We add that the function

$$\sigma^*(x, y) = \operatorname*{Sup}_{m,n} |\sigma_{m,n}(x, y)|$$

satisfies theorems analogous to Theorems 1, 2 and 5. The same may be said of the Abel and (C, α, β) $(\alpha > 0, \beta > 0)$ means.

Corrigenda to the paper „*On the differentiability of multiple integrals*" by A. Zygmund (Fundamenta Mathematicae, vol. 23, p. 143—149).

Prof. Banach kindly called my attention to the fact that the proof of the lemma on p. 145 is incomplete, for the argument on p. 146, line 15, is valid in the case $k = l$ only. The proof may be completed in various ways, and, in particular, as follows.

We have to show, that, given any functions $h = h(x, y)$, $k = k(x, y)$, the expression

$$\mu(u, v) = \int\!\!\int_S L_h(x - u)\, L_k(y - v)\, dx\, dy$$

satisfies an inequality $I_q[\mu] \leqslant A_q$, where A_q depends on q only. In the first place, we observe that, given any function $g = g(x)$, the function $\lambda(u) = \int_0^1 L_g(u - x)\, dx$ belongs to every L^q, and the integral of $\lambda^q(u)$ over $0 \leqslant u \leqslant 1$ does not exceed a constant B_q^q. This is an analogue, for the one-dimensional space, of the result which we have to prove; the proof follows by an argument similar to that of section 4 of the paper. Assuming this, let us consider any of the terms of the sum $\sum_{k,l=1}^{q} \{\prod_{i=1}^{q}{}^{(k)} \ldots \prod_{j=1}^{q}{}^{(l)} \ldots\}$ on p. 146, line 9. Suppose first that $k \neq l$, e. g. $k = 1$, $l = 2$. Integrating first with respect to $x_3, \ldots, x_q, y_3, \ldots, y_q$, and then with respect to x_1, x_2, y_1, y_2, we obtain

$$\int_0^1\!\!\int_0^1 dx_1\, dy_2\, \mu^{q-2}(x_1, y_2) \int_0^1 L_{2h_2}(x_1 - x_2)\, dx_2 \int_0^1 L_{2k_1}(y_2 - y_1)\, dy_1 .$$

Applying Hölder's inequality with the three exponents $q/(q-2), q, q$, we see that the integral does not exceed $I_q^{q-2}[\mu]\, B_q^2$. If $k = l$, e. g. $k = l = 1$, the integral is equal to $\int_0^1\!\int_0^1 \mu^{q-1}(x_1, y_1)\, dx_1\, dy_1 \leqslant I_q^{q-1}[\mu]$. Collecting the terms, we finally obtain

$$I_q^q[\mu] \leqslant C_q \{I_q^{q-1}[\mu] + I_q^{q-2}[\mu]\},$$

where C_q depends on q only. It is plainly sufficient to consider the case when $h(x, y)$ and $k(x, y)$ have a positive minimum. Then $I_q[\mu]$ is finite, and so does not exceed the largest root of the equation $t^q - C_q(t^{q-1} + t^{q-2}) = 0$. This completes the proof.

On the strong derivatives of functions of intervals.

By

S. Saks (Warszawa).

Introduction. Given a set of $2p$ numbers $a_1 \leqslant b_1$, $a_2 \leqslant b_2, \ldots$, $a_p \leqslant b_p$ the set of points $(x_1, x_2, \ldots, x_p)$ such that $a_i \leqslant x_i \leqslant b_i$ for $i = 1, 2, \ldots, p$, will be denoted as the interval $[a_1, b_1; a_2, b_2; \ldots; a_p, b_p]$ of the p-dimensional space $\mathfrak{R}_p$. If $F(I)$ is an addditive function of intervals and I_0 an interval in $\mathfrak{R}_p$, then $\mathrm{V}(F; I_0)$ will denote the total (absolute) variation of F over I_0. If $F(I)$ is a function of intervals of bounded variation then it may be extended as a completely addditive function of sets to the family of all sets measurable [1]) (B); accordingly, in this case, $\mathrm{V}(F; A)$ for any set A measurable (B) will mean the total variation of F over A.

If $(x_1, x_2, \ldots, x_p)$ is a point in the space $\mathfrak{R}_p$ and $F(I)$ a function of intervals, then the lower and upper limits of the quotient $F(I)/\text{meas}\, I$, where I is an arbitrary interval containing $(x_1, x_2, \ldots, x_p)$ and $d(I) \to 0$, will be called the lower and upper strong derivates of $F(I)$ at the point $(x_1, x_2, \ldots, x_p)$, and denoted by $\underline{F}^*(x_1, x_2, \ldots, x_p)$ and $\overline{F}^*(x_1, x_2, \ldots, x_p)$ respectively. In the case when they are equal we shall write $F^*(x_1, x_2, \ldots, x_p)$ for their common value, that will be called the strong derivative [2]) of $F(I)$ at the point considered. $F'(x_1, x_2, \ldots, x_p)$ will as usually denote the derivative of $F(I)$ in the ordinary sense. In the case

[1]) See for instance de la Vallée-Poussin, *Intégrales de Lebesgue, Fonctions d'ensemble, Classes de Baire*, 2e éd., Paris (1934), pp. 88—95; Saks, *Théorie de l'intégrale*, Warszawa (1933), p. 250.

[2]) Some problems concerning the strong derivation of additive functions of intervals have been recently discussed in a series of papers published in these *Fundamenta*; see the list at the end of this note.

$p=1$ both methods of derivation, ordinary and strong, are completely identical. In the case $p\geqslant 2$, however, they essentially differ and the existence of the ordinary derivative does not in general imply that of the strong one. In a former paper [3]) we have set an example of an additive non-negative and absolutely continuous function of intervals $\Phi(I)$ whose upper strong derivate is everywhere equal to ∞. In the first part of this note we attempt to complete this result as follows:

A. *Given an arbitrary function* $\sigma(t)>0$ *in* $[0,\infty]$ *such that* $\liminf\limits_{t\to\infty}\sigma(t)=0$ *there exists in the interval* $S_0=[0,1;0,1;\ldots;0,1]$ *of* $\mathfrak{R}_p$ *a non-negative measurable function* $\varphi(x_1,x_2,\ldots,x_p)$ *such that* $\sigma(|\varphi|)|\varphi|\overset{+}{\log}{}^{p-1}|\varphi|$ *is summable in* S_0 *and that for the indefinite integral* $\Phi(I)$ *of* φ *the equation* $\overline{\Phi}^*(x_1,x_2,\ldots,x_p)=+\infty$ *holds everywhere in* S_0.

In a sense, this is the best possible result. For by a remarkable theorem of Jessen, Marcinkiewicz and Zygmund [4]), if $|\varphi|\overset{+}{\log}{}^{p-1}|\varphi|$ is integrable then the indefinite integral $\Phi(I)$ of φ is strongly derivable almost everywhere and therefore $\Phi^*=\Phi'$ almost everywhere in $\mathfrak{R}_p$. However, from another point of view, it remains yet to decide whether the condition $\overline{\Phi}^*=\infty$ in Theorem A might be replaced by a more complete one, viz. by $\underline{\Phi}^*<\overline{\Phi}^*=\infty$. This may be done rather easily if we merely require the integrability of φ. A more serious difficulty seems to consist in the demand that $\sigma(|\varphi|)|\varphi|\overset{+}{\log}{}^{p-1}|\varphi|$ should be integrable.

