Corollaire I. Si f est une transformation essentielle de A en Q_2, alors $A_1 = f^{-1}(S_1)$ contient un constituant K tel que f est une transformation essentielle de K en S_1, donc $f(K) = S_1$.

Lemme II. *Soit f une transformation essentielle de A en Q_2, $J \subset Q_2$ une ligne simple fermée, H celui des deux domaines déterminés dans le plan par J, qui est contenu dans Q_2. Alors f est une transformation essentielle de l'ensemble $f^{-1}(\overline{H})$ en $\overline{H} = H + J$.*

J'omets la démonstration, qui ne présente aucune difficulté.

Les lemmes I, II et le corollaire I entraînent le

Corollaire II. Si f est une transformation essentielle de A en Q_2, $J \subset Q_2$ une ligne simple fermée, alors A contient un continu K tel que $f(K) = J$.

Théorème I. *Soit f une transformation essentielle de A en Q_2 et $C \subset Q_2$ un continu. Alors il existe un continu $L \subset A$ tel que $f(L) = C$.*

Il existe une suite de lignes simples fermées $J_n \subset Q_2$ telle que $\lim\limits_{n\to\infty} J_n = C$ [5]). D'après le corollaire II, il existe un continu $K_n \subset A$ tel que $f(K_n) = J_n$, $n = 1, 2, \ldots$ De la suite $\{K_n\}$ nous pouvons extraire une suite convergente $\{K_{n_s}\}$, $s = 1, 2, \ldots$ Soit $L = \lim\limits_{s\to\infty} K_{n_s}$. On a $L \subset A$, L est fermé et connexe, enfin en vertu de la continuité de f, on aura: $f(L) = \lim\limits_{s\to\infty} f(K_{n_s}) = \lim\limits_{s\to\infty} J_{n_s} = \lim\limits_{n\to\infty} J_n = C$, c. q. f. d.

Théorème II. *Tout espace A métrique, compact et de dimension $\geqq 2$ contient un continu indécomposable.*

Il existe [6]) une transformation essentielle f de A en Q_2. Soit C_0 un continu indécomposable contenu dans Q_2. D'après le théorème I A contient un continu L_0 tel que $f(L_0) = C_0$. Or, L_0 contient alors un continu indécomposable L_1 tel que $f(L_1) = C_0$ [7]). Le théorème est ainsi démontré.

[5]) Comp. p. ex. Fund. Math. XVI, p. 157 (mutatis mutandis).
[6]) Alexandroff, Math. Ann. 106, p. 170—171.
[7]) Knaster-Mazurkiewicz, Fund. Math. XXI, p. 87—88.

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On the absolute integrability of Fourier transforms.

By

Einar Hille (New Haven) and J. D. Tamarkin (Providence).

1. Introduction. In their important paper [1] [1]) Hardy and Littlewood proved that if $\varphi(\theta)$ belongs to a Lebesgue class L_p over $(-\pi, \pi)$, $p > 1$, and if $\{c_m\}$, $m = \ldots, -2, -1, 0, 1, 2, \ldots$, is the sequence of complex Fourier coefficients of $\varphi(\theta)$ then

$$\sum_{m=-\infty}^{\infty} |c_m|^p (|m|+1)^{p-2} \leqq C_p \int_{-\pi}^{\pi} |\varphi(\theta)|^p \, d\theta,$$

where C_p is a constant depending only on p. This constant tends to ∞ as $p \to 1$, and the result does not hold when $p = 1$. It does hold however in the special case where $\varphi(\theta)$ ist the limit function of a function

$$\psi(w) = \sum_{n=0}^{\infty} a_n w^n$$

analytic in the unit circle $|w| < 1$, and such that

$$\int_{-\pi}^{\pi} |\psi(r e^{i\theta})| \, d\theta$$

is bounded for $0 \leqq r < 1$. In this case the result of Hardy and Littlewood can be stated in the form

$$\sum_{n=0}^{\infty} |a_n|/(n+1) \leqq C_1 \int_{-\pi}^{\pi} |\varphi(\theta)| \, d\theta$$

[1]) The numbers in brackets refer to the list at the end of this paper.

and the hypothesis can be replaced by an equivalent one, viz. that both functions $\varphi(\theta)$ and its conjugate $\tilde{\varphi}(\theta)$ are integrable, where $\tilde{\varphi}(\theta)$ is defined by

$$\tilde{\varphi}(\theta) = -\lim_{\varepsilon\to 0}(2\pi)^{-1}\int_{\varepsilon}^{\pi}[\varphi(\theta+\tau)-\varphi(\theta-\tau)]\cot\frac{\tau}{2}\,d\tau$$

(1.1)

$$= (2\pi)^{-1}\,P.V.\int_{-\pi}^{\pi}\varphi(\tau)\cot\frac{\theta-\tau}{2}\,d\tau,$$

and $P.\ V.$ stands for the Cauchy principal value of the integral in question. As a corollary of this result it follows that if both functions, $\psi(\theta)$ and its conjugate $\tilde{\psi}(\theta)$, are of bounded variation over $(-\pi, \pi)$, then their Fourier series converge absolutely. Almost simultaneously with Hardy and Littlewood, Fejér [1] showed that the best possible value of the constant C_1 is 1/2. An extremely simple proof of Fejér's result was given by Zygmund in his recent book ([1], pp. 157—162). Fejér also gave the following elegant geometric interpretation of the preceding results. Let

$$\psi(w) = \sum_{n=0}^{\infty} b_n w^n$$

be analytic in the unit circle $|w| < 1$, and let $\zeta = \psi(w)$ map conformally the unit circle into a bounded domain of the ζ-plane, not necessarily simply covered, and bounded by a rectifiable curve of length l. Then

$$\sum_{n=0}^{\infty} |b_n| \leqq \tfrac{1}{2} l,$$

the coefficient 1/2 being the best possible.

It is the purpose of the present note to investigate the analogues of the preceding results in the case where Fourier series are replaced by Fourier transforms and mapping of the unit circle is replaced by mapping of a half-plane. Hardy and Littlewood ([1], p. 203) state that if $f(x) \in L_p$ over $(0, \infty)$, $p > 1$, and if $F(x)$ is its cosine or sine Fourier transform then

$$\int_0^{\infty} |F(x)|^p x^{p-2}\,dx \leqq C_p \int_0^{\infty} |f(x)|^p\,dx.$$

This result in general is false when $p = 1$. It holds true however in a case which is entirely analogous to that mentioned above in connection with functions analytic in the unit circle, viz. when $f(x)$ is the limit function of a function $f(z)$ analytic in the half-plane $\Im z > 0$ and such that

$$\int_{-\infty}^{\infty} |f(x+iy)|\,dx \quad \text{is bounded for } y > 0.$$

The proof of this result and of various other analogues of results of Hardy and Littlewood, and of Fejér, are found in the last § 4 of the present note [2]).

The following notation will be used throughout this paper. The class of functions $f(x)$ measurable over $(-\infty, \infty)$ and such that

$$\int_{-\infty}^{\infty} |f(x)|^p\,dx < \infty, \quad p \geqq 1,$$

will be designated by $\mathfrak{L}_p$, the notation L_p being reserved for the analogous class of functions $\varphi(\theta)$ defined over $(-\pi, \pi)$.

