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The determination of representative elements in the residual classes of a Boolean algebra.

By

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1. Introduction.

If A is any abstract ring and $\mathfrak{a}$ is a left- and right-ideal in A, we may consider the problem of selecting a single representative element from each of the residual classes (mod $\mathfrak{a}$) *in such a manner that sums and products of representative elements are themselves representative elements.* If such a selection is possible, the representative elements evidently constitute a subsystem of A which is isomorphic to the quotient-ring $A/\mathfrak{a}$ under the correspondence carrying each residual class (mod $\mathfrak{a}$) into its representative element. In this paper we shall confine our attention to rings in which every element is idempotent — that is, in which the law $aa = a$ obtains. These rings will be seen to have the formal properties of certain algebras of classes, and will therefore be termed *Boolean rings.* A particular case of the representation problem for Boolean rings has previously been discussed by one of us[1]). Here we shall examine the problem on an abstract basis, giving sufficient conditions for the existence of a solution, special cases in which no solution exists, and special cases in which a solution exists if and only if $\aleph_{n+1} = 2^{\aleph_n}$. The sufficient conditions given here can be applied to the particular case mentioned above.

[1]) J. von Neumann, Journal für Mathematik, 165 (1931), pp. 109—115.

2. Boolean Rings[2]).

We shall review briefly some of the principal algebraic properties of rings in which every element is idempotent, showing in particular how they are related to algebras of classes. We first establish the following result:

Theorem 1. *A Boolean ring is necessarily commutative; obeys the two equivalent laws* $a+a=0$, $a=-a$; *and necessarily contains divisors of* 0 *unless it contains at most two elements. Every subring of a Boolean ring is itself a Boolean ring. Every system with double composition homomorphic to a Boolean ring* A *is a Boolean ring isomorphic to a quotient-ring* $A/\mathfrak{a}$ *where* $\mathfrak{a}$ *is an ideal in* A. *If* $\mathfrak{a}$ *is an ideal in a Boolean ring* A, *then the following assertions are equivalent:* $\mathfrak{a}$ *is a prime ideal,* $A/\mathfrak{a}$ *is a ring of exactly two elements,* $\mathfrak{a}$ *is a divisorless ideal.*

From the left- and right-distributive laws for multiplication and the commutative and associative laws for addition, we see that in a Boolean ring

$$a+b=(a+b)(a+b)=(a+b)\,a+(a+b)\,b=(a+ba)+ \\ +(ab+b)=(a+b)+(ba+ab)$$

and hence that $ba+ab=0$. If we put $b=a$, we find at once that $a+a=0$ or, equivalently, $a=-a$. Using this result, we conclude that $ba=-(ab)=ab$. In a Boolean ring with more than two elements, we can choose elements a and b so that $a\neq b$, $a\neq 0$, $b\neq 0$. If $ab=0$, then a and b are both divisors of zero. On the other hand, if $ab\neq 0$, then ab and $a+b$ are both divisors of zero: for, $a+b=0$ would imply $a=-b=b$, contrary to hypothesis; and $ab(a+b)=aba+abb=ab+ab=0$.

Since the law $aa=a$ holds in every subsystem and in every homomorph of a Boolean ring, the assertions of the theorem concerning subrings and homomorphs follow directly from specializations of known propositions of abstract algebra. If now $\mathfrak{a}$ is a prime ideal in a Boolean ring, then, by definition, $A/\mathfrak{a}$ has no divisors of zero; hence, by the results above, $A/\mathfrak{a}$ has exactly two elements. If $A/\mathfrak{a}$ has exactly two elements — that is, if the residual classes (mod $\mathfrak{a}$) are two in number — then $\mathfrak{a}$ is obviously divisorless. Finally, if $\mathfrak{a}$ is divisorless, it is known to be prime.

Theorem 2. *A Boolean ring* A *can be imbedded in a Boolean ring* B *in which there is a unit element; and even in such a way that the elements of* A *constitute a prime ideal in* B[3]).

We introduce first an abstract element ε, different from those of A, and define

$$\varepsilon\varepsilon=\varepsilon,\quad \varepsilon a=a\varepsilon=a,\quad \varepsilon+0=0+\varepsilon=\varepsilon,\quad \varepsilon+\varepsilon=0.$$

We then consider the class of ordered pairs (a,α) where a is in A and $\alpha=0$ or $\alpha=\varepsilon$. For such pairs, we define addition and multiplication by the rules

$$(a,\alpha)+(b,\beta)=(a+b,\ \alpha+\beta),$$
$$(a,\alpha)(b,\beta)=(ab+\alpha b+a\beta,\ \alpha\beta).$$

It is easily verified that under these operations the class of pairs (a,α) is an abstract ring in which every element is idempotent and in which the element $(0,\varepsilon)$ is a unit in accordance with the relation

$$(0,\varepsilon)(a,\alpha)=(0a+\varepsilon a+0\alpha,\ \varepsilon\alpha)=(a,\alpha).$$

The class of pairs $(a,0)$ is a ring isomorphic to A, as can ce seen from the equations

$$(a,0)+(b,0)=(a+b,0),\quad (a,0)(b,0)=(ab,0).$$

It is also an ideal in the ring of all pairs (a,α), since we have

$$(a,0)(b,\beta)=(ab+a\beta,0),$$
$$(a,0)-(b,0)=(a-b,0).$$

Furthermore, it is a prime ideal $\mathfrak{a}$: for the elements (a,α) and (b,β) are congruent (mod $\mathfrak{a}$) if and only if $(a,\alpha)-(b,\beta)=(a,\alpha)+(b,\beta)=$ $=(a+b,\ \alpha+\beta)$ is an element for which $\alpha+\beta=0$; and since $\alpha+\beta=0$ if and only if $\alpha=\beta$, there are exactly two residual

[2]) The results of this section have been announced previously by Stone, Proceedings of the National Academy of Sciences, 20 (March 1934), pp. 197—202; 21 (Febr. 1935), pp. 103—105. A detailed exposition will appear elsewhere. The algebraic background is given by van der Waerden, Moderne Algebra, I (Berlin 1930), Chapter III.

[3]) The unit in B is necessarily unique. If A has no unit and is related to B in this particular way, then every Boolean ring C which has a unit and which is an extension of A contains an isomorph of B.

classes (mod $\mathfrak{a}$) and Theorem 1 is applicable. We are now permitted to replace each pair $(a, 0)$ by the element a itself, so as to obtain the Boolean ring B described in the theorem.

Theorem 3. *If A is a Boolean ring with unit element e, the introduction of a unary operation $'$ and a binary operation $\vee$ through the equations*

(1) $a' = a + e$ (2) $a \vee b = a + b + ab$

converts A into an algebraic system in which

(3) $a \vee b = b \vee a$ (4) $a \vee (b \vee c) = (a \vee b) \vee c$

(5) $(a' \vee b')' \vee (a' \vee b)' = a$

the old operations being expressed in terms of the new through the equations

(6) $a + b = ab' \vee a'b = (a' \vee b'')' \vee (a'' \vee b')'$

(7) $ab = (a' \vee b')'$

On the other hand, if B is an algebraic system with a unary operation $'$ and a binary operation $\vee$ obeying the rules (3), (4) *and* (5), *then the introduction of new operations through equations* (6) *and* (7) *converts B into a Boolean ring with unit e, the old operations being expressed in terms of the new through equations* (1) *and* (2).

If A is a given Boolean ring with unit e, it is easily verified with the help of the special rules set forth in Theorem 1 that the relations (3)—(7) inclusive are algebraic identities in A in the presence of the defining equations (1) and (2). The calculations are simple enough that we shall not give details here. On the other hand, if B is a given system with properties (3), (4) and (5), we can similarly verify with the help of a recent paper of Huntington[4]) that the introduction of new operations through equations (6) and (7) effects the conversion described. Since the algebraic properties of such a system B are not at first sight so familiar as those of an abstract ring, the calculations involved in this verification are not immediately obvious. Huntington's discussion, however, shows that the abstract system B has the formal properties of an algebra of classes, in the sense that the operations $'$ and $\vee$ behave like the operations of forming the complement and union respectively. From this point of view the necessary calculations become essentially simple. As Huntington proves, the element $e = a \vee a'$ is independent of a and serves as the unit required by the statement of the theorem. We may remark that, in establishing the existence of a solution of the equation $x + a = b$, one can avoid some complications by first proving the special rules $a + 0 = a$, $a + a = 0$ for $0 = e'$ and then calling upon the commutative and associative laws for the operation $+$ to verify that $x = a + b$ is a solution.

[4]) Huntington, Trans. Amer. Math. Soc. 35 (1935), pp. 274—304 and pp. 552—558.

By combining Theorems 2 and 3 with the results of Huntington's paper, we can now make the following assertion:

Theorem 4. *In a Boolean ring A, the operations of addition and multiplication behave formally like the operations of forming the symmetric difference (that is, the class of objects belonging to one or the other, but not to both, of two classes) and the intersection of two classes, respectively; and the operation $\vee$ introducedt hrough the equation $a \vee b = a + b + ab$ behaves like the operation of forming the union of two classes. Furthermore, if A contains a unit e, then the operation $'$ introduced through the equation $a' = a + e$ behaves like the operation of forming the complement of a class.*

We can, in fact, go so far as to construct an algebra of classes which is isomorphic to a given Boolean ring A; but this construction is not essential to the considerations of this paper and therefore will not be discussed here. This result serves to show that Boolean rings actually have *all* the formal properties of those algebras of classes in which it is possible to form symmetric differences and (finite) unions at pleasure. Of course, it is now an easy matter to give concrete examples of Boolean rings, either with or without unit, merely by constructing such algebras of classes. The Lebesgue-measurable subsets of the plane, for instance, constitute a Boolean ring with unit; while the Lebesgue-measurable subsets of the plane which have *finite* measure constitute a Boolean ring without unit. Theorem 2 shows us that it is always possible to adjoin a unit to the latter ring; indeed, it is easily seen that the adjunction consists essentially in considering the complements (relative to the plane) of the sets belonging to the ring along with the latter sets.