In the second part which is independent of the first the following generalization of a theorem of Lusin will be established.

B. *If* $\varphi(x_1,x_2,\ldots,x_p)$ *is a measurable function in the interval* $S_0=[0,1;0,1;\ldots;0,1]$, *then for every* $\varepsilon>0$ *there exists an additive continuous function of intervals* $\Phi(I)$ *such that*

(a) $\Phi^*(x_1,x_2,\ldots,x_n)=\varphi(x_1,x_2,\ldots,x_n)$ *almost everywhere in* S_0,

(b) $$V(\Phi;S_0)\leqslant(1+\varepsilon)\int\ldots\int_{S_0}|\varphi|\,dx_1\ldots dx_p.$$

Condition (b) is immaterial if φ is not summable in S_0. In the case, however, when φ is a summable function, it means that $\Phi(I)$ is of a bounded variation. More precisely, Theorem B may in that case be stated as follows:

B[bis]. *If a function* $\varphi(x_1,x_2,\ldots,x_p)$ *is summable in the interval* $S_0=[0,1;0,1;\ldots;0,1]$ *and* $F(I)$ *denotes the indefinite integral of* φ, *then, for every* $\eta>0$, *there exists an additive continuous and singular* [5]) *function of intervals* $\Delta(I)$, *such that* $V(\Delta;S_0)\leqslant\eta$ *and that the function* $\Phi(I)=F(I)+\Delta(I)$ *satisfies the condition* (a) *of Theorem* A.

Indeed, let V_0 denote the integral of $|\varphi|$ over S_0 and let $\varepsilon=\eta/V_0$ (we may obviously admit $V_0>0$). Let $\Phi(I)$ be a continuous additive function of intervals subject to the conditions (a) and (b) of Theorem A, and let $\Delta(I)=\Phi(I)-F(I)$. As $\Delta'=\Phi'-F'=\Phi^*-\varphi=0$ almost everywhere in S_0, the function Δ is a singular one. Thus there exists a set of measure zero E in S_0 such that [6]) $V(\Delta;S_0)=V(\Delta;E)$. Consequently

$$V(\Phi;S_0)=V(\Phi;E)+V(\Phi;S_0-E)=V(\Delta;E)+V(F;S_0)=$$
$$=V(\Delta;S_0)+V_0;$$

and so, in virtue of (b), we get $V(\Delta;S_0)\leqslant\varepsilon V_0=\eta$, which proves the result.

For the case of one variable ($p=1$), Theorem A was established by Lusin [7]). In that case the condition (b) is naturally omitted as a superfluous one, since in $\mathfrak{R}_1$ the condition (a) is satisfied by the indefinite integral $\Phi(x)$ of the given function $\varphi(x)$ whenever the latter is summable in $[0,1]$. The same remark applies to the case of a space $\mathfrak{R}_p$ for arbitrary p provided that the ordinary derivative is considered instead of the strong one.

For the sake of simplicity we shall consider the case of the plane. The reasonings are actually the same in the general case. Thus, in the sequel, by an interval $[a,b;c,d]$ we shall mean the rectangle whose sides are parallel to the coordinate axes and whose opposite corners, lower left hand and upper right-hand, are

[3]) Saks, **6**; see also Busemann u. Feller, **2**.

[4]) Jessen, Marcinkiewicz and Zygmund, **3**.

[5]) An additive function of intervals is said to be singular if it is of bounded variation and has the derivative (in the ordinary sense) almost everywhere equal to zero.

[6]) Cf. e. g. de la Vallée-Poussin, l. c. [1]), pp. 102—112; Saks, l. c. [1]), pp. 256—257.

[7]) Lusin, *Integral and Trigonometrical Series* (in Russian), Moscow (1915), pp. 34—41; *Sur la notion de l'intégrale, Annali di Mat.* (3), t. 26 (1917), pp. 77—129.

(a, c) and (b, d) respectively. Given a function $F(x, y)$ of two variables, the corresponding function of intervals will be denoted by the same letter; i. e. we shall put $F(I) = F(b, d) - F(a, d) - F(c, d) + F(a, b)$ for $I = [a, b; c, d]$.

I.

1. **Bohr construction.** We shall recall some details of a Bohr[8]) construction upon which the proof of Theorem A actually rests.

Let $S = [a, b; c, d]$ be an interval, $\alpha > 1$ an arbitrary number and $N = \mathrm{E}\alpha$. Consider a finite set of subintervals in S

$$(1.1)\quad I_j^{(1)} = \left[a,\ a + j\cdot\frac{b-a}{N};\quad c,\ c + \frac{d-c}{j}\right] \quad \textit{where } j = 1, 2, \ldots, N.$$

Let us put

$$\Delta^{(1)} = \sum_{j=1}^{N} I_j^{(1)}, \qquad \delta^{(1)} = \prod_{j=1}^{N} I_j^{(1)}.$$

We have

$$\text{meas } I_j^{(1)} = N \text{ meas } \delta^{(1)} \quad \textit{for} \quad j = 1, 2, \ldots, N,$$

$$\text{meas } \Delta^{(1)} = N\left(1 + \frac{1}{2} + \ldots + \frac{1}{N}\right) \text{meas } \delta^{(1)} \geqslant \frac{N \log N}{2} \text{meas } \delta^{(1)}.$$

The remaining part of S, viz. $S - \Delta^{(1)}$, will be divided into a finite system of non-overlapping intervals. To each of them we apply the same operation as above to S. We carry out this process a sufficient number of times till the area of the remainder is less than $(\text{meas } S)/(N+1)^2$. Then we divide the remainder into a finite number of arbitrary non-overlapping intervals, $J^{(1)}, J^{(2)}, \ldots, J^{(r)}$, say. So we obtain a subdivision of S into a set of non-overlapping intervals

$$I_1^{(1)}, \ldots, I_N^{(1)};\quad I_1^{(2)}, \ldots, I_N^{(2)}; \ldots;\quad I_1^{(s)}, \ldots, I_N^{(s)};\quad J^{(1)}, J^{(2)}, \ldots, J^{(r)}$$

such that, upon putting

$$\Delta^{(i)} = \sum_{j=1}^{N} I_j^{(i)}, \qquad \delta^{(i)} = \prod_{j=1}^{N} I_j^{(i)} \quad \textit{for} \quad i = 1, 2, \ldots, s,$$

we have

$$(1.2)\quad \text{meas } I_j^{(i)} = N \text{ meas } \delta^{(i)} \quad \textit{for } i = 1, 2, \ldots, s;\ j = 1, 2, \ldots, N;$$

[8]) See Carathéodory, *Vorlesungen über reelle Funktionen*, 2 Aufl., Leipzig (1927), p. 689 - 691.