Let $f(x) \in \mathfrak{L}_p$. The conjugate function $\tilde{f}(x)$ is defined by

$$(1.2)\quad \tilde{f}(x) = -\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\infty}[f(x+t)-f(x-t)]/t\,dt = \frac{1}{\pi}\,P.\ V.\int_{-\infty}^{\infty}\frac{f(t)\,dt}{x-t}.$$

It is known that $\tilde{f}(x)$ exists for almost all x and, in case $p > 1$, $\tilde{f}(x) \in \mathfrak{L}_p$. (See e. g. Zygmund [1], Ch. XII). Let $f(z)$, $z = x + iy$, be analytic in the half-plane $y > 0$. If the limit

$$\lim_{y\to 0} f(x+iy) = f(x)$$

exists for almost all x, $f(x)$ will be called the limit function of $f(z)$.

By $\mathfrak{H}_p$ we denote the class of functions $f(z)$ analytic in the half-plane $y > 0$ and such that the integral

$$J(y; f) = \int_{-\infty}^{\infty} |f(x+iy)|^p\,dx \leqq M^p,$$

where M is a positive constant which depends only on f and p.

[2]) One of these results was stated without proof in our note [1].

Let $g(x)$ be measurable over $(-\infty, \infty)$. We set

(1.3) $$I(z; g) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} g(t)\,\frac{dt}{t-z},$$

(1.4) $$P(z; g) = \frac{1}{\pi}\int_{-\infty}^{\infty} g(t)\,\frac{y\,dt}{(t-x)^2+y^2} = \int_{-\infty}^{\infty} g(t)\,K(t; z)\,dt,$$

(1.5) $$\widetilde{P}(z; g) = -\frac{1}{\pi}\int_{-\infty}^{\infty} g(t)\,\frac{(t-x)\,dt}{(t-x)^2+y^2} = \int_{-\infty}^{\infty} g(t)\,\widetilde{K}(t; z)\,dt,$$

where

(1.6) $$K(t; z) = \frac{1}{\pi}\frac{y}{(t-x)^2+y^2} = \Re\frac{1}{\pi i(t-z)},$$

(1.7) $$\widetilde{K}(t; z) = -\frac{1}{\pi}\frac{t-x}{(t-x)^2+y^2} = \Im\frac{1}{\pi i(t-z)}.$$

All these integrals converge absolutely whenever $g(t)\,\epsilon\,\mathfrak{L}_p$, $p \geqq 1$, $y \neq 0$. We shall call $I(z; g)$ and $P(z; g)$ integrals of Cauchy type and of Poisson type respectively, associated with the function $g(t)$. We observe that, on setting $\bar{z} = x - iy$,

(1.8) $$P(z; g) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} g(t)\left[\frac{1}{t-z} - \frac{1}{t-\bar{z}}\right] dt = I(z; g) - I(\bar{z}; g),$$

while

(1.9) $$2I(z; g) = P(z; g) + i\widetilde{P}(z; g).$$

If $g(x) = f(x)$ is the limit function of a function $f(z)$ analytic for $y > 0$, and such that

$$f(z) = I(z; f) \quad \text{or} \quad f(z) = P(z; f)$$

we shall say that $f(z)$ is represented by its proper Cauchy integral $I(z; f)$, or by its proper Poisson integral $P(z; f)$, omitting the adjective „proper" if no confusion arises.

The linear transformation

(1.10) $$w = \frac{1+iz}{1-iz}, \qquad z = i\,\frac{1-w}{1+w}$$

maps the half-plane $y > 0$ into the interior of the unit circle $|w| < 1$, the correspondence between the boundaries being given by

(1.11) $$x = \tan\theta/2.$$

If $\varphi(w)$ is analytic in $|w| < 1$, we define the limit function of $\varphi(w)$ by

$$\varphi(e^{i\theta}) = \lim_{r\to 1}\varphi(r\,e^{i\theta}),$$

whenever it exists for almost all θ. As usual [3]) H_p will denote the class of functions $\varphi(w)$ analytic in $|w| < 1$ and such that

$$\int_{-\pi}^{\pi} |\varphi(r\,e^{i\theta})|^p\,d\theta \leqq M^p, \quad w = r\,e^{i\theta}.$$

We shall also use properties of integrals of Cauchy type,

(1.12) $$I_c(w; \gamma) = \frac{1}{2\pi i}\int_{-\pi}^{\pi} \gamma(\tau)\,\frac{d(e^{i\tau})}{e^{i\tau}-w}$$

and of Poisson type,

(1.13) $$\begin{aligned} P_c(w; \gamma) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \gamma(\tau)\,\frac{(1-r^2)\,d\tau}{1-2r\cos(\theta-\tau)+r^2} \\ &= \frac{1}{2\pi i}\int_{|\zeta|=1} \gamma(\tau)\left[\frac{1}{\zeta-w} - \frac{1}{\zeta-w^*}\right] d\zeta = I_c(w; \gamma) - I_c(w^*; \gamma), \\ &\qquad \zeta = e^{i\tau}, \quad w^* = 1/\bar{w}, \end{aligned}$$

associated with the given function $\gamma(\tau)$, as well as those of the proper Cauchy and Poisson integrals, $I_c(w; \varphi)$, $P_c(w; \varphi)$ associated with the given function $\varphi(w)$ analytic in $|w| < 1$.

The fundamental notions of the theory of Fourier transforms are of course indispensable for what follows. For these properties we refer to Zygmund [1], Ch. XII. Let $f(x)$ be integrable over every finite range and let

$$\mathfrak{f}_a(x) = (2\pi)^{-1/2}\int_{-a}^{a} f(t)\,e^{-ixt}\,dt.$$

[3]) We refer to the papers by F. Riesz [1], Fichtenholz [1], and Smirnoff [1], concerning various properties of classes H_p and of integrals of Cauchy type and Poisson type which will be used in the sequel.

If $\mathfrak{f}_a(x)$ converges in $\mathfrak{L}_q$, $1 \leqq q \leqq \infty$, to a function $\mathfrak{f}(x)$, so that as $a \to \infty$,

$$\int_{-\infty}^{\infty} |\mathfrak{f}(x) - \mathfrak{f}_a(x)|^q \, dx \to 0,$$

or

$$\mathfrak{f}_a(x) \to \mathfrak{f}(x) \text{ uniformly over } (-\infty, \infty),$$

according as $\infty > q$ or $q = \infty$, then we write

$$\mathfrak{f}(x) = T(x; f),$$

and call $\mathfrak{f}(x)$ the Fourier transform of $f(x)$ in $\mathfrak{L}_q$. It is well known that whenever $f(x) \in \mathfrak{L}_p$, $1 \leqq p \leqq 2$, it has a Fourier transform $\mathfrak{f}(x)$ in $\mathfrak{L}_{p'}$, $p' = p/(p-1)$.

An important tool in investigating various classes of functions analytic in the unit circle is furnished by the classical „factorization theorem" of F. Riesz-Ostrowski, according to which a function $\varphi(w)$ of a given class can be represented as a product of the „Blaschke function" $b_\varphi(w)$, associated with $\varphi(w)$, and of a function of the same class, which does not vanish in the unit circle. Such is for instance the case of functions of the class H_p. An analogous factorization theorem for functions analytic in the half-plane is indispensable for our discussion. A theorem of this kind is proved in the next § 2 for the functions of class $\mathfrak{H}_p$, $p \geqq 1$, together with some other properties of this important class of functions. Such factorization theorems are obtainable for much more general classes of functions analytic in a half-plane (Gabriel [1, 2]). This problem is of considerable importance in itself. We intend to return to this question in another paper, restricting our investigation at present to the class $\mathfrak{H}_p$, $p \geqq 1$, which will be sufficient for our immediate purposes. In this case the simplest procedure consists merely in showing that, under the transformation (1.10) the class $\mathfrak{H}_p$ is transformed into a sub-class of H_p, which is readily done by using an elegant method introduced by Gabriel. Our proof in § 4 appeals also to some properties of conjugate functions which are discussed in § 3.