In the succeeding discussion, we shall often use the operations $\vee$ and $'$ as defined in Theorems 3 and 4 in place of, or in conjunction with, the operations of addition and multiplication given in a Boolean ring. We shall therefore find it convenient to characterize ideals in terms of operations other than those given initially. It will be sufficient to note the following result.

Theorem 5. *In order that a non-void subclass $\mathfrak{a}$ of a Boolean ring A be an ideal, it is necessary and sufficient that the conditions*

(1) ab *belongs to* $\mathfrak{a}$ *whenever* a *belongs to* $\mathfrak{a}$;

(2) $a+b$ *belongs to* $\mathfrak{a}$ *whenever* a *and* b *belong to* $\mathfrak{a}$,

be satisfied; and it is also necessary and sufficient that the conditions (1) *and*

(3) $a \vee b$ *belongs to* $\mathfrak{a}$ *whenever* a *and* b *belong to* $\mathfrak{a}$,

be satisfied.

Since $a+b=a-b$ by Theorem 1, conditions (1) and (2) are essentially a trivial restatement of the conditions imposed in the usual definition of an ideal. Thus we have merely to prove that (2) and (3) are equivalent in the presence of (1). It is obvious that (3) follows from an application of (1) and (2) to the relation $a \vee b = a+b+ab$. On the other hand, it is clear that (2) follows similarly from an application of (1) and (3) to the relations

$$(a \vee b)(a+b) = (a+b+ab)(a+b) = (a+b) + ab(a+b) = a+b.$$

As a consequence of Theorem 5, it becomes especially simple to give illustrations of ideals in Boolean rings. For instance, the class of all subsets of the plane which have measure zero is an ideal in the Boolean ring of all Lebesgue-measurable subsets of the plane.

Finally, it will be convenient for us to introduce on an abstract footing the relation corresponding to that of inclusion. This we may do as follows:

Definition 1. *If a and b are elements of a Boolean ring, then b is said to include, or contain, a and a is said to be included by, or contained in, b if any one of the equivalent conditions*

$$ab=a, \quad a \vee b = b, \quad ab' = 0, \quad a' \vee b = e$$

is valid, the two last being significant only in case the given Boolean ring has a unit.

The equivalence of the various conditions and the usual properties of the relation of inclusion defined thereby hardly need be

discussed in detail here. We shall make particular use of the third condition; and shall, in general. rely upon the entirely sound analogy with the relation of class-inclusion rather than give formal proofs of such simple facts as we need in this connection.

3. Algebraic Aspects of the Representation Problem.

We shall now investigate the algebraic side of the representation problem stated in the introduction. First, let us describe the problem in somewhat more formal language.

Definition 2. *If $\mathfrak{a}$ is a left- and right-ideal in an arbitrary ring A, then the $(A, \mathfrak{a})$ representation problem is the problem of constructing a function $f(a)$ defined over A and assuming values in A with the following properties:*

P 1. $f(a) \equiv a \pmod{\mathfrak{a}}$;
P 2. $a \equiv b \pmod{\mathfrak{a}}$ *implies* $f(a) = f(b)$;
P 3. $f(a+b) = f(a) + f(b)$;
P 4. $f(ab) = f(a) f(b)$.

We could also state the problem in the following equivalent form: to construct a function $g(k)$, defined over $A/\mathfrak{a}$ (the elements of which are the residual classes $k \pmod{\mathfrak{a}}$ in A) and assuming values in A, with the following properties: $g(k)$ is an element in k, $g(k+l) = g(k) + g(l)$, $g(kl) = g(k)g(l)$. For, if $f(a)$ is a function with the properties described in Definition 2, we can define a function $g(k)$ by putting $g(k) = f(a)$ where a is in k; and, conversely, if $g(k)$ is a function with the properties set forth above, we can define a suitable function $f(a)$ by putting $f(a) = g(k)$ when a is in k.

We begin by examining the status of the properties $P3$ and $P4$ of Definition 2.

Theorem 6. *If $f(a)$ is a function defined over a Boolean ring A and assuming values in a Boolean ring B, then $f(a)$ has the properties $P3$, $P4$ if and only if it has the properties*

$$P0. \quad f(0) = 0, \qquad P4., \qquad P5. \quad f(a \vee b) = f(a) \vee f(b).$$

If we put $a=b$ in $P3$ and apply Theorem 1, we obtain $P0$; and $P3$ and $P4$ obviously imply $P5$ in accordance with the relations

$$f(a \vee b) = f(a+b+ab) = f(a) + f(b) + f(a)f(b) = f(a) \vee f(b).$$

On the other hand, $P0$, $P4$ and $P5$ show that, in case $cd=0$ we have

$$f(c+d)=f(c\vee d)=f(c)\vee f(d)=f(c)+f(d)+f(c)f(d)=$$
$$=f(c)+f(d)+f(cd)=f(c)+f(d).$$

Thus the relations $(a+ab)(ab+b)=0$, $(a+ab)ab=0$, and $(ab+b)ab=0$ enable us to write

$$f(a+b)=f((a+ab)+(ab+b))=f(a+ab)+f(ab+b)=$$
$$=[f(a+ab)+f(ab+b)]+[f(ab)+f(ab)]=$$
$$=[f(a+ab)+f(ab)]+[f(b+ab)+f(ab)]=$$
$$=f(a+ab+ab)+f(b+ab+ab)=f(a)+f(b)$$

for arbitrary elements a and b. The theorem is thus established.

Theorem 7. *If $f(a)$ is a function defined over a Boolean ring A with unit and assuming values in a Boolean ring B with unit, then $f(a)$ has the property $P4$ if and only if has the property*

$P6$. $\quad a'bc=0$ *implies* $f'(a)f(b)f(c)=0.$

We note that $a'c=0$ implies $ac=c$. Thus, when $P4$ is valid, we have $f'(a)f(c)=f'(a)f(ac)=f'(a)f(a)f(c)=0$. It is therefore evident that in the presence of $P4$ the relation $a'bc=0$ implies $f'(a)f(b)f(c)=f'(a)f(bc)=0$, On the other hand, if $P6$ is valid, we first observe that the equation $c=ab$ is equivalent to the simultaneous relations $c'(ab)=0$, $a'c=0$, $a'b=0$ and thus conclude that $f(c)=f(ab)$ satisfies the relations

$$f'(c)f(a)f(b)=0,\quad f'(a)f(c)=0,\quad f'(a)f(b)=0$$

and hence also the relation $f(ab)=f(a)f(b)$.

Similarly, we have

Theorem 8. *If $f(a)$ is a function defined over a Boolean ring A with unit and assuming values in a Boolean ring B with unit, then $f(a)$ has the property $P5$ if and only if it has the property*

$P7$. $\quad a'b'c=0$ *implies* $f'(a)f'(b)f(c)=0$

We first note that $a'c=0$ if and only if $a\vee c=a$. Thus, if $P5$ is valid, $a'c=0$ implies $f(a)\vee f(c)=f(a\vee c)=f(a)$, and hence $f'(a)f(c)=0$, Now, replacing a by $a\vee b$ and applying $P5$, we find that $(a\vee b)'c=a'b'c=0$ implies $f'(a)f'(b)f(c)=(f(a)\vee f(b))'f(c)=$ $=f'(a\vee b)f(c)=0$. On the other hand, if $P7$ is valid, we first observe that the equation $c=a\vee b$ is equivalent to the simultaneous relations $a'b'c=(a\vee b)'c=0$, $c'a=0$, $c'b=0$ and thus conclude that $f(c)=f(a\vee b)$ satisfies the relations

$$f'(a)f'(b)f(c)=0,\quad f'(c)f(a)=0,\quad f'(c)f(b)=0$$

and hence also the relation $f(a\vee b)=f(a)\vee f(b)$.

Theorem 9. *If $f(a)$ and $g(a)$ are functions defined over a Boolean ring A with unit, assuming values in a Boolean ring B with unit, and connected by the relation $f(a')=g'(a)$, — then $f(a)$ has the property $P5$ if and only if $g(a)$ has the property $P4$.*

If $f(a)$ has property $P5$, then

$$g(ab)=f'((ab)')=f'(a'\vee b')=(f(a')\vee f(b'))'=f'(a')f'(b')=g(a)g(b);$$

and if $g(a)$ has property $P4$, then

$$f(a\vee b)=f((a'b')')=g'(a'b')=(g(a')g(b'))'=g'(a')\vee g'(b')=f(a)\vee f(b).$$

Theorem 10. *If $f(a)$ is a function defined over a Boolean ring A and assuming values in a Boolean ring B, then the function $g(a)=f(a)+f(0)$ has whatever of the properties $P4$ and $P5$ is valid for $f(a)$ and has also the property $P0$.*

It is immediately evident that $g(0)=f(0)+f(0)=0$. We have

$$g(a)g(b)=[f(a)f(b)+f(0)]+f(a)f(0)+f(b)f(0),$$
$$g(a)\vee g(b)=g(a)+g(b)+g(a)g(b)=[f(a)+f(b)+f(a)f(b)+f(0)]+$$
$$f(a)f(0)+f(b)f(0)=[(f(a)\vee f(b))+f(0)]+f(a)f(0)+f(b)f(0).$$

If $f(a)$ has the property $P4$, then $f(a)f(0)=f(a0)=f(0)$, $f(b)f(0)=f(0)$, and the expression for $g(a)g(b)$ reduces to

$$g(a)g(b)=f(ab)+f(0)=g(ab).$$

Similarly, if $f(a)$ has the property $P5$, we find that

$$f(a)f(0)=f(a\vee 0)f(0)=f(a)f(0)\vee f(0)=f(0),\quad f(b)f(0)=f(0),$$

and hence that the expression for $g(a)\vee g(b)$ reduces to

$$g(a)\vee g(b)=f(a\vee b)+f(0)=g(a\vee b).$$

This completes the proof of the theorem.