$$(1.3)\quad \text{meas } \Delta^{(i)} \geqslant \frac{1}{2} N \log N \text{ meas } \delta^{(i)} \quad \textit{for} \quad i = 1, \ldots, s;$$

$$(1.4)\quad \text{meas } (J^{(1)} + J^{(2)} + \ldots + J^{(r)}) \leqslant \frac{\text{meas } S}{(N+1)^2}.$$

In the above subdivision of S we have started from a set (1.1) of intervals with a common corner at (a, c) and their opposite corners on the curve $(x-a)(y-c) = (b-a)(d-c)/N = (\text{meas } S)/N$. So the total area $\Delta^{(1)}$ of these intervals was approximately equal to

$$\frac{\text{meas } S}{N} \int_{(b-a)/N}^{b-a} \frac{dx}{x} = \frac{\log N}{N} \text{ meas } S.$$

If, however, instead of an interval on the plane we considered an interval $S = [a_1, b_1;\ a_2, b_2; \ldots,\ a_p, b_p]$ in a space $\mathfrak{R}_p$, then we should have to start Bohr's construction from a set of p-dimensional intervals with the common corner $(a_1, a_2, \ldots, a_p)$ and the opposite corners belonging to a $(p-1)$-dimensional variety $(x_1 - a_1)(x_2 - a_2)\ldots(x_p - a_p) = (\text{meas } S)/N^{p-1}$. The total volume of these intervals, as we easily compute, is approximately equal to $\left(\frac{\log N}{N}\right)^{p-1} \cdot \text{meas } S$, while their product is an interval of volume $(\text{meas } S)/N^{p-1}$. Accordingly, passing to a space $\mathfrak{R}_p$ we merely have to replace $N \log N$ in (1.3) by $N \log^{p-1} N$ (with a suitable constant coefficient) and $(N+1)^2$ in (1.4) by $(N+1)^p$. This explains the rôle of the $(p-1)$-st power of $\log|\varphi|$ in the general enunciation of Theorem A.

Now let $\psi(x, y)$ be the function equal to α over the set $\sum_{i=1}^{s} \delta^{(i)} + \sum_{i=1}^{r} J^{(i)}$, and to 0 elsewhere. In virtue of (1.3)

$$\sum_{i=1}^{s} \text{meas } \delta^{(i)} \leqslant \frac{8}{(N+1)\log(N+1)} \sum_{i=1}^{s} \text{meas } \Delta^{(i)} \leqslant \frac{8 \text{ meas } S}{\alpha \log \alpha}.$$

Hence, by (1.4)

$$\int\!\!\int_P \psi \log \psi \, dx\, dy = \alpha \log \alpha \cdot \left(\sum_{i=1}^{s} \text{meas } \delta^{(i)} + \sum_{i=1}^{r} \text{meas } J^{(i)}\right) \leqslant 9 \text{ meas } S.$$

On the other hand, it results from (1.2) and the definition of $\psi(x, y)$ that

$$\int\!\!\int_{I_j^{(i)}} \psi \, dx\, dy = \alpha \text{ meas } \delta^{(i)} \geqslant \text{meas } I_j^{(i)} \quad \textit{where } i = 1, 2, \ldots, s;\ j = 1, 2, \ldots, N;$$

$$\int\!\!\int_{J^{(i)}} \psi \, dx\, dy = \alpha \text{ meas } J^{(i)} \quad \textit{where} \quad i = 1, 2, \ldots, r.$$

From the above discussion there follows

Lemma. *Given an interval S and a number $\alpha > 1$ there exists in S a function $\psi_{S,\alpha}(x,y)$ which enjoys the following properties:*

(A_1) $\psi_{S,\alpha}$ *takes on two values only* 0 *and* α, *each on a finite aggregate of intervals* [9]);

(A_2) $\int\int_P \psi_{S,\alpha}(x,y) \overset{+}{\log} \psi_{S,\alpha}(x,y)\, dx\, dy \leqslant 9 \operatorname{meas} S$;

(A_3) *every point* (x,y) *in* S *belongs to an interval* $I \subset S$ *such that* $\int\int_I \psi_{S,\alpha}(x,y)\, dx\, dy \geqslant \operatorname{meas} I$.

2. **Proof of Theorem A.** Let now $\sigma(t) > 0$ be a function in $[0,1]$ such that $\liminf_{t\to\infty} \sigma(t) = 0$, and let $\{t_i\}$ be an increasing sequence of values of t such that $t_i \to \infty$, $\sigma(t_i) \to 0$. Denote by $\varepsilon(t)$ a continuous, non-decreasing function in $[0,1]$, which coincides with $\sigma(t)$ at the points t_i. We can obviously assume that $\varepsilon(t) < 1$.

We shall define a sequence of non-negative functions $\{\psi_k(x,y)\}$ in the square $S_0 = [0,1;0,1]$ so as to satisfy the following conditions:

(C_1) ψ_k *takes on a finite number of values only, say* $0 = \alpha_0^{(k)} < \alpha_1^{(k)} < \ldots < \alpha_{n_k}^{(k)}$, *each of which over an aggregate of a finite number of intervals;* $\psi_0 \equiv 3$ *identically;*

$$(C_2) \qquad \varepsilon(\alpha_1^{(k)}) < 1/2^k, \quad \alpha_1^{(k)} > \sum_{j=1}^{k-1} \sum_{i=1}^{n_j-1} \alpha_i^{(j)} \quad \textit{for} \quad k \geqslant 1:$$

(C_3) *the sum*

$$\frac{\psi_0(x,y)}{[\varepsilon(\alpha_1^{(0)})]^{1/2}} + \frac{\psi_1(x,y)}{[\varepsilon(\alpha_1^{(1)})]^{1/2}} + \ldots + \frac{\psi_k(x,y)}{[\varepsilon(\alpha_1^{(k)})]^{1/2}}$$

admits only the values which belong to the set $\{t_i\}$;

$$(C_4) \qquad \int\int_{S_0} \psi_k\, dx\, dy \leqslant \int\int_{S_0} \psi_k \overset{+}{\log} \psi_k\, dx\, dy \leqslant 9\,;$$

(C_5) *every point* (x,y) *in* S *belongs to an interval* $I \subset S_0$ *such that* $d(I) < 1/k$ *and* $\int\int_I \psi_k\, dx\, dy \geqslant \operatorname{meas} I$.

[9]) We neglect sets of points of measure zero.

For suppose that ψ_k are defined and satisfy the above conditions for $k \leqslant h-1$. Divide S_0 into a finite number of intervals $I_1, I_2, \ldots, I_{n_h}$ so that the sum $\sum_{j=0}^{h-1} \psi_j(x,y)/[\varepsilon(\alpha_1^{(j)})]^{1/2}$ has a constant value over each of them, say β_n over I_n for $n = 1, 2, \ldots, n_h$. This may obviously be done in virtue of the condition (C_1) which by assumption is satisfied for $k = 1, 2, \ldots, h-1$. We can admit that I_n are of diameters less than $1/h$.