2. Functions of class $\mathfrak{H}_p$. In this paragraph we shall discuss some general properties of functions of class $\mathfrak{H}_p$, which are important for the theory of Fourier transforms and also are interesting in themselves. The discussion will be based on a few preliminary lemmas. Unless explicitly stated to the contrary, it will be understood that $1 \leqq p < \infty$.

Lemma 2.1. *If $U(w)$ is $\geqq 0$ and subharmonic in the interior of a circle Γ, and is continuous in the closed area (Γ) bounded by Γ, then for any circle C in (Γ)*

$$\int_C U(w)\,|dw| \leqq 2 \int_\Gamma U(w)\,|dw|. \tag{2.1}$$

The proof is found in Gabriel [3].

Lemma 2.2. *Let $g(t) \in \mathfrak{L}_p$ over $(-\infty, \infty)$ and let*

$$\gamma(\tau) = g(t) = g\left(\tan \frac{\tau}{2}\right)$$

be its transform on the unit circle. The integrals of Cauchy type and of Poisson type associated with $g(t)$ are transformed under the transformation

$$w = \frac{1 + iz}{1 - iz}, \qquad z = i\,\frac{1 - w}{1 + w} \tag{2.2}$$

according to the formulas

$$I(z; g) = I_c(w; \gamma) + C_0, \qquad C_0 = I(-i; g), \tag{2.3}$$

$$P(z; g) = P_c(w; \gamma). \tag{2.4}$$

The constant C_0 vanishes when $I(z; g)$ is the proper Cauchy integral of a function of class $\mathfrak{H}_p$.

Lemma 2.3. *If $f(z)$ is analytic in the half-plane $y > 0$ and has a limit function $f(x) \in \mathfrak{L}_p$ then whenever $f(z)$ is represented by its proper Cauchy integral, it is also represented by its proper Poisson integral and vice versa.*

The proof of these lemmas is found in Hille and Tamarkin [2].

Lemma 2.4. *A function $f(z) \in \mathfrak{H}_p$ tends uniformly to zero when z tends to infinity in any closed half-plane $y \geqq \varepsilon > 0$, where ε is arbitrarily small but fixed.*

The proof is based on the representation of $f(z)$ by means of its Cauchy and Poisson integrals related to the half-plane $y \geqq y_0 > 0$. Such a representation was proved by Bochner [1] in the case $p = 1$ and by Paley and Wiener [1] in the case $p = 2$.

In our proof we follow the line of argument used by Paley and Wiener.

On applying Cauchy's formula to the rectangle with vertices at the points $(\pm T+iy_0)$, $(\pm T+iY)$ we have

$$2\pi i f(z) = \int_{-T}^{T}\frac{f(t+iy_0)}{t+iy_0-z}\,dt - \int_{-T}^{T}\frac{f(t+iY)}{t+iY-z}\,dt +$$

$$+\, i\int_{y_0}^{Y}\frac{f(T+i\eta)}{T+i\eta-z}\,d\eta - i\int_{y_0}^{Y}\frac{f(-T+i\eta)}{-T+i\eta-z}\,d\eta =$$

$$= I_1(T,y_0) - I_1(T,Y) + I_2(T) - I_2(-T), \quad 0<y_0<y<Y.$$

Choose X such that $2|x|<X$. Then

$$\frac{1}{X}\int_{X}^{2X}|I_2(\pm T)|\,dT \leqq \frac{1}{X}\int_{X}^{2X}dT\int_{y_0}^{Y}\frac{|f(\pm T+i\eta)|}{|\pm T+i\eta-z|}\,d\eta \leqq$$

$$\leqq \frac{2}{X^2}\int_{y_0}^{Y}d\eta\int_{X}^{2X}|f(\pm T+i\eta)|\,dT \leqq$$

$$\leqq \frac{2}{X^2}\int_{y_0}^{Y}d\eta\, X^{1/p'}\left[\int_{X}^{2X}|f(\pm T+i\eta)|^p\,dT\right]^{1/p} \leqq 2MX^{1/p'-}(Y-y_0)$$

where

$$\int_{-\infty}^{\infty}|f(t+iy)|^p\,dt \leqq M^p,$$

and, in case $p=1$, $p'=\infty$, the factor $X^{1/p'}$ is to be replaced by 1. On keeping y_0 and Y fixed and allowing $X\to\infty$ we get

$$2\pi i f(z) = \lim_{X\to\infty}\frac{1}{X}\int_{X}^{2X}[I_1(T,y_0)-I_1(T,Y)]\,dT.$$

Since the limits

$$I_1(\infty,y_0) = \lim_{T\to\infty}I_1(T,y_0) = \int_{-\infty}^{\infty}\frac{f(t+iy_0)}{t+iy_0-z}\,dt = I_1(y_0),$$

$$I_1(\infty,Y) = \lim_{T\to\infty}I_1(T,Y) = \int_{-\infty}^{\infty}\frac{f(t+iY)}{t+iY-z}\,dt = I_1(Y)$$

exist as absolutely convergent integrals it is seen that

$$2\pi i f(z) = I_1(y_0) - I_1(Y).$$

Now let $Y\to\infty$. In case $p=1$ we have

$$|I_1(Y)| \leqq \int_{-\infty}^{\infty}\frac{|f(t+iY)|}{Y-y}\,dt \leqq M(Y-y)^{-1} = O(1/Y),$$

while, in case $p>1$,

$$|I_1(Y)| \leqq \left[\int_{-\infty}^{\infty}|f(t+iY)|^p\,dt\right]^{1/p}\left[\int_{-\infty}^{\infty}\frac{dt}{\{(t-x)^2+(Y-y_0)^2\}^{p'/2}}\right]^{1/p'} =$$

$$= O\left(Y^{-1-\frac{1}{p'}}\right).$$

Hence $I_1(Y)\to 0$ in either case, and

(2.5) $$f(z) = (2\pi i)^{-1}\int_{-\infty}^{\infty}\frac{f(t+iy_0)}{t+iy_0-z}\,dt, \quad 0<y_0<y.$$

The same argument shows that

(2.6) $$0 = (2\pi i)^{-1}\int_{-\infty}^{\infty}\frac{f(t+iy_0)}{t+iy_0-z'}\,dt, \quad z'=x+iy', \quad y'<y_0.$$

Let $\varepsilon>0$ be given. In formulas (2.5) and (2.6) put

$$y_0=\frac{\varepsilon}{2}, \quad z'=x+i(2y_0-y), \quad y>y_0$$

and subtract (2.6) from (2.5). Thus we obtain the desired representation

(2.7) $$f(z) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t+iy_0)(y-y_0)}{(t-x)^2+(y-y_0)^2}\,dt = \int_{-\infty}^{\infty}f(t+iy_0)K(t;z-iy_0)\,dt.$$

To prove our lemma assume $y\geqq\varepsilon=2y_0$ and first consider the case $p=1$. A positive δ being given choose T_0 so large that

$$\int_{-\infty}^{-T_0}|f(t+iy_0)|\,dt + \int_{T_0}^{\infty}|f(t+iy_0)|\,dt < \delta.$$

Since $K(t;z-y_0)\leqq(y-y_0)^{-1}<y_0^{-1}$ the contribution of the corresponding range of integration in (2.7) will not exceed δ/y_0. After

T_0 has been so fixed, the contribution of the remaining range $(-T_0, T_0)$ is obviously $O(1/z)$ uniformly in the half-plane $y \geqq \varepsilon$.