Theorem 11. *If $f(a)$ is a function defined over a Boolean ring A with unit and assuming values in a Boolean ring B with unit, then $f(a)$ has the properties $P2$, $P3$, $P4$ if and only if it has the property*

$P.$ $$a_1 \dots a_m b_1' \dots b_n' \equiv 0 \pmod{\mathfrak{a}} \text{ implies}$$
$$f(a_1) \dots f(a_m) f'(b_1) \dots f'(b_n) = 0 \quad \text{for} \quad m \geqq 1 \quad \text{and} \quad n \geqq 0$$

where the same ideal $\mathfrak{a}$ *enters in* $P2$ *and* P. *In order that* $f(a)$ *shall have the additional property that* $f(e) = e$, *where* e *denotes indifferently the units in* A *and* B, *it is necessary and sufficient that the property* P *be widened to the property*

$P^*.$ $$a_1 \dots a_m b_1' \dots b_n' \equiv 0 \pmod{\mathfrak{a}} \text{ implies}$$
$$f(a_1) \dots f(a_m) f'(b_1) \dots f'(b_n) = 0 \quad \text{for} \quad m + n \geqq 1.$$

If $f(a)$ has properties $P3$, $P4$, then it also has property $P5$ by Theorem 6; hence we have, for $m \geqq 1$ and $n \geqq 0$,

$$\begin{aligned} f(a_1) \dots f(a_m) f'(b_1) \dots f'(b_n) &= f(a_1 \dots a_m) f'(b_1 \vee \dots \vee b_n) = \\ &= f(a_1 \dots a_m) + f(a_1 \dots a_m) f(b_1 \vee \dots \vee b_n) = \\ &= f(a_1 \dots a_m + a_1 \dots a_m (b_1 \vee \dots \vee b_n)) = \\ &= f(a_1 \dots a_m (b_1 \vee \dots \vee b_n)') = \\ &= f(a_1 \dots a_m b_1' \dots b_n'). \end{aligned}$$

Consequently, if $f(a)$ also has property $P2$, we see that

$$a_1 \dots a_m b_1' \dots b_n' \equiv 0 \pmod{\mathfrak{a}}$$

implies

$$f(a_1) \dots f(a_m) f'(b_1) \dots f'(b_n) = f(0)$$

for $m \geqq 1$, $n \geqq 0$. Since $P3$ implies $f(0) = 0$ in accordance with Theorem 6, we thus find that $P2$, $P3$, and $P4$ imply P.

If, on the other hand, $f(a)$ has the property P, it is at once evident that $f(a)$ has properties $P6$ and $P7$. Hence $f(a)$ has properties $P4$ and $P5$ in accordance with Theorems 7 and 8 respectively. It is also evident that P implies $P0$ — in other words, that $a = 0$ yields $f(a) = 0$. Consequently, by virtue of Theorem 6, we see that P implies $P3$ and $P4$. It remains to establish $P2$. Since $a \equiv b \pmod{\mathfrak{a}}$ implies $ab' \equiv 0 \pmod{\mathfrak{a}}$ and $a'b \equiv 0 \pmod{\mathfrak{a}}$, we conclude with the help of the property P that $a \equiv b \pmod{\mathfrak{a}}$ implies $f(a) f'(b) = 0$, $f'(a) f(b) = 0$, and hence $f(a) = f(b)$.

The property P^* differs from the property P only by the adjunction of the requirement that $b' \equiv 0 \pmod{\mathfrak{a}}$ imply $f'(b) = 0$ — in other words, the requirement that $b \equiv e \pmod{\mathfrak{a}}$ imply $f(b) = e$. In view of the property $P2$, this requirement is satisfied if and only if $f(e) = e$.

By a comparison of Definition 2 and Theorem 11, we at once obtain the following fundamental theorem, on which we base our subsequent study of the representation problem.

Theorem 12. *If* A *is a Boolean ring with unit and* $\mathfrak{a}$ *is an ideal in* A, *then the* $(A, \mathfrak{a})$ *representation problem has a solution if and only if there exists a function* $f(a)$, *defined over* A *and assuming values in* A, *which has properties* $P1$ *and* P; *if such a function* $f(a)$ *exists, it is a solution of the representation problem. In order that this solution have the additional property that* $f(e) = e$, *where* e *is the unit in* A, *it is necessary and sufficient that the property* P *be widened to the property* P^*.

Theorem 12 applies only to the case of a Boolean ring with unit. In order to discuss the general case, we combine Theorem 2, according to which every Boolean ring has an extension with unit, and the following result:

Theorem 13. *If the Boolean ring* A *is imbedded in a Boolean ring* B *in such a manner as to be a prime ideal in* B, *then every ideal* $\mathfrak{a}$ *in* A *is also an ideal in* B; *and the* $(A, \mathfrak{a})$ *representation problem has a solution, if and only if the* $(B, \mathfrak{a})$ *representation problem has a solution.*

Let $f(a)$ be a solution of the $(B, \mathfrak{a})$ representation problem. Since $\mathfrak{a}$ is contained in A, it is evident that $a \equiv b \pmod{\mathfrak{a}}$ implies $a \equiv b \pmod{A}$, whatever the elements a and b in B. In particular, if a belongs to A, we find that $f(a) \equiv a \pmod{\mathfrak{a}}$, $f(a) \equiv a \pmod{A}$, and $f(a) \in A$. Consequently, if we restrict a to be an element of A, the function $f(a)$ has all the properties set forth in Definition 2 and is therefore a solution of the $(A, \mathfrak{a})$ representation problem. On the other hand, let $f(a)$ be a solution of the $(A, \mathfrak{a})$ representation problem. We now have to extend f over B. To do so we put $f(a + \alpha) = f(a) + \alpha$, where a is any element in A and either $\alpha = 0$ or $\alpha = e$ (the unit in B). If $a + \alpha \equiv b + \beta \pmod{\mathfrak{a}}$, we have $a - b \equiv \beta - \alpha \pmod{\mathfrak{a}}$, $a - b \equiv \beta - \alpha \pmod{A}$; thus, if a and b are in A, both $a - b$ and $\beta - \alpha$ are in A; and hence $\beta - \alpha = 0$, $\alpha = \beta$, $a \equiv b \pmod{\mathfrak{a}}$. We have thus proved that $a + \alpha \equiv b + \beta \pmod{\mathfrak{a}}$ implies $f(a + \alpha) = f(a) + \alpha = f(b) + \beta = f(b + \beta)$. By putting $\alpha = 0$, we see that the extended function coincides with $f(a)$ over A. It remains for us to show that the

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extended function is defined for every element in B. If b is any element in B but not in A, we know from the characterization of prime ideals given in Theorem 1 that $b \equiv e \pmod{A}$. Hence $b' \equiv 0 \pmod{A}$ and $b = a + \alpha$ where $a = b'$ is in A and $\alpha = e$. Thus $f(b)$ is defined. To summarize, we may say that $f(a)$ is defined over B, assumes values in B, and has property $P2$. To show that $f(a)$ is a solution of the $(B, \mathfrak{a})$ representation problem, we have only to observe the further relations

$$\begin{aligned} f(a+\alpha) &= f(a) + \alpha \equiv a + \alpha \pmod{\mathfrak{a}}, \\ f(a+\alpha) + f(b+\beta) &= [f(a) + f(b)] + [\alpha + \beta] = \\ &= f(a+b) + (\alpha + \beta) = f((a+\alpha) + (b+\beta)), \\ f(a+\alpha)\, f(b+\beta) &= f(a)\, f(b) + \alpha\, f(b) + f(a)\, \beta + \alpha\beta = \\ &= f(a\,b) + f(\alpha\, b) + f(\beta\, a) + \alpha\,\beta = \\ &= f(ab + \alpha\, b + \beta\, a) + \alpha\,\beta = \\ &= f((a+\alpha)\,(b+\beta)). \end{aligned}$$

In establishing the third property, we have made use of the fact that $f(0) = 0$ by Theorem 6 and of the fact that $\alpha = 0$ or $\alpha = e$ in order to write $\alpha f(b) = f(\alpha b)$ and, similarly, $\beta f(a) = f(\beta a)$. We note that $f(e) = f(0 + e) = f(0) + e = e$ in accordance with our definition of the extended function.

Finally, we shall show that, in considering the $(A, \mathfrak{a})$ representation problem where A is a Boolean ring with unit, we may impose the condition that the solution $f(a)$ have the property $f(e) = e$. To show this, we combine Theorem 13 with the fact that it is always possible, when $\mathfrak{a}$ does not coincide with A, to construct a prime ideal $\mathfrak{p}$ containing $\mathfrak{a}$: for the existence of a solution of the $(A, \mathfrak{a})$ representation problem implies the existence of a solution of the $(\mathfrak{p}, \mathfrak{a})$ representation problem; and this in turn leads to the existence of a solution of the $(A, \mathfrak{a})$ representation problem with $f(e) = e$, by the construction given in Theorem 13. We shall first prove the existence of a prime ideal $\mathfrak{p}$ containing $\mathfrak{a}$, by a suitable application of Theorem 11 [5]). The argument employed will reappear, with variations, in establishing the sufficient conditions of the following section and should therefore be carefully noted.