Now a number $\alpha_1^{(h)}$ will be chosen sufficiently large so as to satisfy the condition (C_2) for $k = h$. Since $t/[\varepsilon(t)]^{1/2}$ is continuous and tends with t to $+\infty$, the value of $\alpha_1^{(h)}$ may be determined in such manner that $\beta_1 + \alpha_1^{(h)}/[\varepsilon(\alpha_1^{(h)})]^{1/2}$ belong to the set $\{t_i\}$. Next the numbers $\alpha_n^{(h)}$ for $n = 2, \ldots, n_h$ will be defined so that $\alpha_1^{(h)} < \alpha_2^{(h)} < \ldots < \alpha_{n_h}^{(h)}$ and that the values $\beta_n + \alpha_n^{(h)}/[\varepsilon(\alpha_1^{(h)})]^{1/2}$ all belong to the set $\{t_i\}$. Thus upon putting $\psi_h(x,y) = \psi_{I_n,\alpha_n^{(h)}}(x,y)$ on I_n for $n = 1, 2, \ldots, n_h$, where $\psi_{I_n,\alpha_n^{(h)}}$ is a function satisfying the conditions of the lemma of § 1 with $S = I_n$ and $\alpha = \alpha_n^{(h)}$, it is readily seen that $\psi_h(x,y)$ and $\alpha_1^{(h)}, \alpha_2^{(h)}, \ldots, \alpha_{n_h}^{(h)}$ verify all the conditions (C_1), (C_2), (C_3), (C_4) and (C_5) for $k = h$.

We now turn to the proof of Theorem A. Let us set

$$\varphi(x,y) = \sum_{k=0}^{\infty} \frac{\psi_k(x,y)}{[\varepsilon(\alpha_1^{(k)})]^{1/2}}$$

and let $\varphi_n(x,y)$ denote the nth partial sum of $\varphi(x,y)$. Let

$$R_n = \underset{(x,y)}{E}\, [\varphi_n(x,y) > \varphi_{n-1}(x,y)] = \underset{(x,y)}{E}\, [\psi_n(x,y) \geqslant \alpha_1^{(n)}].$$

It results from (C_1) and (C_2) that over R_n we have

$$\varphi_{n-1}(x,y) \leqslant \sum_{k=0}^{n-1} \frac{\alpha_{n_k}^{(k)}}{[\varepsilon(\alpha_1^{(n)})]^{1/2}} < \frac{\alpha_1^{(n)}}{[\varepsilon(\alpha_1^{(n)})]^{1/2}} \leqslant \frac{\psi_n(x,y)}{[\varepsilon(\alpha_1^{(n)})]^{1/2}};$$

whence $\varphi_n(x,y) \leqslant 2\,\psi_n(x,y)/[\varepsilon(\alpha_1^{(n)})]^{1/2}$ for $(x,y) \,\varepsilon\, R_n$, and in virtue of ($C_4$)

$$(2.1) \quad \int\int_{R_n} \varphi_n \log \varphi_n\, dx\, dy \leqslant \frac{2}{[\varepsilon(\alpha_1^{(n)})]^{1/2}} \Big\{ \int\int_{R_n} \psi_n \log \psi_n\, dx\, dy + \log \frac{2}{[\varepsilon(\alpha_1^{(n)})]^{1/2}} \int\int_{R_n} \psi_n\, dx\, dy \Big\} \leqslant \frac{18}{[\varepsilon(\alpha_1^{(n)})]^{1/2}} \Big(1 + \log \frac{2}{[\varepsilon(\alpha_1^{(n)})]^{1/2}}\Big).$$

Thus, since by (C_3) the values of $\varphi_n(x,y)$ belong to the set $\{t_i\}$ we have $\sigma[\varphi_n(x,y)] = \varepsilon[\varphi_n(x,y)] \leqslant \varepsilon(\alpha_1^{(n)})$ for $(x,y)\,\epsilon\, R_n$, and therefore, it follows from (2.1) that

$$\text{(2.2)} \qquad \iint\limits_{R_n} \varepsilon(\varphi_n)\, \varphi_n \log \varphi_n \, dx\, dy \leqslant \varepsilon_n,$$

where ε_n has been written for $18\, [\varepsilon(\alpha_1^{(n)})]^{1/2} (1 + \log 2/[\varepsilon(\alpha_1^{(n)})]^{1/2})$. In virtue of the first of the relations (C_2) we have

$$\text{(2.3)} \qquad \sum_{n=1}^{\infty} \varepsilon_n < +\infty.$$

Now let $E_n = \underset{(x,y)}{\mathrm{E}} [\varphi(x,y) = \varphi_n(x,y) > \varphi_{n-1}(x,y)]$. As it results from ($C_4$) and ($C_1$) we have $\psi_k(x,y) = 0$ everywhere in S_0 with the exception at most over a set of measure less than $9/\alpha_1^{(k)}$. Hence, since by the second of the relations (C_2) the series $\sum\limits_k \frac{1}{\alpha_1^{(k)}}$ apparently converges, we have $S_0 = E + \sum\limits_n E_n$ where E is a set of measure zero. As $E_n \subset R_n$, from (2.2) and (2.3) we infer thus

$$\text{(2.4)} \qquad \iint\limits_{S_0} \varepsilon(\varphi)\, \varphi \log \varphi \, dx\, dy \leqslant \sum_{n=1}^{\infty} \iint\limits_{E_n} \varepsilon(\varphi_n)\, \varphi_n \log \varphi_n \, dx\, dy \leqslant \sum_{n=1}^{\infty} \varepsilon_n < \infty.$$

On the other hand, it follows from (C_5) that for any k there corresponds to every point (x,y) in S_0 an interval $I \subset S_0$ such that $(x,y)\,\epsilon\, I$, $\mathrm{d}(I) < 1/k$ and

$$\iint\limits_I \varphi \, dx\, dy \geqslant \frac{1}{\varepsilon(\alpha_1^{(k)})} \iint\limits_I \psi_k \, dx\, dy \geqslant \frac{\text{meas}\, I}{\varepsilon(\alpha_1^{(k)})}.$$

Since $\varepsilon(\alpha_1^{(k)}) \to 0$ as $k \to \infty$, this means that the upper strong derivate of the indefinite integral of $\varphi(x,y)$ is ∞ everywhere in S_0. Hence in view of (2.4) the function $\varphi(x,y)$ satisfies both conditions of Theorem A.

II.

1. **Lemmas.** The proof of Theorem B will rest on a few elementary lemmas.