If $p > 1$ then, by the convexity property,

$$|f(z)|^p \leqq \int_{-\infty}^{\infty} |f(t+iy_0)|^p \; K(t; z - iy_0)\, dt \tag{2.8}$$

and the preceding argument can be applied without modifications to the integral of the right-hand member of (2.8).

Lemma 2.5. *Under the transformation* (2.2) *the class* $\mathfrak{H}_p$ *is transformed into a sub-class of* H_p.

Consider any half-plane $y > \varepsilon > 0$. Its boundary $y = \varepsilon$ is mapped by (2.2) into a circle Γ_ε in the w-plane, tangent from the inside to the circle $|w| = 1$ at $w = -1$. The half-plane itself is transformed into the interior of Γ_ε. Let $f(z) \epsilon \mathfrak{H}_p$. The function $\varphi(w) = f(z)$ is analytic in $|w| < 1$, hence $|\varphi(w)|^p$ is subharmonic in the interior of Γ_ε and by Lemma 2.4, is continuous in the closed area (Γ_ε). Let C be any given circle $|w| = r < 1$. If ε is sufficiently small, C will be in (Γ_ε). Then, by Lemma 2.1,

$$\int_{-\pi}^{\pi} |\varphi(re^{i\theta})|^p \, d\theta = \frac{1}{r}\int_C |\varphi(w)|^p \, |dw| \leqq \frac{2}{r}\int_{\Gamma_\varepsilon} |\varphi(w)|^p \, |dw| =$$

$$= \frac{2}{r}\int_{-\infty}^{\infty} |f(t+i\varepsilon)|^p \, \frac{2\,dt}{(1+\varepsilon)^2 + t^2} < \frac{4}{r} M^p$$

which shows that $\varphi(w) \epsilon H_p$.

We now pass on to the main theorems of this paragraph.

Theorem 2.1. (i) *A function* $f(z) \epsilon \mathfrak{H}_p$ *for almost all* x *has a limit function* $f(x) \epsilon \mathfrak{L}_p$ *to which it tends along any nontangential path.*

(ii) *Any* $f(z) \epsilon \mathfrak{H}_p$ *is represented by its proper Cauchy and Poisson integrals. In terms of the real part of the limit function* $f(x)$ *we also have*

$$f(z) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \Re f(t)\, \frac{dt}{t-z} = 2\, I(z; \Re f) = \tag{2.9}$$

$$= P(z; \Re f) + i\, \tilde{P}(z; \Re f).$$

(iii) *Any* $f(z) \epsilon \mathfrak{H}_p$ *tends to its limit function* $f(x)$ *in the mean of order* p,

$$\int_{-\infty}^{\infty} |f(x+iy) - f(x)|^p \, dx \to 0 \quad \textit{as} \quad y \to 0.$$

Moreover, as $y \downarrow 0$,

$$T(y; f) = \int_{-\infty}^{\infty} |f(x+iy)|^p \, dx \uparrow T(0; f) = \int_{-\infty}^{\infty} |f(x)|^p \, dx.$$

(iv) *If* $f(x) \epsilon \mathfrak{L}_p$ *and*

$$f(z) = P(z; f)$$

is analytic for $y > 0$, *then* $f(z) \epsilon \mathfrak{H}_p$ *and therefore is represented by its proper Poisson and Cauchy integrals* $P(z; f)$, $I(z; f)$ *respectively, as well as by* (2.9).

We observe that the analogous properties of functions of class H_p are well known. Let now $f(z) \epsilon \mathfrak{H}_p$ and let $\varphi(w)$ be the transform of $f(z)$ under the transformation (2.2). By Lemma 2.5, $\varphi(w) \epsilon H_p$, hence $\varphi(w)$ has a limit function $\varphi(e^{i\theta}) \epsilon L_p$ to which it tends along any non-tangential path. Thus the limit function $f(x)$ of $f(z)$ exists almost everywhere along any nontangential path. Moreover, by Fatou's, theorem

$$\int_{-\infty}^{\infty} |f(x)|^p \, dx \leqq \lim_{y \to 0} \int_{-\infty}^{\infty} |f(x+iy)|^p \, dx \leqq M^p$$

so that $f(x) \epsilon \mathfrak{L}_p$. This proves statement (i) of our theorem. To prove statement (ii) construct the integral $P(z; f)$. By Lemma 2.2 we have

$$P(z; f) = P_c(w; \varphi) = \varphi(w) = f(z)$$

since $\varphi(w) \epsilon H_p$ is represented by its proper Poisson integral. Thus $f(z)$ is represented by its Poisson and by Lemma 2.3 by its Cauchy integral. To complete the proof of (ii) we observe that $f(z) = P(z; f)$ implies in view of (1.6)

$$\Re f(z) = P(z; \Re f) = \Re\{2\, I(z; \Re f)\}.$$

Since $f(z)$ and $2\, I(z; \Re f)$ are analytic for $y > 0$, and tend to zero when $y \to \infty$, (2.9) follows. To prove statement (iii) we observe that, by (ii),

$$f(z) - f(x) = \int_{-\infty}^{\infty} [f(t) - f(x)]\, K(t; z)\, dt,$$

whence, by the convexity property,

$$|f(x+iy)-f(x)|^p \leqq \int_{-\infty}^{\infty} |f(t)-f(x)|^p K(t;z)\,dt =$$
$$= \int_{-\infty}^{\infty} |f(x+t)-f(x)|^p K(t;iy)\,dt,$$

and

$$\int_{-\infty}^{\infty} |f(x+iy)-f(x)|^p dx \leqq \int_{-\infty}^{\infty} dt\, K(t;iy) \int_{-\infty}^{\infty} |f(x+t)-f(x)|^p dx.$$

Since the function

$$F(t) = \int_{-\infty}^{\infty} |f(x+t)-f(x)|^p dx$$

is continuous at $t=0$, statement (iii) becomes a consequence of the classical property of the Poisson integral for

$$\int_{-\infty}^{\infty} F(t)\,K(t;iy)\,dt \to F(0)=0 \quad \text{as} \quad y\to 0.$$

The fact that $T(y;f)$ increases when y decreases is well known and is readily derived from (2.8). The same applies to statement (iv).