[5]) Other proofs are known. See Stone, Proceedings of the National Academy of Sciences, 20 (1934), pp. 197—202.

Theorem 14. *If A is a Boolean ring with unit and $\mathfrak{a}$ is any ideal in A which is a proper subclass of A, then there exists a function $\alpha(a)$ defined over A, assuming only the two values 0 and e in A, and possessing property P^*. The class $\mathfrak{p}$ of all elements a such that $\alpha(a) = 0$ is a prime ideal containing $\mathfrak{a}$. Conversely, if $\mathfrak{p}$ is any prime ideal containing $\mathfrak{a}$, then the function $\alpha(a)$ which is equal to 0 or to e according as a is in $\mathfrak{p}$ or not, has the property P^*; any such function has also the properties $P0$, $P2$, $P3$, $P4$, and $\alpha(e) = e$.*

We shall give an inductive construction based upon the property P^* and a suitable well-ordering of the residual classes $(\text{mod } \mathfrak{a})$. We choose an ordinal number ω so that the residual classes $(\text{mod } \mathfrak{a})$ can be put in one-to-one correspondence with the class of ordinals γ such that $\gamma < \omega$. We have to use the Zermelo hypothesis at this point, of course. We may suppose (though we do not need to do so in the present instance) that ω is the *first* ordinal number available for this purpose. Under this supposition the ordinals γ such that $\gamma < \delta < \omega$ constitute a class with cardinal number *less* than that of the quotient algebra $A/\mathfrak{a}$ (that is, the class of all residual classes $(\text{mod}\, \mathfrak{a})$). We may further suppose that the class $\mathfrak{a}$ is put in correspondence with the ordinal number 1. Having assigned an ordinal number $\gamma, \gamma < \omega$, to each residual class $(\text{mod } \mathfrak{a})$ in this manner, we may interpret the resulting correspondence as an assignment of ordinal numbers to the elements of A. Accordingly, we shall speak hereafter of elements with the ordinal γ (in this correspondence).

We initiate our inductive construction by putting $\alpha(a) = 0$ for all a in $\mathfrak{a}$. If we denote by P_2^* the property obtained from P^* by restricting the elements $a_1, \ldots, a_m$, $b_1, \ldots, b_n$ which it involves to be elements with the ordinal less than 2 — that is, to be elements in $\mathfrak{a}$ — it is obvious that this property is valid for the function $\alpha(a)$. Now let us suppose that the construction has been carried to the point that $\alpha(a)$ is defined for all elements with ordinals γ, where $\gamma < \delta < \omega$, assumes only the values 0 and e in A, and has the property P_δ^* obtained by restricting the elements $a_1, \ldots, a_m$, $b_1, \ldots, b_n$ involved in P^* to be elements with ordinals preceding δ. On this assumption we have to define $\alpha(x)$, where x is an arbitrary element with ordinal δ, so that $\alpha(x) = 0$ or $\alpha(x) = e$ and so that the property $P_{\delta+1}^*$ is valid. If we agree to determine the value $\alpha(x)$

so that $x_1 \equiv x_2 \pmod{\mathfrak{a}}$ implies $\alpha(x_1) = \alpha(x_2)$ — that is, so that $\alpha(x)$ depends only on the residual class (mod $\mathfrak{a}$) with ordinal δ — we see that the restrictions imposed upon our choice of $\alpha(x)$ by those conditions involved in $P^*_{\delta+1}$ but not in P^*_δ reduce to

(1) $\qquad a_1 \dots a_m b'_1 \dots b'_n x \equiv 0 \pmod{\mathfrak{a}}$ implies

$$\alpha(a_1) \dots \alpha(a_m)\, \alpha'(b_1) \dots \alpha'(b_n)\, \alpha(x) = 0 \quad \text{for } m + n \geqq 1,$$

(2) $\qquad c_1 \dots c_p d'_1 \dots d'_q x' \equiv 0 \pmod{\mathfrak{a}}$ implies

$$\alpha(c_1) \dots \alpha(c_p)\, \alpha'(d_1) \dots \alpha'(d_q)\, \alpha'(x) = 0 \quad \text{for } p + q \geqq 1,$$

where all elements other than x have ordinals preceding δ. In order to eliminate conditions involving *more* than one element with ordinal δ, we have made use of the fact that $x_1 \equiv x_2 \pmod{\mathfrak{a}}$ implies $x_1 x_2 \equiv x_1 \pmod{\mathfrak{a}}$, $x'_1 x'_2 \equiv x'_1 \pmod{\mathfrak{a}}$, $x_1 x'_2 \equiv 0 \pmod{\mathfrak{a}}$ and also of the requirement that $x_1 \equiv x_2 \pmod{\mathfrak{a}}$ shall imply $\alpha(x_1) = \alpha(x_2)$ and hence $\alpha(x_1)\ \alpha(x_2) = \alpha(x_1)$, $\alpha'(x_1)\ \alpha'(x_2) = \alpha'(x_1)$, $\alpha(x_1)\ \alpha'(x_2) = 0$. In order to eliminate conditions in which $m + n = 0$ or $p + q = 0$, we write them in the form: $0' x \equiv 0 \pmod{\mathfrak{a}}$ implies $\alpha'(x)\,\alpha(0) = 0$, $0' x' \equiv 0 \pmod{\mathfrak{a}}$ implies $\alpha'(0)\,\alpha'(x) = 0$, by virtue of the fact that $0' = e$, $\alpha(0) = 0$. Now let us consider the class $\mathfrak{L}$ of all elements $\alpha(a_1) \dots \alpha(a_m)\ \alpha'(b_1) \dots \alpha'(b_n)$ involved under (1); and the class $\mathfrak{D}$ of all elements $\alpha'(c_1) \dots \alpha(c_p)\,\alpha'(d_1) \dots \alpha'(d_q)$ involved under (2). It is evident that $\mathfrak{L}$ and $\mathfrak{D}$ are both non-void classes, containing the element 0 but no elements other than 0 and e. Now the product of any element in $\mathfrak{L}$ and any element in $\mathfrak{D}$ is equal to 0: for we have

$$\begin{aligned}(a_1 \dots a_m b'_1 \dots b'_n)\ (c_1 \dots c_p d_1 \dots d_q)' &= \\ = (a_1 \dots a_m b'_1 \dots b'_n)\ (c_1 \dots c_p d'_1 \dots d'_q)\ (x + x') &= \\ = (a_1 \dots a_m b'_1 \dots b'_n x)\ (c_1 \dots c_p d'_1 \dots d'_q) &+ \\ + (a_1 \dots a_m b'_1 \dots b'_n)\ (c_1 \dots c_p d'_1 \dots d'_q x') &\equiv 0 \pmod{\mathfrak{a}}.\end{aligned}$$

and can therefore appeal to the property P^*_δ to conclude that

$$\alpha(a_1) \dots \alpha(a_m)\, \alpha'(b_1) \dots \alpha'(b_n)\, \alpha(c_1) \dots \alpha(c_p)\, \alpha'(d_1) \dots \alpha'(d_q) = 0.$$

Consequently, we see that $\mathfrak{L}$ and $\mathfrak{D}$ cannot both contain the element e. In case $\mathfrak{L}$ contains e, therefore, we must put $\alpha(x) = 0$; and upon doing so, we see that conditions (1) and (2) are both satisfied. In case $\mathfrak{D}$ contains e, we must similarly put $\alpha'(x) = 0$, $\alpha(x) = e$; and, upon doing so, we see that conditions (1) and (2) are both satisfied.

In case neither $\mathfrak{L}$ nor $\mathfrak{D}$ contains e, we may put $\alpha(x) = 0$ or $\alpha(x) = e$; and, whichever we choose to do, we see that conditions (1) and (2) are both satisfied. Thus we can so define $\alpha(x)$ for all x with the ordinal δ that the conditions $P^*_{\delta+1}$ are verified. The principle of transfinite induction therefore establishes the existence of the desired function $\alpha(a)$, defined over A and having the property P^*.

The class $\mathfrak{p}$ of all elements a for which $\alpha(a) = 0$ is an ideal: for, with the help of Theorem 11, we see that $\alpha(a) = 0$ implies $\alpha(ab) = \alpha(a)\,\alpha(b) = 0$; and that $\alpha(a) = \alpha(b) = 0$ implies $\alpha(a + b) = \alpha(a) + \alpha(b) = 0$. The ideal $\mathfrak{p}$ contains $\mathfrak{a}$, since we have so determined $\alpha(a)$ that $\alpha(a) = 0$ for all a in $\mathfrak{a}$. By Theorem 11, we know that $\alpha(e) = e$; and if a and b are elements of A not in $\mathfrak{p}$, we have $\alpha(a - b) = \alpha(a + b) = \alpha(a) + \alpha(b) = e + e = 0$, $a \equiv b \pmod{\mathfrak{p}}$. Hence there are exactly two residual classes (mod $\mathfrak{p}$), and $\mathfrak{p}$ is a prime ideal.

On the other hand, if $\mathfrak{p}$ is a prime ideal containing $\mathfrak{a}$, it is easily verified that the function $\alpha(a)$ equal to 0 or to e, according as a is in $\mathfrak{p}$ or not, must have all the properties $P2$, $P3$, $P4$, and P^*. The property $P2$ is valid because the residual classes (mod $\mathfrak{a}$) are contained in the residual classes (mod $\mathfrak{p}$). The two-element Boolean ring $A/\mathfrak{p}$ is evidently isomorphic to the subring of A consisting of the elements 0 and e alone; and hence the function $\alpha(a)$ defines a homomorphism from A to this subring, thereby automatically possessing the properties $P3$ and $P4$. Finally Theorem 11 shows that $\alpha(a)$ must also have the property P^*.