Lemma 1. *Given a linear function $F(x) = px + q$ in an interval $[a,b]$ and positive numbers ε, $\eta < 1$, there exist a monotonic continuous function $\Phi(x)$ in $[a,b]$ and a set P of measure $\eta\cdot(b-a)$ such that*

$$\text{(1.1)} \quad |\Phi(x) - F(x)| < \varepsilon \quad \textit{in} \quad [a,b], \qquad \Phi(a) = F(a), \quad \Phi(b) = F(b);$$

$$\text{(1.2)} \quad \Phi'(x) = 0 \quad \textit{almost everywhere in } [a,b];$$

$$\text{(1.3)} \quad |\Phi(x') - \Phi(x)| \leqslant \frac{4|p|}{\eta} |x'-x| \quad \textit{for} \quad x'\,\epsilon\,[a,b], \quad x\,\epsilon\,[a,b]-P.$$

Proof. Divide $[a,b]$ into n intervals $I_1, I_2, \ldots, I_n$ of equal length $\delta = (b-a)/n < \varepsilon/|p|$. Next, divide each interval $I_k = [a+(k-1)\delta, a+k\delta]$ into five sub-intervals, viz.

$$J_k^{(1)} = \left[a+(k-1)\delta,\ a+\left(k-1+\frac{\eta}{4}\right)\delta\right],$$
$$J_k^{(2)} = \left[a+\left(k-1+\frac{\eta}{4}\right)\delta,\ a+\left(k-1+\frac{\eta}{2}\right)\delta\right],$$
$$J_k = \left[a+\left(k-1+\frac{\eta}{2}\right)\delta,\ a+\left(k-\frac{\eta}{2}\right)\delta\right],$$
$$J_k^{(3)} = \left[a+\left(k-\frac{\eta}{2}\right)\delta,\ a+\left(k-\frac{\eta}{4}\right)\delta\right],$$
$$J_k^{(4)} = \left[a+\left(k-\frac{\eta}{4}\right)\delta,\ a+k\delta\right].$$

The intervals $J_k^{(j)}$ (where $k = 1, 2, \ldots, n$; $j = 1, 2, 3, 4$) are of equal length $\eta\delta/4$, and so, upon denoting their sum by P, we have meas $P = 4n\eta\delta/4 = \eta\cdot(b-a)$.

Now let $\Phi(x)$ denotes a continuous and monotonic function in $[a,b]$ which has the derivative $\Phi'(x)$ almost everywhere equal to 0, coincides with $F(x)$ at the middle- and end-points of the intervals I_k and is constant in the intervals $J_k^{(2)} + J_k + J_k^{(3)}$. Since the oscillation of $\Phi(x) - F(x)$ over each interval I_k is less than, or equal to, $|p|\delta < \varepsilon$ we see at once that the condition (1.1) is satisfied. In order to prove (1.3) we may evidently suppose that both points x' and x belong to the same interval I_k, i. e. that $x \in J_k$ and $x' \in I_k$. Then if $x' \in J_k^{(2)} + J_k + J_k^{(3)}$ we have $\Phi(x') = \Phi(x)$ and the inequality in (1.3) is obvious. If, however, $x' \in J_k^{(1)} + J_k^{(4)}$ then $|x' - x| \geqslant \eta\delta/4$ and, since $|\Phi(x') - \Phi(x) \leqslant |p|\delta$, we again obtain the inequality in (1.3).

Lemma 2. *Let $\psi(x)$ be a measurable function in $[a, b]$ such that $|\psi(x)| \leqslant M < \infty$. Then for any pair of positive numbers $\varepsilon, \eta < 1$ there exist a continuous function $\Psi(x)$ in $[a, b]$ and a set P of measure $\eta \cdot (b - a)$ such that*

$$|\Psi(x)| \leqslant \varepsilon \quad \textit{in} \quad [a, b],$$

$$\Psi'(x) = \psi(x) \quad \textit{almost everywhere in} \quad [a, b],$$

$$|\Psi(x') - \Psi(x)| \leqslant \frac{5M}{\eta}|x' - x| \quad \textit{whenever } x' \in [a, b] \textit{ and } x \in [a, b] - P,$$

$$\mathrm{V}(\Psi; a, b) \leqslant 2\int_a^b |\psi(x)|\, dx.$$

Proof. Let $G(x)$ be an indefinite integral of $\psi(x)$ in $[a, b]$. Divide $[a, b]$ into a finite number of intervals $I_1, I_2, \ldots, I_n$ so that the oscillation of $G(x)$ be less than $\varepsilon/2$ on each of then. Let $F(x)$ denote the function which is linear in each of these intervals and coincides with $G(x)$ at their end-points. The angular coefficient of $F(x)$ in every interval I_k is less than, or equal to M. Hence, by applying the preceding lemma to the function $F(x)$ in each of the intervals I_k, we obtain a set P of measure $\eta\cdot(b-a)$ and a function $\Phi(x)$, continuous in $[a, b]$ and monotonic in each I_k, such that

(1.4) $$|\Phi(x) - F(x)| < \varepsilon/2 \quad \text{in} \quad [a, b], \qquad \mathrm{V}(\Phi; I_k) = \left|\int_{I_k} \psi(x)\, dx\right|,$$

(1.5) $$\Phi'(x) = 0 \quad \textit{almost everywhere in } [a, b],$$

(1.6) $$|\Phi(x') - \Phi(x)| \leqslant \frac{4M}{\eta}|x' - x| \quad \textit{whenever } x' \in [a, b] \textit{ and } x \in [a, b] - P.$$

Let $\Psi(x) = G(x) - \Phi(x)$. Since $|G(x) - F(x)| \leqslant \varepsilon/2$ throughout the interval $[a, b]$, and $|G(x'') - G(x')| \leqslant M \cdot |x'' - x'|$ for any pair of points x', x'' in $[a, b]$, we conclude at once from (1.4), (1.5) and (1.6) that the function $\Psi(x)$ together with the set P satisfy the required conditions of the lemma.

Lemma 3. *Let $f(x, y)$ be a measurable function, E a closed set in the square $S_0 = [0, 1; 0, 1]$ and $\sigma < 1$ a positive number. Then there exist in S_0 an additive continuous function of intervals $F(I)$ and a measurable set Q such that*

(i) $$\text{meas } Q \leqslant \sigma,$$

(ii) $$F^*(x, y) = 0 \quad \textit{at every point } (x, y) \in E,$$

(iii) $$\iint_{S_0 - E} |F^*(x, y) - f(x, y| \, dx\, dy < \sigma^2,$$

(iv) $$|F(I)| \leqslant \sigma \cdot \text{meas } I \quad \textit{whenever} \quad I \cdot (E - Q) \neq 0,$$

(v) $$\mathrm{V}(F; S_0) \leqslant 4 \iint_{S_0 - E} |f(x, y)|\, dx\, dy,$$

(vi) $$|F(I)| < \sigma \quad \textit{for any interval} \quad I \subset S_0.$$

Proof. We can approximate $f(x, y)$ by a function

$$f_0(x, y) = \sum_{k=1}^{n} g_k(x)\, h_k(y)$$

where $h_k(x)$ and $g_k(x)$ are continuous functions in $[0, 1]$, so that [10])

(1.7) $$\iint_{S_0} |f(x,y) - f_0(x,y)|\, dx\, dy < \frac{\sigma^2}{2},$$

(1.8) $$\sum_{k=1}^{n} \iint_{S_0} |g_k(x)\, h_k(y)|\, dx\, dy \leqslant \iint_{S_0} |f(x, y)|\, dx\, dy.$$

On the other hand, an aggregate of non-overlapping intervals $I_1, I_2, \ldots, I_m$ may be determined in $S_0 - E$ so that

(1.9) $$\iint_{S_0 - E - R} |f(x, y)|\, dx\, dy < \frac{\sigma^2}{2} \qquad \textit{where} \quad R = \sum_{k=1}^{m} I_k.$$

[10]) In order to satisfy the condition (1.8) it is convenient to set the function $f(x, y)$ in the form $f = f_1 - f_2$, where $f_1 \geqslant 0$, $f_2 \geqslant 0$ and $f_1 \cdot f_2 = 0$, and then to apply the well-known methods of approximation to f_1 and f_2 separately.