Theorem 2.2. *A function $f(z)\in\mathfrak{H}_p$ can be represented as a product*

(2.10) $$f(z) = b_f(z)\,h(z),$$

where

$$b_f(z) = \prod_{(\nu)} \frac{z-z_\nu}{z-\bar z_\nu}\,\frac{\bar z_\nu - i}{z_\nu + i}$$

is the Blaschke function associated with $f(z)$ and $h(z)\in\mathfrak{H}_p$, but does not vanish in the half-plane $y>0$. Here $\{z_\nu\}$ is the sequence of zeros of $f(z)$ in the half-plane $y>0$ and the condition

(2.11) $$\sum_{(\nu)} y_\nu/(1+x_\nu^2+y_\nu^2) < \infty$$

must be satisfied. If $f(z)$ does not vanish for $y>0$, $b_f(z)\equiv 1$, $h(z)=f(z)$. Otherwise

$$|b_f(z)|<1, \quad y>0, \quad |b_f(x)|=1 \quad \textit{almost everywhere.}$$

The limit function $h(x)$ of $h(z)$ satisfies the condition

(2.12) $$|h(x)| = |f(x)| \quad \textit{almost everywhere}$$

so that

(2.13) $$\int_{-\infty}^{\infty} |h(z)|^p dx \leqq \int_{-\infty}^{\infty} |h(x)|^p dx = \int_{-\infty}^{\infty} |f(x)|^p dx \leqq M^p.$$

The analogous theorem for functions of class H_p is well known. Now, if $f(z)\in\mathfrak{H}_p$ and $\varphi(w)=f(z)$, then by Lemma 2.5, $\varphi(w)\in H_p$. Hence we have the representation

$$\varphi(w) = b_\varphi(w)\,\eta(w),$$

where $b_\varphi(w)$ is the Blaschke function relative to the unit circle associated with $\varphi(w)$, and $\eta(w)\in H_p$ but does not vanish in $|w|<1$. We set $b_f(z)=b_\varphi(w)$, $h(z)=\eta(w)$. The properties of $b_f(z)$ stated in the theorem are readily derived from the known properties of $b_\varphi(w)$. As to $h(z)$, we have by Lemma 2.2,

$$P(z;h) = P_c(w;\eta) = \eta(w) = h(z)$$

and the properties of $h(z)$ now are proved by using Theorem 2.1. Finally, condition (2.11) is derived from the corresponding condition for the roots $\{w_\nu\}$ of $\varphi(w)$, viz. that the infinite product $\prod_{(\nu)}|w_\nu|$ must converge[4])

3. Conjugate functions. The properties of conjugate functions which we discuss in the present paragraph are partially known, at least under more restrictive assumptions. For the reader's convenience we state explicitly, in form of lemmas, those properties we need. It will be understood again that $1\leqq p<\infty$.

Lemma 3.1. *Let $g(x)\in\mathfrak{L}_p$ and let its conjugate function*

$$\tilde g(x) = \frac{1}{\pi}\, P.V. \int_{-\infty}^{\infty} \frac{g(t)\,dt}{x-t}$$

4) The results stated in Theorem 2.2 are not essentially new. An analogous theorem was proved by Hardy, Ingham and Pólya [1], Theorem 1, under conditions which are not equivalent to ours. Formula (2.10) could also be derived from a theorem by Gabriel [1, 2]. It is not at all obvious, however, that the functions of class $\mathfrak{H}_p$ satisfy the conditions required by Gabriel; that they actually do so follows from our Lemma 2.4. The proof and application of this lemma is therefore the only essential novelty of our discussion. Our method obviously breaks down if $p<1$.

exist for a given value of x, this being the case for almost all values of x. Then (see (1.5), (1.7))

$$\tilde{P}(z; g) = \int_{-\infty}^{\infty} g(t)\tilde{K}(t; z)\,dt \to \tilde{g}(x) \quad as \quad y \to 0. \tag{3.1}$$

There is no loss of generality in assuming $x = 0$. Then

$$\tilde{P}(z; g) = \tilde{P}(iy; g) = -\frac{1}{\pi}\int_0^{\infty}[g(t) - g(-t)]\frac{t\,dt}{t^2 + y^2}.$$

But

$$\frac{t}{t^2 + y^2} - \frac{1}{t} = \frac{y^2}{t^2 + y^2},$$

and by hypothesis

$$\tilde{g}(0) = -\lim_{\varepsilon \to 0}\frac{1}{\pi}\int_{\varepsilon}^{\infty}[g(t) - g(-t)]\,\frac{dt}{t}$$

exists. Hence

$$\tilde{P}(iy; g) = \tilde{g}(0) + \lim_{\varepsilon \to 0}\frac{1}{\pi}\int_{\varepsilon}^{\infty}\frac{g(t) - g(-t)}{t}\,\frac{y^2}{t^2 + y^2}\,dt \equiv \tilde{g}(0) + T(y).$$

We consider $T(y)$ as an improper integral and write

$$T(y) = \int_0^{\eta} + \int_{\eta}^{\infty} \equiv T_1(y) + T_2(y),$$

where η will be chosen sufficiently small but fixed. In the integral $T_2(y)$ the integrand is dominated by a fixed integrable function $|[g(t) - g(-t)]/t|$ and almost everywhere tends to 0 as $y \to 0$. Hence

$$T_2(y) \to 0 \quad \text{as} \quad y \to 0.$$

As to $T_1(y)$, on introducing the improper integral

$$G(t) = \frac{1}{\pi}\int_0^{t}[g(\tau) - g(-\tau)]\,\frac{d\tau}{\tau}$$

and integrating by parts, we have

$$T_1(y) = G(\eta)\,\frac{y^2}{y^2 + \eta^2} - \int_0^{\eta} G(t)\,d_t\,\frac{y^2}{t^2 + y^2}.$$

Since $G(\eta) \to 0$ as $\eta \to 0$, the integrated term here is $o(1)$ as $\eta \to 0$, and the second term is

$$o\left(-\int_0^{\eta} d_t\frac{y^2}{t^2 + y^2}\right) = o(1), \quad \text{uniformly in } y.$$

This proves lemma 3.1. We observe that the relation

$$P(z; g) \to g(x) \quad \text{as} \quad y \to 0 \tag{3.2}$$

for almost all x is well known.

Theorem 3.1. *Let $g(x)$ and $\tilde{g}(x) \in \mathfrak{L}_p$* [5]*. The function*

$$f(z) = 2I(z; g) = P(z; g) + i\tilde{P}(z; g) \tag{3.3}$$

is analytic in the half-plane $y > 0$ and is representable by its proper Cauchy and Poisson integrals. Furthermore its limit function $f(x)$ is such that

$$f(x) = g(x) + i\tilde{g}(x). \tag{3.4}$$

Conversely if $f(z)$ is representable by its proper Cauchy or Poisson integral with the limit function $f(x) \in \mathfrak{L}_p$, then

$$\Im f(x) = \tilde{\Re f}(x). \tag{3.5}$$

Let $f(z)$ be given by (3.3). By Lemma 3.1 its limit function $f(x) = g(x) + i\tilde{g}(x)$. On the other hand $f(z)$ is represented by an integral of Cauchy type; by Lemma 2.2. (2.3), its transform $\varphi(w)$ in the w-plane is represented also by an integral of Cauchy type; since, however, the limit function $\varphi(e^{i\theta}) \in L_p$, $\varphi(w)$ is also represented by its proper Cauchy or Poisson integrals; again by Lemma 2.2, (2.4), it follows that $f(z)$ is represented by its proper Poisson integral, and, by Lemma 2.3, by its proper Cauchy integral.

To prove the converse let

$$f(z) = I(z; f) = P(z; f), \qquad f(x) \in \mathfrak{L}_p.$$

On the other hand by Theorem 2.1, (iv), we also have

$$f(z) = P(z; \Re f) + i\tilde{P}(z; \Re f).$$

[5]) If $p > 1$ and $g(x) \in \mathfrak{L}_p$, then also $\tilde{g}(x) \in \mathfrak{L}_p$. In this case the results stated in Lemma 3.2 were obtained by M. Riesz [1].