Theorem 15. *If A is a Boolean ring with unit e and if $\mathfrak{a}$ is an ideal which is a proper subclass of A, the existence of a solution $f(a)$ of the $(A, \mathfrak{a})$ representation problem implies the existence of a solution $g(a)$ for which $g(e) = e$; the solution $g(a)$ can be obtained from $f(a)$ through the equation $g(a) = f(a) \vee \alpha(a) f'(a')$, where $\alpha(a)$ is any function with the properties described in Theorem 14.*

The argument leading to this result has already been sketched in the remarks preceding Theorem 14. In order to establish the explicit formula for $g(a)$, we note that it is identical with that obtained by restricting $f(a)$ to the prime ideal $\mathfrak{p}$ for which $\alpha(a) = 0$ and then extending this function to A by the construction given in the proof of Theorem 13. In fact, we see that, if a is in $\mathfrak{p}$, then $g(a)=f(a)$; and that, if a is not in $\mathfrak{p}$, then $\alpha(a)=e$ $a=a'+\alpha(a)$ and

$$g(a)=f(a)\vee f'(a')=f(a)+(e+f(a'))+f(a)\,(e+f(a'))=$$
$$=f(a')+e+f(a)\,f(a')=f(a')+\alpha(a)+f(a\,a')=f(a')+\alpha(a)$$

where a' is in $\mathfrak{p}$. This completes the proof. We may remark that the excluded case where $\mathfrak{a}=A$ is one in which the representation problem obviously has one and only one solution, namely that given by putting $f(a)=0$ for all a in A.

4. Sufficient Conditions.

We shall now give three sets of sufficient conditions for the existence of a solution of the $(A, \mathfrak{a})$ representation problem. One of these is a comparatively trivial set of purely algebraic nature. In the other two cases, we apply processes of construction akin to that used in the proof of Theorem 14. It is necessary for us, at each advanced stage of the construction, to effect *infinitely* many changes in certain chosen elements; and hence it is also necessary for us to provide a process for combining these changes in a single *algebraic* operation. Accordingly, we impose conditions upon the ideal $\mathfrak{a}$ which, expressed in intuitive language, empower us to form the union of *infinitely* many elements in $\mathfrak{a}$. It seems clear that an inductive construction could not be carried through in the absence of conditions of this character; and hence that the distinction between those cases where the representation problem has a solution and those where it does not, cannot in general be based upon criteria of an unquestionably algebraic nature.

Our first criterion is the following:

Theorem 16. *If $\mathfrak{a}$ is a principal ideal in a Boolean ring A, then the $(A, \mathfrak{a})$ representation problem has a solution.*

The principal ideal generated by an element a is easily seen to be the class $\mathfrak{a}$ of all products ab, where b is in A: for the element $a=aa$ is in $\mathfrak{a}$, and $\mathfrak{a}$ is obviously the smallest ideal containing a. If $\mathfrak{a}$ is a principal ideal in A and A is imbedded as a prime ideal in a Boolean ring B with unit, it is clear that $\mathfrak{a}$ remains a principal ideal in B. Hence, by virtue of Theorem 13, we may confine our attention to the case where A has a unit. Here the function $f(b)=a'b$, where a is the generating element of $\mathfrak{a}$, is evidently a solution of the $(A, \mathfrak{a})$ representation problem. In the first place, we note that $a'\equiv e \pmod{\mathfrak{a}}$ and hence that $f(b)\equiv b \pmod{\mathfrak{a}}$. In the second place, we observe that $b\equiv c \pmod{\mathfrak{a}}$ implies $b-c=ad$, $a'b-a'c=0$, and $f(b)=f(c)$. Finally, we remark that $f(b+c)=$ $=f(b)+f(c)$, $f(bc)=f(b)\,f(c)$. Of course we do not have $f(e)=e$ unless $a=0$; but by an application of Theorem 15, we can replace $f(b)$ by a solution which does have the latter property.

Theorem 17. *If $\mathfrak{a}$ is an ideal in a Boolean ring A and if $\mathfrak{a}$ has the property*

(1) *whenever $\mathfrak{b}$ is a non-void subclass of $\mathfrak{a}$ with cardinal number less than that of the quotient-ring $A/\mathfrak{a}$, there exists an element b_0 in $\mathfrak{a}$ such that*

(i) *if b is in $\mathfrak{b}$, then $b\,b_0=b$;*

(ii) *if $bc=b$ for every b in $\mathfrak{b}$, then $b_0\,c=b_0$;*

then the $(A, \mathfrak{a})$ representation problem has a solution.

The condition (1) means that every class $\mathfrak{b}$ of the kind described has a „union" b_0 — that is, determines a „least" element in A containing all the elements in $\mathfrak{b}$; and that this „union" is itself an element in $\mathfrak{a}$. The latter part of the condition is independent of the first. If we imbed the Boolean ring A as a prime ideal in a Boolean ring B with unit, the quotient ring $B/\mathfrak{a}$ has cardinal number twice that of $A/\mathfrak{a}$. Hence, if $A/\mathfrak{a}$ is infinite, $B/\mathfrak{a}$ has the same cardinal number as $A/\mathfrak{a}$; and the condition (1) remains valid when $\mathfrak{a}$ is considered in B. If $A/\mathfrak{a}$ is finite, then so is $B/\mathfrak{a}$; and the condition (1) is automatically satisfied by the ideal $\mathfrak{a}$, whether it be considered in A or in B. These remarks, taken in combination with Theorem 13, permit us to confine our attention to the case where A has a unit. In this case we can replace (i) and (ii) by the equivalent statements (i) if b is in $\mathfrak{b}$, then $b\,b_0'=0$, (ii) if $bc'=0$ for every b in $\mathfrak{b}$, then $b_0\,c'=0$. We shall refer to these statements alone in the sequel.

We assign ordinal numbers γ, $\gamma<\omega$, to the elements of A in the manner described in the proof of Theorem 14, and we start the inductive construction of a solution $f(a)$ by putting $f(a)=0$ for all a in $\mathfrak{a}$. It is then evident that $f(a)\equiv a \pmod{\mathfrak{a}}$ and that the property P^* holds for all elements with the ordinal number 1 — that is, all elements in $\mathfrak{a}$. Now let us suppose that the construction has been carried to the point that $f(a)$ is defined for all elements with ordinals γ, where $\gamma < \delta < \omega$, in such a manner that $f(a) \equiv a \pmod{\mathfrak{a}}$ and that the property P_δ^* (obtained by restricting the elements $a_1, \ldots, a_m, b_1, \ldots, b_n$ involved in P^* to have ordinals preceding δ) is valid. On this assumption we have to define $f(x)$ for all elements x with the ordinal δ so that $f(x) \equiv x \pmod{\mathfrak{a}}$ and so that the property $P_{\delta+1}^*$ is valid. If we agree to determine $f(x)$ so that $x_1 \equiv x_2 \pmod{\mathfrak{a}}$ implies $f(x_1) = f(x_2)$ — that is, so that $f(x)$ depends only on the residual class (mod $\mathfrak{a}$) with the ordinal δ — we see that the restrictions imposed upon our choice of $f(x)$ by those conditions involved in $P_{\delta+1}^*$ but not P_δ^* reduce, as in the case considered in Theorem 14, to

$$(1) \qquad a_1 \ldots a_m b_1' \ldots b_n' x \equiv 0 \pmod{\mathfrak{a}} \quad \text{implies}$$
$$f(a_1) \ldots f(a_m) f'(b_1) \ldots f'(b_n) f(x) = 0 \quad \text{for } m + n \geqq 1,$$

$$(2) \qquad c_1 \ldots c_p d_1' \ldots d_q' x' \equiv 0 \pmod{\mathfrak{a}} \quad \text{implies}$$
$$f(c_1) \ldots f(c_p) f'(d_1) \ldots f'(d_q) f'(x) = 0 \quad \text{for } p + q \geqq 1,$$

where all elements other than x have ordinals preceding δ. We now select a fixed element x_0 with the ordinal δ and inquire what modifications we must make in x_0 on account of the conditions (1) and (2) if we attempt to convert x_0 into $f(x) = f(x_0)$. In order to do so we consider the class $\mathfrak{L}$ of all elements $f(a_1) \ldots f(a_m) f'(b_1) \ldots f'(b_n) x_0$ arising from (1); and the class $\mathfrak{D}$ of all elements $f(c_1) \ldots f(c_p) f'(d_1) \ldots f'(d_q) x_0'$ arising from (2). We must, so to speak, suppress all elements of $\mathfrak{L}$ from x_0 and adjoin all elements of $\mathfrak{D}$ to x_0 in passing from x_0 to $f(x_0)$. One sees immediately that both $\mathfrak{L}$ and $\mathfrak{D}$ have cardinal numbers which are finite if that of the class of ordinals γ where $\gamma < \delta < \omega$ is finite; and at most equal to that of the class of ordinals γ where $\gamma < \delta < \omega$, if this is infinite. The assignment of ordinal numbers to the elements of A has been made in such a manner that we can therefore state: The cardinal numbers of $\mathfrak{L}$ and $\mathfrak{D}$ are either both *finite*, or both *less* than that of $A/\mathfrak{a}$. Furthermore, $\mathfrak{L}$ and $\mathfrak{D}$ are subclasses of $\mathfrak{a}$ by virtue of the fact that

$$f(a_1) \ldots f(a_m) f'(b_1) \ldots f'(b_n) \equiv a_1 \ldots a_m b_1' \ldots b_n' \equiv 0 \pmod{\mathfrak{a}},$$
$$f(c_1) \ldots f(c_p) f'(d_1) \ldots f'(d_q) \equiv c_1 \ldots c_p d_1' \ldots d_q' \equiv 0 \pmod{\mathfrak{a}}.$$