Let M denote the upper bound of the values of $|g_k(x)|$ and $|h_k(x)|$ in $[0, 1]$ and let $l = \mathrm{d}(E, R) > 0$ be the distance of the sets E and R. Then, in virtue of Lemma 2 a system of functions $G_k(x)$, $H_k(x)$, where $k = 1, 2, \ldots, n$, and a set P of linear measure less than $\sigma/2$, may be defined in $[0, 1]$ so as to satisfy the following conditions:

$$(1.10) \qquad |G_k(x)| < \frac{l\sigma^2}{160\, M n^2 m}, \qquad |H_k(x)| < \frac{l\sigma^2}{160\, M n^2 m} \qquad \textit{in } [0, 1];$$

$$(1.11) \qquad G_k'(x) = g_k(x), \quad H_k'(x) = h_k(x) \quad \textit{almost everywhere in } [0, 1];$$

$$(1.12) \qquad \begin{cases} |G_k(x') - G_k(x)| \leqslant \dfrac{20\, M n\, |x' - x|}{\sigma}, \\ |H_k(x') - H_k(x)| \leqslant \dfrac{20\, M n\, |x' - x|}{\sigma}, \\ \textit{whenewer} \quad x' \in [0, 1] \quad \textit{and} \quad x \in [0, 1] - P; \end{cases}$$

$$(1.13) \qquad \mathrm{V}(G_k; 0, 1) \leqslant 2 \int_0^1 |g_k(x)|\, dx, \qquad \mathrm{V}(H_k; 0, 1) \leqslant 2 \int_0^1 |h_k(x)|\, dx.$$

Let us now put

$$(1.14) \qquad \Psi(x, y) = \sum_{k=1}^{n} G_k(x)\, H_k(y), \qquad F(I) = \sum_{k=1}^{m} \Psi(I \cdot I_k),$$

and let Q denote the plane set of points (x, y) whose at least one coordinate belongs to P. It will be shown that the function $F(I)$ and the set Q satisfy the conditions (i—vi).

Indeed, we have $\text{meas}\, Q \leqslant 2\, \text{lin meas}\, P < \sigma$ i. e. the condition (i). Next, by (1.14), the function $F(I)$ vanishes for any interval outside of R, whence $F^*(x, y) = 0$ identically over $S_0 - R$, and in particular over the set E. Further, by (1.14) and (1.11)

$$\Psi^*(x,y) = \sum_{k=1}^{n} G_k'(x)\, H_k'(y) = \sum_{k=1}^{n} g_k(x) h_k(y) = f_0(x,y) \textit{ almost everywhere;}$$

and therefore, on account of (1.14), (1.9) and (1.7), we have

$$\iint_{S_0 - E} |F^*(x, y) - f(x, y)|\, dx\, dy \leqslant \iint_{S_0 - E} |F^*(x, y) - f_0(x, y)\, dx\, dy \,+$$

$$+ \iint_{S_0} |f_0(x, y) - f(x, y)|\, dx\, dy \leqslant \iint_{R} |\Psi^*(x, y) - f_0(x, y)|\, dx\, dy \,+$$
$$+ \iint_{S_0 - E - R} |\Psi^*(x, y)|\, dx\, dy + \iint_{S_0} |f_0(x, y) - f(x, y)|\, dx\, dy < \sigma^2,$$

which means the condition (iii).

In order to prove the condition (iv) consider an arbitrary interval $I = [a, b; c, d]$ such that $I \cdot (E - Q) \neq 0$. The inequality $F(I) \leqslant \sigma \cdot \text{meas}\, I$ is obvious in case $I \cdot R = 0$, for then $F(I) = 0$. Hence suppose that $I \cdot R \neq 0$. Then $\mathrm{d}(I) \geqslant \mathrm{d}(E, R) = l$, and therefore one at least of the numbers $b - a$, $c - d$ is greater than $l/2$. Suppose that $b - a \geqslant l/2$. We may also suppose that one of the corners of I, say the point (a, c), belongs to $E - Q$; for, otherwise, we might divide I into four, or two, intervals so that their common corner should belong to $E - Q$.

Now let $J = [\alpha, \beta; \gamma, \delta]$ be an arbitrary sub-interval of I. As, by assumption the point (a, c) does not belong to Q, its ordinate c certainly does not belong to the set P and, in view of (1.12), we have

$$|H_k(\delta) - H_k(\gamma)| \leqslant |H_k(\delta) - H_k(c)| + |H_k(\gamma) - H_k(c)| \leqslant \frac{40\, M n (d - c)}{\sigma}$$

for $k = 1, 2, \ldots, n$. On the other hand, in virtue of (1.10)

$$|G_k(\beta) - G_k(\alpha)| \leqslant \frac{l\sigma^2}{80\, M n^2 m} \leqslant \frac{\sigma^2 \cdot (b - a)}{40\, M n^2 m}.$$

Hence, for every interval $J = [\alpha, \beta; \gamma, \delta] \subset I$,

$$\Psi(J) \leqslant \sum_{k=1}^{n} |G_k(\beta) - G_k(\alpha)| \cdot |H_k(\delta) - H_k(\gamma)| \leqslant$$
$$\leqslant n \cdot \frac{\sigma^2 \cdot (b - a)}{40\, M n^2 m} \cdot \frac{40\, M n (d - c)}{\sigma} \leqslant \frac{\sigma\, \text{meas}\, I}{m}.$$

Thus $|F(I)| \leqslant \sum_{k=1}^{m} |\Psi(I \cdot I_k)| \leqslant \sigma\, \text{meas}\, I$, and so the condition (iv) is established. Further from (1.14), (1.13) and (1.8) we derive

$$\mathrm{V}(F; S_0) = \mathrm{V}(\Psi; R) \leqslant \mathrm{V}(\Psi; S_0) \leqslant \sum_{k=1}^{n} \mathrm{V}(G_k; 0, 1) \cdot \mathrm{V}(H_k; 0, 1) \leqslant$$
$$\leqslant 4 \sum_{k=1}^{n} \int_0^1 |g_k(x)|\, dx \cdot \int_0^1 |h_k(x)|\, dx \leqslant 4 \iint_{S_0} |f(x, y)|\, dx\, dy$$

which proves the condition (v). Finally, (vi) apparently follows from (1.14) and (1.10).