Since $\Re f \in \mathfrak{L}_p$, Lemma 3.1 shows that for almost all x,

$$P(z; \Re f) \to \Re f(x), \qquad \widetilde{P}(z; f) \to \widetilde{\Re} f(x).$$

Consequently for almost x

$$f(x) = \Re f(x) + i \Im f(x) = \Re f(x) + i \widetilde{\Re} f(x),$$

and (3.5) follows.

Corollary. *If $g(x)$ and $\widetilde{g}(x)$ both $\in L_p$, $p \geqq 1$, then $-g(x)$ is the conjugate of $\widetilde{g}(x)$ and the reciprocity relations*

$$\text{(3.6)} \quad \widetilde{g}(x) = \frac{1}{\pi} P.\,V. \int_{-\infty}^{\infty} \frac{g(t)\,dt}{x-t}, \qquad g(x) = -\frac{1}{\pi} P.\,V. \int_{-\infty}^{\infty} \frac{\widetilde{g}(t)\,dt}{x-t}$$

hold [6].

Indeed, we have

$$f(z) = 2I(z; g) = I(z; g) + iI(z; \widetilde{g}) = 2iI(z; \widetilde{g}),$$

so that

$$P(z; g) + i\widetilde{P}(z; g) = iP(z; \widetilde{g}) - \widetilde{P}(z; \widetilde{g})$$

and we have only to apply Lemma 3.1. If $p > 1$ it is sufficient to assume only that either $g(x)$ or $\widetilde{g}(x) \in \mathfrak{L}_p$.

Theorem 3.2. *Assume that the functions $g(x)$ and $\widetilde{g}(x)$ both $\in \mathfrak{L}_p$ and that they both are of bounded variation over $(-\infty, \infty)$. Under these assumptions we have*

(i) *$g(x)$ and $\widetilde{g}(x)$ are absolutely continuous so that the derivatives $g'(x)$ and $\widetilde{g}'(x)$ both $\in \mathfrak{L}_1$.*

(ii) *The function $\widetilde{g}'(x)$ is the conjugate of $g'(x)$.*

(iii) *If*

$$f(z) = 2I(z; g) = I(z; g + i\widetilde{g}) = I(z; f), \qquad f(x) = g(x) + i\widetilde{g}(x),$$

then the derivative $f'(z) \in \mathfrak{H}_1$ and thus is represented by its Cauchy and Poisson integrals. The limit function of $f'(z)$ is

$$f'(x) = g'(x) + i\widetilde{g}'(x).$$

By Theorem 3.1 $f(z)$ is representable by its Cauchy and Poisson integrals and has the limit function $f(x) = g(x) + i\widetilde{g}(x)$. By assumption this function is of bounded variation over $(-\infty, \infty)$. Hence its transform $\varphi(e^{i\theta})$ on the unit circle $|w| = 1$, which is the limit function of the transform $\varphi(w) = f(z)$, is also of bounded variation over $(-\pi, \pi)$. It is well known that under these conditions $\varphi(e^{i\theta})$ is absolutely continuous in θ, hence $f(x)$ is absolutely continuous in x. Thus statement (i) is proved.

[6] In case $p > 1$ a proof was given by M. Riesz [1].

To prove statement (iii) we write

$$f(z) = P(z; f) = \int_{-\infty}^{\infty} f(t)\, K(t; z)\, dt,$$

$$f'(z) = \frac{\partial f(z)}{\partial x} = \int_{-\infty}^{\infty} f(t) \frac{\partial}{\partial x} K(t; z)\, dt$$

$$= -\int_{-\infty}^{\infty} f(t) \frac{\partial}{\partial t} K(t; z)\, dt = \int_{-\infty}^{\infty} f'(t)\, K(t; z)\, dt.$$

The operations of differentiation under integral sign and at integrating by parts are obviously permissible; the integrated terms vanish since $K(t; z)$ vanishes for $t \to \pm\infty$, while $f(t)$ is bounded. It follows that $f'(x)$ is the limit function of $f'(z)$ and, by Theorem 2.1, iv, $f'(z) \in \mathfrak{H}_1$. Thus statement (iii) is proved. The proof of (ii) is now obtained by an easy application of Theorem 3.1.

4. Applications to the theory of Fourier transforms.

We shall need some additional lemmas whose analogues in the theory of power series are quite trivial.

Lemma 4.1. *Let $f(x) \in \mathfrak{L}_p$, $1 \leqq p < \infty$. If $f(x)$ has a Fourier transform $\mathfrak{f}(x)$ in some $\mathfrak{L}_q$, $1 \leqq q \leqq \infty$, then in order that $f(x)$ be the limit function of a function $f(z) \in \mathfrak{H}_p$ (or, which is the same, of a function analytic in the half-plane $y > 0$ and representable by its Cauchy or Poisson integrals) it is necessary and sufficient that $\mathfrak{f}(x) = 0$ for $x < 0$.*

Lemma 4.2. *If $f(x) \in \mathfrak{L}_p$, $1 \leqq p \leqq \infty$ and has a Fourier transform $\mathfrak{f}(x)$ in $\mathfrak{L}_q$, $1 \leqq q \leqq \infty$, then the Poisson integral associated with $f(x)$ can be written in the form*

$$\text{(4.1)} \quad P(z; g) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{y\,dt}{(t-x)^2 + y^2} = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{ixt} e^{-y|t|} \mathfrak{f}(t)\, dt, \quad y > 0.$$

If $f(x)$ is the limit function of a function $f(z)\in\mathfrak{H}_p$, $1\leq p<\infty$, then

$$P(z;f)=(2\pi)^{-1/2}\int_0^\infty e^{ixt}e^{-yt}\mathfrak{f}(t)\,dt,\quad y>0 \text{ [7]}. \tag{4.2}$$

Lemma 4.3. *If $f(x)$ and $g(x)\in\mathfrak{L}_2$ and $\mathfrak{f}(x)$, $\mathfrak{g}(x)$ are their Fourier transforms in $\mathfrak{L}_2$, then*

$$\int_0^\infty dy\int_0^\infty e^{-yt}|\mathfrak{f}(t)|\,dt\int_0^\infty e^{-ys}|\mathfrak{g}(s)|\,ds\leq \tag{4.3}$$

$$\leq\pi\left[\int_{-\infty}^\infty|f(x)|^2\,dx\int_{-\infty}^\infty|g(x)|^2\,dx\right]^{1/2}.$$

Indeed the left-hand member of (4.3) is equal to

$$\int_0^\infty\int_0^\infty|\mathfrak{f}(t)\,\mathfrak{g}(s)|/(t+s)\,dt\,ds$$

and by Hilbert's inequality (Hardy, Ingham and Pólya [1]) does not exceed

$$\pi\left[\int_0^\infty|\mathfrak{f}(t)|^2\,dt\int_0^\infty|\mathfrak{g}(s)|^2\,ds\right]^{1/2}\leq\pi\left[\int_{-\infty}^\infty|\mathfrak{f}(t)|^2\,dt\int_{-\infty}^\infty|\mathfrak{g}(s)|^2\,ds\right]^{1/2}=$$

$$=\pi\left[\int_{-\infty}^\infty|f(x)|^2\,dx\int_{-\infty}^\infty|g(x)|^2\,dx\right]^{1/2}.$$

In passing to our main results we first give a new proof of a theorem due to M. Riesz [1].

