C_y + a z b) = r + s$. This is impossible, because the arc $a r + rus + s b$ spans W in C but clearly does not separate x and y in C, contrary to what we have just shown. Hence $C_x + C_y = C - a z b$, so that $a z b$ separates C irreducibly. Accordingly, C is a 2-cell bounded by W, and (iii) is proven.

Now to prove (6.2), we define the set B to be equal to J plus all cyclic elements C of Z such that $C \cdot J$ contains more than one point. Then B is an A-set[1]) in Z and, by (iii) every true cyclic element E of B is a 2-cell bounded by a true cyclic element of J, and the interior of E is open in B. Thus B is a base set bounded by J. Since B is an A-set in Z, every component S of $Z - B$ has just one boundary point x in B; and, by (i), $S + x$ is a cactoid. Therefore Z is a hemicactoid with base set B bounded by J, and our theorem is proven.

(6.3). *If, as in* (6.2), *the 2-cells $Z_n \to Z$ regularly relative to 0- and 1-cycles, then the hemicactoid Z reduces to its base set B.*

Proof. Recalling the proof of (6.2), this is equivalent to showing that no such component S of $Z - B$ as in (i) can exist. If such a component S does exist, its boundary $\bar{S} - S$ is a single point x; and just as in the proof of (i) under (6.2), we can set up a sequence of 2-cells S_i such that $S_i \subset Z_{n_i}$, $S_i \to K \supset S$, and S_i is bounded by a simple closed curve C_i, where $C_i \to x$. But clearly this is impossible, since by regular convergence relative to 1-cycles, the 2-cells S_i bounded by the curves C_i would have to approach 0 in diameter (since $\delta(C_i) \to 0$). Thus S cannot exist and hence $Z = B$.

[1]) See Kuratowski and Whyburn, Fund. Math., vol. 16 (1930), p. 309.

(6.4). *If, as in* (6.2), *the* 2-*cells* $Z_n \to Z$ *regularly relative to* 0- *and* 1-*cycles and if* $J_n \to J$ *regularly relative to* 0-*cycles, then* Z *is a* 2-*cell bounded by* J.

For by (6.2), Z is a hemicactoid with base set B bounded by J; by (6.3), it follows that under these conditions, Z reduces to its base set B; and finally, by (6.1) [or (3.2)], J is a simple closed curve. Thus $B = Z$ is a 2-cell bounded by J, and our theorem is proven.

In conclusion, we call attention to the fact that in all cases considered in this paper, 0-regular convergence for a sequence of sets of type A has yielded as limiting set a type of set B which can be obtained by an upper semi-continuous decomposition [1]) of A into continua, or in other words, a type of set B which can be the image of A under a „monotone" transformation in the sense of C. B. Morrey (loc. cit). This suggests that our results above may be approached from an „analytic" or transformation point of view. This is indeed the case, and a study of this method of approach will be made in an article which is to follow the present one.

[1]) See R. L. Moore, *Foundations of point set theory*, Amer. Math. Soc. Colloquium Publications, 1932, Ch. V.

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Über die Abbildungen von Sphären auf Sphären niedrigerer Dimension.

Von

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Die Frage, für welche Dimensionszahlen N und n mit $N > n$ es möglich ist, die Sphäre S^N wesentlich auf die Sphäre S^n abzubilden [1]), ist meines Wissens bisher nur in zwei Fällen beantwortet: 1) Für jedes $N > 1$ ist es unmöglich, die Sphäre S^N wesentlich auf den Kreis S^1 abzubilden; 2) es ist möglich, die Sphäre S^3 auf die Kugelfläche S^2 wesentlich abzubilden [2]).

Die Frage scheint mir aus verschiedenen Gründen der weiteren Untersuchung wert zu sein. Erstens, versagt bei der Behandlung der Abbildungen der S^N in die S^n die übliche Methode des Abbildungsgrades; denn in S^N ist jeder n-dimensionale Zyklus homolog Null und wird daher mit dem Grade 0 abgebildet; infolgedessen zwingt unsere Frage dazu, nach neuen Methoden zu suchen. Zweitens, weisen eine Reihe bekannter Sätze darauf hin, daß sich in der Existenz einer wesentlichen Abbildung eines Raumes R auf die S^n

[1]) Eine stetige Abbildung f_0 des Raumes A auf den Raum B heißt „wesentlich", wenn bei jeder Abbildung f_1, in welche sich f_0 stetig überführen läßt, das Bild $f_1(A)$ der ganze Raum B ist. Ist B eine Sphäre, so bedeutet die Unwesentlichkeit von f_0, daß sich f_0 in eine solche Abbildung f_1 stetig überführen läßt, bei welcher $f_1(A)$ ein einziger Punkt ist.

[2]) H. Hopf, *Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche*, Math. Ann. 104. Die Kenntnis dieser Arbeit wird für das folgende vorausgesetzt. Einen neuen Beweis für die wesentliche Abbildbarkeit der S^3 auf die S^2 hat W. Hurewicz gegeben: *Beiträge zur Topologie der Deformationen*, Proceed. Amsterdam XXXVIII („Anwendungen", S. 117).