If the cardinal numbers of $\mathfrak{L}$, $\mathfrak{D}$ are less than that of $A/\mathfrak{a}$, then our hypothesis concerning the ideal $\mathfrak{a}$ shows that there exist in $\mathfrak{a}$ elements c_0 and d_0 which are the „unions" of all the elements in $\mathfrak{L}$ and in $\mathfrak{D}$ respectively. If they are finite, then the c_0 and d_0 can be constructed directly: If $\mathfrak{L} = (a_1^{(1)} \ldots a^{(r)})$ and $\mathfrak{D} = (b_1^{(1)} \ldots, b^{(s)})$ then put

$$c_0 = a^{(1)} \vee \ldots \vee a^{(r)}, \qquad d_0 = b^{(1)} \vee \ldots \vee b^{(s)}.$$

Now, just as in the proof of Theorem 14, we can show that

$$f(a_1) \ldots f(a_m) f'(b_1) \ldots f'(b_n) f(c_1) \ldots f(c_p) f'(d_1) \ldots f'(d_q) = 0$$

for all elements with ordinals preceding δ which are involved in (1) and (2). Since this equation holds after multiplication by x_0 or by x_0', we see that the „unions" c_0 and d_0 have the properties

$$f(c_1) \ldots f(c_p) f'(d_1) \ldots f'(d_q)\, c_0 = 0,$$
$$f(a_1) \ldots f(a_m) f'(b_1) \ldots f'(b_n)\, d_0 = 0$$

for the elements in question. These two properties show that, if we define $f(x) = x_0 c_0' \vee d_0$ for all x with the ordinal δ, then properties (1) and (2) hold. For we have

$$f(a_1) \ldots f(a_m) f'(b_1) \ldots f'(b_n) f(x) =$$
$$= (f(a_1) \ldots f(a_m) f'(b_1) \ldots f'(b_n) x_0)\, c_0' \vee$$
$$\vee f(a_1) \ldots f(a_m) f'(b_1) \ldots f'(b_n)\, d_0 = 0,$$

$$f(c_1) \ldots f(c_p) f'(d_1) \ldots f'(d_q) f'(x) =$$
$$= (f(c_1) \ldots f(c_p) f'(d_1) \ldots f'(d_q) x_0')\, d_0' \vee$$
$$\vee (f(c_1) \ldots f(c_p) f'(d_1) \ldots f'(d_q) c_0)\, d_0 = 0,$$

by virtue of these properties and the definition of c_0 and d_0. Now, since $x \equiv x_0 \pmod{\mathfrak{a}}$, $c_0' \equiv e \pmod{\mathfrak{a}}$, $d_0 \equiv 0 \pmod{\mathfrak{a}}$, it is evident that $f(x) \equiv x \pmod{\mathfrak{a}}$. Thus the indicated definition for $f(x)$ satisfies the requirements we have laid down. The principle of transfinite induction therefore establishes the existence of a function $f(a)$, defined over A, which has the property $f(a) \equiv a \pmod{\mathfrak{a}}$ and the property P^*. According to Theorem 12, this function is a solution of the $(A, \mathfrak{a})$ representation problem.

As an illustration of the manner in which this theorem can be applied, let us consider the case where A is the class of all Lebesgue-measurable subsets of the plane and $\mathfrak{a}$ is the ideal of all sets of measure zero. It is a familiar fact that every Lebesgue-measurable set is congruent (mod $\mathfrak{a}$) to some Borel set and hence that $A/\mathfrak{a}$ has the cardinal number of the continuum. *If we assume that the cardinal number of the continuum is* $\aleph_1$, then we see that $\mathfrak{a}$

satisfies the hypothesis of Theorem 17, the union of a *countable* class of sets of measure zero being itself a set of measure zero; and we conclude that the $(A, \mathfrak{a})$ representation problem has a solution. We shall see at the end of this §, that the continuum hypothesis is not really needed in this problem.

The following theorem is an abstract formulation of the scheme previously used by one of us to treat the exemple of the preceding paragraph [6]).

Theorem 18. *Let A be a Boolean ring with unit in which there exists a function $F(a)$ assuming values in A and having the properties $P1$, $P2$ and one, but not necessarily both, of the properties $P4$ and $P5$. Let $\mathfrak{a}$ be an ideal with the property: if $\mathfrak{L}$ and $\mathfrak{D}$ are non-void subclasses of $\mathfrak{a}$ with cardinal numbers less than that of $A/\mathfrak{a}$ such that $c'd = 0$ for all c in $\mathfrak{L}$ and all d in $\mathfrak{D}$, then there exists an element a_0 in A such that $c' a_0 = a_0 d = 0$ for all c in $\mathfrak{L}$ and all d in $\mathfrak{D}$. Then the $(A, \mathfrak{a})$ representation problem has a solution.*

We first show that we can confine our attention to the case where the given function $F(a)$ has not only properties $P1$, $P2$, and $P5$ but also the properties $F(0) = 0$, $F(e) = e$. If $F(a)$ has property $P4$, we can replace it by $F'(a')$ in accordance with Theorem 9 so as to obtain a function with the property $P5$, if we note that $F'(a')$ has properties $P1$ and $P2$ together with $F(a)$. If $F(a)$ has properties $P1$, $P2$ and $P5$ but not the property $F(0) = 0$, we can replace it by $F(a) + F(0)$ in accordance with Theorem 10, if we note that $F(0) \equiv 0 \pmod{\mathfrak{a}}$, $F(a) + F(0) \equiv a \pmod{\mathfrak{a}}$. Finally, if $F(a)$ has the properties $P1$, $P2$, $P5$ and $P0$ but not the property $F(e) = e$, we can replace it by $F(a) \vee \alpha(a) F'(e)$, where $\alpha(a)$ is a function of the type constructed in Theorem 14. To justify this replacement, we need only remark the relations

$$F'(e) \equiv 0 \pmod{\mathfrak{a}}, \quad F(a) \vee \alpha(a) F'(e) \equiv a \pmod{\mathfrak{a}},$$
$$F(a \vee b) \vee \alpha(a \vee b) F'(e) = (F(a) \vee \alpha(a) F'(e)) \vee (F(b) \vee \alpha(b) F'(e)),$$
$$F(0) \vee \alpha(0) F'(e) = 0, \quad F(e) \vee \alpha(e) F'(e) = F(e) \vee F'(e) = e.$$

Hence we can always replace $F(a)$ by a function which has the various properties desired.

[6]) See J. von Neumann, Journal für Mathematik, 165 (1931), pp. 109—115.

The element a_0 described in the condition imposed upon the ideal $\mathfrak{a}$ is evidently a kind of Dedekind cut „between“ the classes $\mathfrak{L}$ and $\mathfrak{D}$: for every element in $\mathfrak{L}$ contains every element in $\mathfrak{D}$, while a_0 contains every element in $\mathfrak{L}$ and is, in turn, contained in every element in $\mathfrak{D}$. Furthermore, it is clear that a_0 is not only an element in A but also an element in $\mathfrak{a}$: for, taking c as an arbitrary element in $\mathfrak{L}$, we have $c \equiv 0 \pmod{\mathfrak{a}}$, $a_0 \equiv a_0 c' = 0 \pmod{\mathfrak{a}}$. We have not required that a_0 be uniquely determined by the classes $\mathfrak{L}$ and $\mathfrak{D}$.

We now proceed to a construction similar to that effected in the preceding theorem. We assign ordinal numbers to the residual classes (mod $\mathfrak{a}$) just as in Theorems 14 and 17. Here, however, we propose to determine the function $f(a)$ so that it has the property $f(a) \equiv a \pmod{\mathfrak{a}}$ and the property P^{**}

$$P^{**}. \qquad f(a_1)\dots f(a_m) f'(b_1)\dots f'(b_n) F'(a_1 \dots a_m b'_1 \dots b'_n) = 0, \quad m + n \geqq 1.$$

for *all* elements in A. By virtue of the properties of $F(a)$ we see that P^{**} implies P^*: for if $a_1 \dots a_m b'_1 \dots b'_n \equiv 0 \pmod{\mathfrak{a}}$, then $F(a_1 \dots a_m b'_1 \dots b'_n) = F(0) = 0$; and P^{**} therefore yields $f(a_1)\dots f(a_m) f(b_1)\dots f(b_n) = 0$, $m + n \geqq 1$. Thus the function $f(a)$, if it can be constructed, will be a solution of the $(A, \mathfrak{a})$ representation problem in accordance with Theorem 12. We start the construction by putting $f(a) = 0$ for all elements a in $\mathfrak{a}$ — that is, for all elements a with the ordinal number 1. The conditions imposed by P^{**} when all the elements concerned are in $\mathfrak{a}$ are now seen to reduce to the single condition $F'(e) = 0$, which is satisfied in accordance with our initial remarks concerning the function $F(a)$. Now let us suppose that the construction has been carried to the point that $f(a)$ is defined for all elements with ordinals γ, where $\gamma < \delta < \omega$, in such a manner that $f(a) \equiv a \pmod{\mathfrak{a}}$ and that the property P^{**}_{δ} (obtained by restricting the elements involved in P^{**} to have ordinals preceding δ) is valid. On this assumption we have to define $f(x)$ for all elements x with the ordinal δ so that $f(x) \equiv x \pmod{\mathfrak{a}}$ and so that the property $P^{**}_{\delta+1}$ is valid. If we agree to determine $f(x)$ so that $x_1 \equiv x_2 \pmod{\mathfrak{a}}$ implies $f(x_1) = f(x_2)$ and if we make use of the fact that $F(a)$ has the property $P2$, we see that the restrictions imposed upon our choice of $f(x)$ by those conditions involved in $P^{**}_{\delta+1}$ but not P^{**}_{δ} reduce to

$$(1) \qquad f(a_1)\dots f(a_m) f'(b_1)\dots f'(b_n) f(x) F'(a_1 \dots a_m b'_1 \dots b'_n x) = 0, \quad m + n \geqq 1,$$
$$(2) \qquad f(c_1)\dots f(c_p) f'(d_1)\dots f'(d_q) f'(x) F'(c_1 \dots c_p d'_1 \dots d'_q x') = 0, \quad p + q \geqq 1.$$