Theorem 4.1. *If $f(z)\in\mathfrak{H}_p$, $1\leq p<\infty$, then for each x,*

$$\int_0^\infty|f(x+iy)|^p\,dy\leq\frac{1}{2}\int_{-\infty}^\infty|f(t)|^p\,dt \text{ [8]}. \tag{4.4}$$

[7]) The proof of these lemmas is found in Hille and Tamarkin [2]. The discussion given there was concerned with Fourier transforms in a certain generalized sense, but is valid without any modifications in the case of Fourier transforms in $\mathfrak{L}_q$.

[8]) This is an analogue of a classical theorem by Fejér and F. Riesz according to which if $\varphi(w)\in H_p$ and D is any diameter of the unit circle Γ, then

$$\int_D|\varphi(w)|^p\,|dw|\leq\frac{1}{2}\int_\Gamma|\varphi(w)|^p\,|dw|.$$

We start with the case $p=2$. Then $f(x)$ has a Fourier transform $\mathfrak{f}(x)\in\mathfrak{L}_2$ which vanishes for $x\leqslant 0$, so that

$$\int_{-\infty}^\infty|f(x)|^2\,dx=\int_0^\infty|\mathfrak{f}(x)|^2\,dx.$$

By Lemma 4.2 we have

$$f(z)=P(z;f)=(2\pi)^{-1/2}\int_0^\infty e^{ixt}e^{-yt}\mathfrak{f}(t)\,dt,\quad y>0,$$

whence, by Lemma 4.3,

$$\int_0^\infty|f(x+iy)|^2\,dy\leq(2\pi)^{-1}\int_0^\infty dy\left[\int_0^\infty e^{-yt}|\mathfrak{f}(t)|\,dt\right]^2$$

$$\leq\frac{1}{2}\int_0^\infty|\mathfrak{f}(t)|^2\,dt=\frac{1}{2}\int_{-\infty}^\infty|f(x)|^2\,dx.$$

Thus (4.4) is proved in the case $p=2$. In the general case, if $f(z)$ has no zeros in the half-plane $y>0$, we apply the preceding result to the function $[f(z)]^{p/2}\in\mathfrak{H}_2$. Finally, if $f(z)$ has zeros in $y>0$, we use Theorem 2.2 with the result

$$\int_0^\infty|f(x+iy)|^p\,dy\leq\int_0^\infty|h(x+iy)|^p\,dy\leq\frac{1}{2}\int_{-\infty}^\infty|h(x)|^2\,dx=$$

$$=\frac{1}{2}\int_{-\infty}^\infty|f(x)|^2\,dx.$$

On the basis of the preceding results we now can prove our main.

Theorem 4.2. *If both $g(x)$ and its conjugate $\tilde{g}(x)\in\mathfrak{L}_1$, and if $\mathfrak{f}(x)$ is the Fourier transform (in $\mathfrak{L}_\infty$) of the function*

$$f(x)=g(x)+i\,\tilde{g}(x)$$

then

$$\int_0^\infty|\mathfrak{f}(t)|/t\,dt\leq\left(\frac{\pi}{2}\right)^{1/2}\int_{-\infty}^\infty|f(x)|\,dx, \tag{4.5}$$

the constant $\left(\frac{\pi}{2}\right)^{1/2}$ being the best possible.

We first observe that, by Theorem 3.1, $f(x)$ is the limit function of a function $f(z) \in \mathfrak{H}_1$. Hence by Lemma 4.1, $\mathfrak{f}(x)=0$ for $x<0$ If now $\mathfrak{f}(x)$ is real-valued and $\geqq 0$, the proof of Theorem 4.2 is almost trivial. Indeed under this assumption we have in view of Lemma 4.2 and Theorem 4.1,

$$(4.6) \qquad \int_0^\infty \mathfrak{f}(t)/t\,dt = \int_0^\infty dy \int_0^\infty e^{-yt}\mathfrak{f}(t)\,dt = (2\pi)^{1/2}\int_0^\infty f(iy)\,dy \leqq$$
$$\leqq \left(\frac{\pi}{2}\right)^{1/2}\int_{-\infty}^{\infty} |f(x)|\,dx.$$

In general, however, this argument is not valid, and we have to use a device analogous to that used by Zygmund in the case of the power series ([1], pp. 158–159). If $f(z)$ has zeros in the half-plane $y>0$, we use Theorem 2.2 to write

$$f(z) = b_f(z)\,h(z),$$

$h(z) \neq 0$ in $y>0$, $h(z) \in \mathfrak{H}_1$, $|h(x)| = |f(x)|$ almost everywhere,

$$|b_f(z)| < 1 \quad \text{for} \quad y>0.$$

We set

$$f(z) = f_1(z)\,f_2(z), \quad f_1(z) = b_f(z)\,[h(z)]^{1/2}, \quad f_2(z) = [h(z)]^{1/2},$$

so that $f_k(z) \in \mathfrak{H}_2$, $k=1,2$, while

$$\int_{-\infty}^{\infty} |f_k(x)|^2\,dx = \int_{-\infty}^{\infty} |h(x)|\,dx = \int_{-\infty}^{\infty} |f(x)|\,dx, \quad k=1,2.$$

If $f(z)$ has no zeros in $y>0$ the situation is even simpler, but for the uniformity of notation we shall put

$$h(z) = f(z), \quad f_1(z) = f_2(z) = [f(z)]^{1/2}.$$

Now introduce the Fourier transforms $\mathfrak{f}_k(x)$ (in $\mathfrak{L}_2$) of $f_k(x)$. Again, since $\mathfrak{f}_k(x)=0$ for $x<0$, we have

$$\int_0^\infty |\mathfrak{f}_k(x)|^2\,dx = \int_{-\infty}^{\infty} |\mathfrak{f}_k(x)|^2\,dx = \int_{-\infty}^{\infty} |f_k(x)|^2\,dx = \int_{-\infty}^{\infty} |f(x)|\,dx.$$

The Fourier transform $\mathfrak{f}(x)$ of $f(x) = f_1(x)\,f_2(x)$ can be expressed in terms of $\mathfrak{f}_k(x)$ by means of the well known „Faltung" rule (e. g. Wiener [1], pp. 70–71),

$$(4.7) \quad \mathfrak{f}(t) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} \mathfrak{f}_1(u)\,\mathfrak{f}(t-u)\,du = (2\pi)^{-1/2}\int_0^t \mathfrak{f}_1(u)\,\mathfrak{f}_2(t-u)\,du.$$

This yields

$$\int_0^\infty |\mathfrak{f}(t)|/t\,dt = \int_0^\infty dy \int_0^\infty e^{-yt}|\mathfrak{f}(t)|\,dt \leqq$$
$$\leqq (2\pi)^{-1/2}\int_0^\infty dy \int_0^\infty e^{-yt}\,dt \int_0^t |\mathfrak{f}_1(u)|\,|\mathfrak{f}_2(t-u)|\,du =$$
$$= (2\pi)^{-1/2}\int_0^\infty dy \int_0^\infty e^{-yu}|\mathfrak{f}_1(u)|\,du \int_0^\infty e^{-yv}|\mathfrak{f}_2(v)|\,dv \leqq$$
$$\leqq \left(\frac{\pi}{2}\right)^{1/2}\left[\int_0^\infty |\mathfrak{f}_1(u)|^2\,du \int_0^\infty |\mathfrak{f}_2(u)|^2\,du\right]^{1/2} = \left(\frac{\pi}{2}\right)^{1/2}\int_0^\infty |f(x)|\,dx,$$

which is the desired result.