2. **Proof of Theorem B.** Let $\varphi(x, y)$ be a measurable function in the square $S_0 = [0, 1; 0, 1]$ and $\varepsilon < 1$ a positive number. Let $V_0 = \iint_{S_0} |\varphi(x, y)|\, dx\, dy$. For the sequel we may clearly suppose that $32\varepsilon < V_0$. We shall define a non-decreasing sequence of closed sets $\{E_n\}$, a sequence of mesurable sets $\{Q_n\}$ and a sequence of continuous additive functions of intervals $\{F_n(I)\}$ so as to satisfy the following conditions:

$$(\mathrm{P}_1) \qquad \operatorname{meas}(S_0 - E_n) < \frac{1}{2^{n-1}};$$

$$(\mathrm{P}_2) \qquad \operatorname{meas} Q_n < \frac{1}{2^n};$$

$$(\mathrm{P}_3) \qquad |F_n(I)| < \frac{1}{2^n} \quad \textit{for } n > 1 \textit{ and every interval } I \subset S_0;$$

$$(\mathrm{P}_4) \qquad F_n(I) \leqslant \frac{\operatorname{meas} I}{2^n} \quad \textit{whenever } n > 1 \textit{ and } I \cdot (E_{n-1} - Q_n) \neq 0;$$

$$(\mathrm{P}_5) \qquad \mathrm{V}(F_n; S_0) \leqslant \frac{\varepsilon V_0}{2^n} \quad \textit{for} \quad n > 1; \quad \mathrm{V}(F_1; S_0) \leqslant V_0;$$

$$(\mathrm{P}_6) \qquad \left|\sum_{k=1}^{n} F_k^*(x,y) - \varphi(x,y)\right| < \varepsilon^2 \textit{ on } E_n;\ F_n^*(x,y) = 0 \textit{ on } E_{n-1} \textit{ for } n > 1;$$

$$(\mathrm{P}_7) \qquad \iint_{S_0 - E_n} \left\{ |\varphi(x, y)| + \sum_{k=1}^{n} |F_k^*(x, y)| \right\} dx\, dy \leqslant \frac{\varepsilon V_0}{2^{n+3}}.$$

We proceed by induction. First, let E_0 be a closed set in S_0 such that $\varphi(x,y)$ is continuous over E_0 and that $\iint_{S_0 - E_0} |\varphi|\, dx\, dy \leqslant \varepsilon V_0/33$; let $F_1(I) = \iint_{E_0 \cdot I} \varphi(x, y)\, dx\, dy$. Since φ is bounded over E_0 we have $F_1^*(x, y) = \varphi(x, y)$ at almost every point [11] $(x, y) \in E_0$. Let E_1 be a closed subset of E_0 such that $F_1^*(x, y) = \varphi(x, y)$ e v e r y w h e r e on E_1 and that $\iint_{S_0 - E_1} |\varphi|\, dx\, dy \leqslant \varepsilon V_0/32$. On putting $Q_1 = 0$, we see at once that E_1, $F_1(I)$ and Q_1, as defined above, satisfy the conditions $(\mathrm{P}_1 - \mathrm{P}_7)$ for $n = 1$.

[11]) See for instance Riesz, F., **5**; Busemann u. Feller, **2**, pp. 247—250; Saks, l. c. [1]), p. 232. For more general results see Besicovitch **1**; Ward, **7**.

Now let us suppose that E_n, $F_n(I)$ and Q_n are defined and satisfy these conditions for $n \leqslant p - 1$. Put

$$f_p(x, y) = \varphi(x, y) - \sum_{k=1}^{p-1} F_k^*(x, y).$$

In virtue of (P_7) for $n = p - 1$ we have

$$(2.1) \qquad \iint_{S_0 - E_{p-1}} |f_p(x, y)|\, dx\, dy \leqslant \frac{\varepsilon V_0}{2^{p+2}}.$$

Next, let $\sigma_p < \varepsilon^2/2^p$ be a positive number such that, for any measurable set A in S_0,

$$(2.2) \qquad \iint_{A} \left\{ |\varphi(x, y)| + \sum_{k=1}^{p-1} |F_k^*(x, y)| \right\} dx\, dy \leqslant \frac{\varepsilon V_0}{2^{p+5}} \textit{ if } \operatorname{meas} A \leqslant \sigma_p.$$

Let $Q_p = Q$ be a set and $F_p = F$ an additive continuous function, satisfying the conditions (i—vi) of Lemma 3 with $\sigma = \sigma_p$, $E = E_{p-1}$ and $f(x, y) = f_p(x, y)$. So we have

$$(2.3) \qquad \iint_{S_0 - E_{p-1}} |F_p^*(x, y) - f_p(x, y)|\, dx\, dy < \sigma_p^2,$$

and therefore the inequality

$$(2.4) \qquad \left| \sum_{k=1}^{p} F_k^*(x, y) - \varphi(x, y) \right| = |F_p^*(x, y) - f_p(x, y)| < \sigma_p < \varepsilon^2$$

holds everywhere on $S_0 - E_{p-1}$, except at most on a subset of measure less than σ_p. Let H_p denote a closed subset of $S_0 - E_{p-1}$ such that (2.4) holds on it everywhere and that

$$\operatorname{meas}(S_0 - E_{p-1} - H_p) \leqslant \sigma_p.$$

Hence on putting $E_p = E_{p-1} + H_p$ we have

$$(2.5) \qquad \operatorname{meas}(S_0 - E_p) \leqslant \sigma_p.$$

We shall show that E_p, Q_p and F_p satisfy the required conditions $(\mathrm{P}_1 - \mathrm{P}_7)$ for $n = p$. Indeed, the condition (P_1) is already contained in (2.5). Next, for $\sigma = \sigma_p$, $Q = Q_p$, $f = f_p$ the conditions (i),

(vi) and (iv) of Lemma 3 include the conditions (P_2), (P_3) and (P_4) respectively, while from (v), on account of (2.1), there follows

$$V(F_p; S_0) \leqslant 4 \int\int_{S_0 - E_{p-1}} |f_p(x,y)|\, dx\, dy \leqslant \frac{\varepsilon V_0}{2^p}$$

which is the condition (P_5). Further, the equation $F_p^*(x,y) = 0$ and the inequality (2.4) are everywhere satisfied over the sets E_{p-1} and $H_p = E_p - E_{p-1}$ respectively; hence, as the relations (P_6) hold by assumption for $n = p-1$, they are again true for $n = p$. Finally, by (2.5) and (2.2), we have

$$\int\int_{S_0 - E_p} \left\{ |\varphi(x,y)| + \sum_{k=1}^{p-1} |F_k^*(x,y)| \right\} dx\, dy \leqslant \frac{\varepsilon V_0}{2^{p+5}}; \tag{2.6}$$

whence, in virtue of (2.3) and $\sigma_p^2 < \varepsilon^4/2^{2p} < \varepsilon V_0/2^p \cdot 32$,

$$\int\int_{S_0 - E_p} |F_p^*(x,y)|\, dx\, dy \leqslant \int\int_{S_0 - E_p} |f_p(x,y)|\, dx\, dy + \sigma_p^2 \leqslant \frac{\varepsilon V_0}{2^{p+4}}. \tag{2.7}$$

By adding the inequalities (2.6) and (2.7) we obtain the relation (P_7) for $n = p$.

Let us now put

$$F(I) = \sum_{k=1}^{\infty} F_k(I) \tag{2.8}$$

$$\psi(x,y) = \varphi(x,y) - \sum_{k=1}^{\infty} F^*(x,y), \qquad \Psi(I) = \int\int_I \psi\, dx\, dy,$$

$$\Phi(I) = F(I) + \Psi(I),$$

$$A = \sum_{k=1}^{\infty} E_k - \prod_{n=1}^{\infty} \sum_{k=n}^{\infty} Q_k.$$

In virtue of (P_3) the series (2.8) converges uniformly in S_0, and by (P_5) we have

$$\mathrm{V}(F; S_0) \leqslant \mathrm{V}(F_1; S_0) + \sum_{n=2}^{\infty} \mathrm{V}(F_n; S_0) \leqslant \left(1 + \frac{\varepsilon}{2}\right) \cdot V_0. \tag{2.9}$$

On the other hand, it follows from (P_6) and (P_1) that almost everywhere in S_0 the series $\sum_k F_k^*(x,y)$ converges and $|\psi(x,y)| \leqslant \varepsilon^2$.