The reduction is similar to those indicated in the proofs of Theorems 14 and 17. It should be noted that the conditions where $m+n=0$, $p+q=0$, which should apparently be explicitly included here, are actually obtained by putting $m=0$, $n=1$, $b_1=0$ and $p=0$, $q=1$, $d_1=0$, respectively. The conditions in these special cases are merely $f(x)F'(x)=0$, $f'(x)F'(x')=0$. Thus we must so choose $f(x)$ that it contains $F'(x')$ and is contained in $F(x)$. Our problem is therefore that of finding an element a contained in $F(x)F(x')$ so that it is possible to set $f(x)=F'(x')\vee a$. To this end we consider the class $\mathfrak{L}$ of all elements

$$[f(a_1)\dots f(a_m)f'(b_1)\dots f'(b_n)F'(a_1\dots a_m b_1'\dots b_n' x)]' F(x)F(x'),$$

and the class $\mathfrak{D}$ of all elements

$$[f(c_1)\dots f(c_p)f'(d_1)\dots f'(d_q)F'(c_1\dots c_p d_1'\dots d_q' x')]' F(x)F(x'),$$

where the argument elements are those involved in (1) and (2) respectively. Since $F(x)F(x')\equiv xx'=0 \pmod{\mathfrak{a}}$, it is evident that $\mathfrak{L}$ and $\mathfrak{D}$ are subclasses of $\mathfrak{a}$. It is also evident that $\mathfrak{L}$ and $\mathfrak{D}$ are non-void classes either both finite, or both with cardinal numbers *less* than that of $A/\mathfrak{a}$, in accordance with our assignment of ordinals to the residual classes (mod $\mathfrak{a}$). In order to apply our hypothesis concerning the ideal $\mathfrak{a}$ to the classes $\mathfrak{L}$ and $\mathfrak{D}$, it is necessary for us to show that every element in $\mathfrak{L}$ contains every element in $\mathfrak{D}$. For this purpose, it is evidently sufficient to show that every element

$$f(c_1)\dots f(c_p)f'(d_1)\dots f'(d_q)F'(c_1\dots c_p d_1'\dots d_q' x')$$

is contained in every element

$$(f(a_1)\dots f(a_m)f'(b_1)\dots f'(b_n)F'(a_1\dots a_m b_1'\dots b_n' x))';$$

in other words, to show that

$$(f(a_1)\dots f(a_m)f'(b_1)\dots f'(b_n)F'(a_1\dots a_m b_1'\dots b_n' x))\cdot$$
$$\cdot(f(c_1)\dots f(c_p)f'(d_1)\dots f'(d_q))F'(c_1\dots c_p d_1'\dots d_q' x'))=0.$$

Since the element

$$f(a_1)\dots f(a_m)f(c_1)\dots f(c_p)f'(b_1)\dots f'(b_n)f'(d_1)\dots f'(d_q)$$

is contained in the element $F(a_1\dots a_m c_1\dots c_p b_1'\dots b_n' d_1'\dots d_q')$ in accordance with the property P_δ^{**}, it is even sufficient to prove that

$$F(a_1\dots a_m c_1\dots c_p b_1'\dots b_n' d_1'\dots d_q')\cdot F(a_1\dots a_m b_1'\dots b_n' x)\cdot$$
$$\cdot F'(c_1\dots c_p d_1'\dots d_q' x')=0. \tag{3}$$

This result follows at once if we can show that

$$F(ab)F'(ax)F'(bx')=0$$

for *all* a, b, x in A. Since $(ab)(ax)'(bx')'=(ab)(a'\vee x')(b'\vee x)= =abx'(b'\vee x)=0$, we can make use of Theorem 8 to obtain the desired result as a new form of the property $P7$. We are thus in a position to apply our hypothesis concerning the ideal $\mathfrak{a}$ if the cardinal numbers of $\mathfrak{L}$, $\mathfrak{D}$ are less than that of $A/\mathfrak{a}$. If they are finite, we can form the „union" of all the elements of $\mathfrak{D}$ directly, as in the proof of Theorem 17. In both cases we obtain a „cut" between the classes $\mathfrak{L}$ and $\mathfrak{D}$. We take a_0 as such a „cut" between the classes $\mathfrak{L}$ and $\mathfrak{D}$, and put $f(x)=F'(x')\vee a_0$. It is evident that $f(x)\equiv F'(x')\equiv x \pmod{\mathfrak{a}}$, since a_0 is in $\mathfrak{a}$ and $F(x)$ has the property $P1$; and also that $x_1\equiv x_2 \pmod{\mathfrak{a}}$ implies $f(x_1)=f(x_2)$, since $F(a)$ has the property $P2$. To verify that $f(x)$ satisfies the conditions (1) and (2) is all that remains. Condition (1) becomes

$$f(a_1)\dots f(a_m)f'(b_1)\dots f'(b_n)F'(x')F'(a_1\dots a_m b_1'\dots b_n' x)\vee$$
$$\vee[f(a_1)\dots f(a_m)f'(b_1)\dots f'(b_n)F'(a_1\dots a_m b_1'\dots b_n' x)]\,a_0=0.$$

By P_δ^{**}, the first term here is contained in

$$F(a_1\dots a_m b_1'\dots b_n')F'(x')F'(a_1\dots a_m b_1'\dots b_n' x)$$

which is equal to 0 as a special case of (3) above. The second term is contained in the element

$$[f(a_1)\dots f(a_m)f'(b_1)\dots f'(b_n)F'(a_1\dots a_m b_1'\dots b_n' x)\vee F'(x)\vee F'(x')]\,a_0=$$
$$=([f(a_1)\dots f(a_m)f'(b_1)\dots f'(b_n)F(a_1\dots a_m b_1'\dots b_n' x)]' F(x)F(x'))'\,a_0$$

which is equal to 0 by virtue of the definition of the element a_0. Thus the condition (1) is satisfied. Similarly, condition (2) becomes

$$f(c_1)\dots f(c_p)f'(d_1)\dots f'(d_q)F'(x)a_0'F'(c_1\dots c_p d_1'\dots d_q' x')=0.$$

Since $F(x)\vee F'(x)=e$, we can write this condition in the form

$$[f(c_1)\dots f(c_p)f'(d_1)\dots f'(d_q)F'(c_1\dots c_p d_1'\dots d_q' x')F(x)F(x')]\,a_0'\vee$$
$$\vee f(c_1)\dots f(c_p)f'(d_1)\dots f'(d_q)F'(x)F'(c_1\dots c_p d_1'\dots d_q' x')a_0'=0.$$

The first term here is equal to 0 by the definition of the element a_0. By P_δ^{**}, the second term is contained in the element

$$F(c_1\dots c_p d'_1\dots d'_q)\, F'(x)\, F'(c_1\dots c_p d'_1\dots d'_q x')$$

which is equal to 0 as a special case of (3) above. Hence condition (2) is satisfied. The principle of transfinite induction now brings the discussion to a close.

This theorem can be applied to the problem discussed in illustration of the preceding theorem; it leads to the same result without making use of the continuum hypothesis. If $\mathfrak{a}$ is the ideal of all sets of measure zero in the Boolean ring of all Lebesgue-measurable subsets of the plane, then it evidently has the property demanded in the theorem: for the union of the sets in the class $\mathfrak{D}$ is a cut of the type desired and, being contained in those sets of measure zero which belong to $\mathfrak{L}$, is itself a set of measure zero, belonging accordingly to $\mathfrak{a}$. In addition, it is possible to construct a function $F(a)$, defined for all Lebesgue-measurable sets, assuming Lebesgue-measurable sets (even Borel sets) as values, and possessing properties $P1$, $P2$ and $P5$: for this purpose, one defines $F(a)$ as that set of points at which the Lebesgue-measurable set a has inferior density 1 [7]).

5. Some Special Cases.

Here we shall give cases in which the representation problem has no solution or in which it has a solution if and only if $\aleph_{\eta+1} = 2^{\aleph_\eta}$. It will be convenient to consider Boolean rings with classes as elements.