It remains only to show that the coefficient $\left(\frac{\pi}{2}\right)^{1/2}$ in the right-hand member of (4.5) is the best possible. By considering the conformal mapping of an appropriate ellipse into the unit circle it is easy to construct a function $\psi(w)$, analytic in the closed unit circle $|w| \leqq 1$, real valued for real values of w, and such that

$$\int_0^1 \psi(w)\,dw > \left(\frac{1}{2}-\varepsilon\right)\int_{|w|=1} |\psi(w)|\,|dw|\ ^{9)},$$

where ε can be taken arbitrarily small but fixed. Using the transformation

$$w = \frac{1+iz}{1-iz}, \qquad z = i\,\frac{1-w}{1+w},$$

consider the function

$$f(z) = -i\,\psi(w)\,\frac{dw}{dz} = \psi(w)\,\frac{2}{(1-iz)^2}.$$

[9]) Cf. Fejér [1], p. 118, footnote.

Since $\psi(w)$ is bounded in $|w| \leqq 1$, $f(z) \in \mathfrak{H}_1$ and

$$\int_{|w|=1} |\psi(w)|\,|dw| = \int_{-\infty}^{\infty} |f(x)|\,dx.$$

Now the segment $(0, 1)$ of the w-plane is mapped into the segment $(0, i)$ of the y-axis in the z-plane and a simple computation shows that

$$\int_0^1 \psi(w)\,dw = \int_0^1 f(iy)\,dy > \left(\frac{1}{2} - \varepsilon\right) \int_{-\infty}^{\infty} |f(x)|\,dx.$$

Thus in view of (4.2) we have for this particular function $f(z)$

$$\int_0^{\infty} |\mathfrak{f}(t)|/t\,dt = \int_0^{\infty} dy \int_0^{\infty} e^{-yt} |\mathfrak{f}(t)|\,dt \geqq (2\pi)^{1/2} \int_0^1 |P(iy; f)|\,dy \geqq$$

$$\geqq (2\pi)^{1/2} \int_0^1 P(iy; f)\,dy = (2\pi)^{1/2} \int_0^1 f(iy)\,dy > \left\{\left(\frac{\pi}{2}\right)^{1/2} - \delta\right\} \int_{-\infty}^{\infty} |f(x)|\,dx$$

where $\delta = \varepsilon(2\pi)^{1/2}$. Since δ can be made as small as we please the proof of Theorem 4.2 is completed.

The following theorem appears as in immediate corollary of Theorem 4.2.

Theorem 4.3. *Assume that $g(x)$ and its conjugate $\tilde{g}(x)$ are both of bounded variation over $(-\infty, \infty)$, and in addition $\in \mathfrak{L}_p$, $1 \leqq p < \infty$. Then $f(x) = g(x) + i\tilde{g}(x)$ is absolutely continuous and the Fourier transform $\mathfrak{f}(x)$ of $f(x)$, defined by*

$$\mathfrak{f}(x) = (2\pi)^{-1/2} \lim_{a\to\infty} \int_{-a}^{a} f(t)\, e^{-itx}\,dt, \tag{4.8}$$

exists for all $x \neq 0$ and is continuous for $x \neq 0$.

Furthermore $\mathfrak{f}(x) = 0$ for $x < 0$ and is absolutely integrable over $(0, \infty)$. More precisely

$$\int_0^{\infty} |\mathfrak{f}(t)|\,dt \leqq \left(\frac{\pi}{2}\right)^{1/2} \int_{-\infty}^{\infty} |f'(x)|\,dx,$$

the constant $\left(\frac{\pi}{2}\right)^{1/2}$ being the best possible.

By Theorem 3.2 $f(x)$ is absolutely continuous and is the limit function of a function $f(z)$ analytic in the half-plane $y > 0$ and such that $f'(z) \in \mathfrak{H}_1$, with the limit function $f'(x)$. By Theorem 4·2 then

$$\int_0^{\infty} |\mathfrak{f}_1(t)|/t\,dt \leqq \left(\frac{\pi}{2}\right)^{1/2} \int_{-\infty}^{\infty} |f'(x)|\,dx,$$

where $\mathfrak{f}_1(x)$ is the Fourier transform (in $\mathfrak{L}_\infty$) of $f'(x)$. On the other hand since under our assumptions $f(x) \to 0$ as $x \to \pm\infty$, we have

$$(2\pi)^{-1/2} \int_{-a}^{a} f(t)\, e^{-ixt}\,dt = (2\pi)^{-1/2} \left\{ \frac{e^{-ixt}}{-ix} f(t) \Big]_{-a}^{a} + \frac{1}{ix} \int_{-a}^{a} f'(t)\, e^{-ixt}\,dt \right\}$$

$$\to \frac{1}{ix} \mathfrak{f}_1(x),$$

whence

$$\mathfrak{f}(x) = \frac{1}{ix} \mathfrak{f}_1(x), \qquad x \neq 0.$$

It should be observed that whenever $f(x)$ has a Fourier transform in some $\mathfrak{L}_q$, this Fourier transform will coincide with $\mathfrak{f}(x)$ defined by (4.8). This remains true even when $f(x)$ has a Fourier transform in the sense of various more general definitions.

We may finally state the following geometric interpretation of Theorem 4.2.

Theorem 4.4. *Let $\zeta = F(z)$ map the half-plane $y > 0$ into a domain $\mathfrak{D}$ (not necessarily simply covered) of the ζ-plane, in such a way that the lengths L_{y_0} of the images of the lines $y = y_0$ are bounded. Then the boundary of $\mathfrak{D}$ is a rectifiable curve, of length $L = \lim_{y_0\to 0} L_{y_0}$; the function $F'(z) = f(z) \in \mathfrak{H}_1$, its limit function $f(x) \in \mathfrak{L}_1$ and the Fourier transform $\mathfrak{f}(x)$ of $f(x)$ vanishes for $x < 0$ and is such that*

$$\int_0^{\infty} |\mathfrak{f}(t)|/t\,dt \leqq \left(\frac{\pi}{2}\right)^{1/2} L.$$

The proof of this theorem does not offer any difficulties and may be left to the reader.

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The determination of representative elements in the residual classes of a Boolean algebra.

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1. Introduction.

If A is any abstract ring and $\mathfrak{a}$ is a left- and right-ideal in A, we may consider the problem of selecting a single representative element from each of the residual classes (mod $\mathfrak{a}$) *in such a manner that sums and products of representative elements are themselves representative elements.* If such a selection is possible, the representative elements evidently constitute a subsystem of A which is isomorphic to the quotient-ring $A/\mathfrak{a}$ under the correspondence carrying each residual class (mod $\mathfrak{a}$) into its representative element. In this paper we shall confine our attention to rings in which every element is idempotent — that is, in which the law $aa = a$ obtains. These rings will be seen to have the formal properties of certain algebras of classes, and will therefore be termed *Boolean rings.* A particular case of the representation problem for Boolean rings has previously been discussed by one of us[1]). Here we shall examine the problem on an abstract basis, giving sufficient conditions for the existence of a solution, special cases in which no solution exists, and special cases in which a solution exists if and only if $\aleph_{n+1} = 2^{\aleph_n}$. The sufficient conditions given here can be applied to the particular case mentioned above.

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