Thus

$$\Psi(I) \leqslant \varepsilon^2 \operatorname{meas} I \quad \textit{whence} \quad \mathrm{V}(\Psi; S_0) \leqslant \varepsilon^2 \leqslant \frac{\varepsilon}{2} \cdot V_0, \tag{2.10}$$

and in view of (2.9) we have $\mathrm{V}(\Phi; S_0) \leqslant (1+\varepsilon) \cdot V_0$, which is the relation (b) of Theorem B. Also from the uniform convergence of the series (2.8) and from the first relation (2.10), it follows that $\Phi(I)$ is a continuous additive function of intervals.

It remains to prove that

$$\Phi^*(x,y) = \varphi(x,y) \quad \textit{almost everywhere in } S_0. \tag{2.11}$$

For this purpose, let (x,y) be an arbitrary point in A. Since $\{E_k\}$ is a non-decreasing sequence, there exists a positive integer k_0 such that for $k \geqslant k_0$ the point (x,y) belongs to any set E_k while to no set Q_k. Therefore, in view of (P_4), for every $k \geqslant k_0$ and every interval I which contains (x,y)

$$\left| \frac{F(I)}{\operatorname{meas} I} - \sum_{n=1}^{k} \frac{F_n(I)}{\operatorname{meas} I} \right| \leqslant \left| \sum_{n=k+1}^{\infty} \frac{F_n(I)}{\operatorname{meas} I} \right| \leqslant \frac{1}{2^k};$$

passing to the limit as $\mathrm{d}(I) \to 0$ and $k \to \infty$, we receive thus

$$F^*(x,y) = \sum_{n=1}^{\infty} F_n^*(x,y) = \varphi(x,y) - \psi(x,y) \tag{2.12}$$

for any point (x,y) in A, and so, on account of (P_1) and (P_2), almost everywhere in S_0. On the other hand, as $\psi(x,y)$ is bounded in S_0 (except perhaps on a set of measure zero), we have $\Psi^*(x,y) = \psi(x,y)$ almost everywhere [11]; which together with (2.12) establishes the relation (2.11) and completes the proof of Theorem B.

3. Remarks. To the theorem established in the preceding section a few remarks may be added. It is easily seen that the function $\Phi(x,y)$ subject to the conditions (a) and (b) of Theorem B may be defined so as to satisfy some more conditions of regularity; e. g. it may be supposed to be a sum of two functions $\Phi_1(x,y)$ and $\Phi_2(x,y)$ of which $\Phi_1(x,y)$ is an indefinite integral of a bounded function, and $\Phi_2(x,y)$ has continuous partial derivatives of the first order [12] $\partial\Phi_2/\partial x$, $\partial\Phi_2/\partial y$ everywhere in S_0 and those of the

[12]) It may be even supposed that $\Phi_2(x,y)$ has continuous partial derivatives $\partial^k \Phi_2/\partial x^k$, $\partial^k \Phi_2/\partial y^k$ of all orders.

second order $\partial^2\Phi_2/\partial x\,\partial y = \partial^2\Phi_2/\partial y\,\partial x$ almost everywhere. Then the function $\Phi(x, y)$ almost everywhere possesses the derivatives $\partial\Phi/\partial x$ and $\partial\Phi/\partial y$, as well the derivatives $\partial^2\Phi/\partial x\,\partial y$, $\partial^2\Phi/\partial y\,\partial x$, with the restriction, however, that the latter are to be understood with respect to the set only on which $\partial\Phi/\partial x$ and $\partial\Phi/\partial y$ exist; so the condition (a) of Theorem B may be replaced by

$$(3.1)\quad \Phi^*(x, y) = \frac{\partial^2 \Phi}{\partial x\,\partial y} = \frac{\partial^2 \Phi}{\partial y\,\partial x} = \varphi(x, y) \quad \textit{almost everywhere in } S_0.$$

We shall also observe that, given an arbitrary enumerable set N in S_0, the function Φ may be so constructed that the relation in (3.1) be satisfied, in particular, at every point of the set[13]) N.

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[13]) Cf. Eilenberg et Saks, *Sur la dérivation des fonctions dans des ensembles dénombrables, Fund. Math.*, this vol., p. 264—265.

Über die stetigen Abbildungen der Strecke.

Von

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Bezeichnungen. Sind a, b reelle Zahlen, so bezeichnen wir mit (a, b) das offene, mit $[a, b]$ das abgeschlossene Intervall mit den Endpunkten a, b. R^k bezeichnet den k-dimensionalen cartesischen Raum, $\sigma(P, Q)$ die Entfernung zweier Punkte P, Q aus R^k. Sind A und B zwei Punktmengen, so ist $\sigma(A, B) = \inf \sigma(P, Q)$, wo $P \in A$ und $Q \in B$, $\delta(A)$ der Diameter der Punktmenge $A \subset R^k$; ist $A \subset R^k$, $\lambda > 0$, so bezeichnet $U(A, \lambda)$ die Menge aller Punkte $P \in R^k$, die der Ungleichung $\sigma(P, A) < \lambda$ genügen. C^k bezeichnet die Menge der in $[0, 1]$ definierten und stetigen Funktionen f, die der Bedingung $f(t) \in R^k$ genügen. C^k wird durch die Formel: $\varrho(f, g) = \operatorname*{Max}_{0 \leqq t \leqq 1} \sigma(f(t), g(t))$; $f \in C^k$, $g \in C^k$ metrisiert. Wenn $f \in C^k$, $0 \leqq \alpha \leqq \beta \leqq 1$, so ist $f[\alpha, \beta]$ die Menge der Punte $f(t)$ mit $t \in [\alpha, \beta]$. Wenn $P \in f[0, 1]$, $Q \in f[0, 1]$, so bezeichnet $\sigma_f(P, Q)$ die relative Entfernung von P und Q in Bezug auf $f[0, 1]$[1]).

H. Jarník hat folgenden Satz bewiesen[2]):

Es gibt in C^2 eine Residualmenge W^2, so dass jedes $f \in W^2$ folgende Eigenschaft besitzt: ist $0 \leqq \alpha < \beta \leqq 1$, so ist jeder Punkt der Kurve $f[\alpha, \beta]$ ein irregulärer Punkt dieser Kurve.

H. Knaster hat mir die Vermutung mitgeteilt, dass folgender Satz richtig ist.

Satz. *Es gibt in C^2 eine Residualmenge W^*, so dass für $f \in W^*$ die Kurve $f[0, 1]$ mit der Universalkurve von Sierpiński homöomorph ist.*

[1]) Fund. Math. I, p. 167—169.

[2]) Jarník, Monatsh. f. Math. Phys. 41, p. 408—423, insb. p. 408, Satz 3, 2 und p. 417—423.