Theorem 19. *Let certain subclasses $a, b, c, \dots$ of a fixed class e constitute a Boolean ring A with cardinal number $\aleph_A$; in particular let the class e and every subclass of e with cardinal number, less than $\aleph_B$, where $\aleph_B$ is an infinite cardinal number, belong to A. Then the class $\mathfrak{a}$ of all subclasses of e with cardial number less than $\aleph_B$ is an ideal in A. The $(A, \mathfrak{a})$ representation problem has a solution if $\aleph_B \geqq \aleph_A$, provided that a cardinal number $\aleph_{B'}$, immediately preceding $\aleph_B$, exists; it has no solution if $\aleph_B < \aleph_A$, provided that the following*

7) See J. von Neumann, Journal für Mathematik, 165 (1931), pp. 109—115; Saks, Théorie de l'intégrale (Monogr. Matem. 2, Warsaw 1933), pp. 53—55.

additional conditions are fulfilled: There exist two subclasses $\mathcal{X}$ and $\mathcal{Y}$ of e with the properties

(1) *$\mathcal{X}$ has cardinal number not less than $\aleph_B$;*

(2) *$\mathcal{Y}$ has cardinal number $\aleph_A$;*

(3) *if x_1 and x_2 are elements in $\mathcal{X}$, then $x_1 = x_2$ or $x_1 \cdot x_2 = 0$;*

(4) *if y_1 and y_2 are elements in $\mathcal{Y}$, then $y_1 = y_2$ or $y_1 \cdot y_2 = 0$;*

(5) *if x is an element in $\mathcal{X}$ and y is an element in $\mathcal{Y}$, then xy is a class with cardinal number 1.*

It is obvious that $\mathfrak{a}$ is an ideal.

If $\aleph_B \geqq \aleph_A$, the ideal $\mathfrak{a}$ has the property postulated in Theorem 17; for if $\mathfrak{L}$ is any subclass of $\mathfrak{a}$ with cardinal number less than that of $A/\mathfrak{a}$, then the cardinal number of $\mathfrak{L}$ is a fortiori less than of A, which is $\aleph_A$. Thus it is $< \aleph_B$, and therefore $\leqq \aleph_{B'}$. Similarly every element of $\mathfrak{L}$ has cardinal number $< \aleph_B$, and therefore $\leqq \aleph_{B'}$. The union of all elements of $\mathfrak{L}$ has therefore cardinal number $\leqq \aleph_{B'} \cdot \aleph_{B'} = \aleph_{B'} < \aleph_B$, and thus belongs to $\mathfrak{a}$. Hence we see that the $(A, \mathfrak{a})$ representation problem has a solution, by Theorem 17.

If $\aleph_B < \aleph_A$ and the subclasses $\mathcal{X}$ and $\mathcal{Y}$ of A with the properties (1)—(5) exist, we assume the existence of a solution $f(a)$ of the representation problem (or merely of a function $f(a)$ with properties $P1$, $P2$, $P4$) and infer a contradiction. Let $\mathcal{X}^*$ be the class of all elements xy, where x is in $\mathcal{X}$, y is in $\mathcal{Y}$, and $xyf(x) = 0$; and let $\mathcal{Y}^*$ be the class of all elements xy, where $xyf(y) = 0$. In view of (5) we must have either $xyf(x) = 0$ or $xyf(x) = xy$, either $xyf(y) = 0$ or $xyf(y) = xy$; and we must also have $f(xy) = 0$ since xy is in $\mathfrak{a}$. Now we see that we cannot have $xy = xyf(x) = xyf(y)$ since $f(x)f(y) = f(xy) = 0$, and we conclude that xy must belong to at least one of the classes $\mathcal{X}^*$ and $\mathcal{Y}^*$. By (1), $\mathcal{X}$ contains a subclass $\mathcal{X}_0$ with cardinal number $\aleph_B$. Let $\mathcal{Y}_0$ be the class of all those elements y in $\mathcal{Y}$ such that xy is in $\mathcal{X}^*$ for some element x in $\mathcal{X}_0$. Since $xy \,\varepsilon\, \mathcal{X}^*$ implies $xyf(x) = 0$, we see that xy is contained in $f'(x)x \equiv x'x = 0 \pmod{\mathfrak{a}}$. By virtue of this fact and (4) above, we conclude that those elements y such that $xy \,\varepsilon\, \mathcal{X}^*$ for any fixed x constitute a class with cardinal number less than or equal to $\aleph_B$. Hence the class $\mathcal{Y}_0$ has cardinal number at most $\aleph_B \cdot \aleph_B = \aleph_B$. By (2) there exists an element y_0 in $\mathcal{Y}$ but not in $\mathcal{Y}_0$. Since xy_0 is not in $\mathcal{X}^*$ for any x in $\mathcal{X}_0$, xy_0 must belong to $\mathcal{Y}^*$ for every x in $\mathcal{X}_0$. By (3), the elements xy_0, where x is in $\mathcal{X}_0$ constitute a class with cardinal number $\aleph_B$. On the other hand, since $xy_0 f(y_0) = 0$, we see that every element xy_0 is contained in the element $f'(y_0)y_0 \equiv y_0' y_0 = 0 \pmod{\mathfrak{a}}$ and hence that the class of all elements xy_0 has cardinal number less than $\aleph_B$. This is the contradiction sought.

An interesting application of this theorem obtains as follows:

If, in particular, we take $\aleph_{B'} = \aleph_\eta$, $\aleph_B = \aleph_{\eta+1}$, $\aleph_A = 2^{\aleph_\eta}$ then we have $\aleph_{B'} < \aleph_A$, therefore $\aleph_B \leqq \aleph_A$, and so we obtain a case in which the $(A, \mathfrak{a})$ representation problem has a solution if and only if $\aleph_B = \aleph_A$, that is

$$\aleph_{\eta+1} = 2^{\aleph_\eta}$$

(The generalized continuum hypothesis). It is, of course, necessary to show how to form a Boolean ring A which fulfills the auxiliary conditions set forth in the second statement of the theorem.

Let e be a set of cardinal nunber $\aleph_A = 2^{\aleph_\eta}$. We define $\mathfrak{a}$ as in the statement of the theorem: it is the set of all subsets of e of cardinal number less than $\aleph_B = \aleph_{\eta+1}$ that is, less than or equal to $\aleph_{B'} = \aleph_\eta$. Therefore $\mathfrak{a}$ has the cardinal number $\aleph_A^{\aleph_\eta} = 2^{(\aleph_\eta^2)} = 2^{\aleph_\eta} = \aleph_A$.

As $\aleph_\eta$ is infinite, $\aleph_A^2 = 2^{2\aleph_\eta} = 2^{\aleph_\eta} = \aleph_A$, so there exists a one-to-one mapping of all ordered pairs (α, β), α, β elements in e, on all elements γ of e:

$$(\alpha, \beta) \rightleftarrows \gamma = \varphi(\alpha, \beta).$$

Define now x_α as the set of all $\varphi(\alpha, \beta)$, β in e, and $\mathfrak{X}$ as the set of all x_α, α in e; and similarly y_β as the set of all $\varphi(\alpha, \beta)$, α in e, and $\mathfrak{Y}$ as the set of all y_β, β in e. It is evident that the classes $\mathfrak{X}$ and $\mathfrak{Y}$ have the properties (1)—(5) of the theorem, and that both have cardinal number $\aleph_A$.

We can now take A as the smallest Boolean ring containing e and the elements of $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{a}$, this ring being obtained by forming all polynomials in terms of the indicated elements. Obviously the cardinal number of A is not less than $\aleph_A$ and not more than $\aleph_0(1 + \aleph_A + \aleph_A^2 + \ldots) = \aleph_0(1 + \aleph_A + \aleph_A + \ldots) = \aleph_A$; so it is equal to $\aleph_A$.

Further interesting examples are the following. Let e be the plane, $\mathfrak{X}$ and $\mathfrak{Y}$ two distinct pencils of parallel lines. If A is the class of all Borel sets, $\mathfrak{a}$ the class of all finite sets, we have $\aleph_A = 2^{\aleph_0}$, $\aleph_B = \aleph_0$. The $(A, \mathfrak{a})$ representation problem has no solution. If A is again the class of all Borel sets, $\mathfrak{a}$ the class of all countable sets, we have $\aleph_A = 2^{\aleph_0}$, $\aleph_B = \aleph_1$, $\aleph_{B'} = \aleph_0$. The $(A, \mathfrak{a})$ representation problem has a solution if and only if $\aleph_1 = 2^{\aleph_0}$, (the continuum hypothesis).

Généralisations du théorème des probabilités totales.

Par

Maurice Fréchet (Paris).

I. La formule de M. Charles Jordan.

Considérons des événements fortuits $H_1, \ldots, H_n$ de probabilités respectives $p_1, \ldots, p_n$ et soit P la probabilité pour que l'un au moins des événements H_j se réalise.

D'après le théorème des probabilités totales, on a

$$P = p_1 + \ldots + p_n$$

quand les événements H_j sont *incompatibles*.

Poincaré a donné à la page 60 de son traité de „*Calcul des Probabilités*“ une formule permettant de calculer P *dans le cas général*, quand, *en outre des* p_j, on connait les probabilités $p_{ik}, \ldots, p_{ik\ldots l}, \ldots$ où, en général, $p_{ik\ldots l}$ est la probabilité du concours de $H_i, H_k, \ldots, H_l$, à savoir

$$(1) \qquad P = \sum_i p_i - \sum_{ij} p_{ij} + \sum_{ijk} p_{ijk} - \ldots + (-1)^{n-1} p_{12\ldots n}.$$

M. Charles Jordan a établi, entre autres, dans un récent mémoire [1]), une formule généralisant la formule (1) de Poincaré. Nous voulons montrer que la formule de M. Jordan, démontrée par lui directement, peut aussi être considérée comme une conséquence immédiate de la même formule (1) de Poincaré.

[1]) *Le théorème de probabilité de Poincaré généralisé au cas de plusieurs variables dépendantes*, Acta Scientiarum Mathematicarum Szeged, t. VII, 1934. Nous visons ici la formule (11), page 108 de cet article.