We add that the function
\[ \sigma^*(x, y) = \sup_{a, n} |\tau_{a, n}(x, y)| \]
satisfies theorems analogous to Theorems 1, 2 and 5. The same may be said of the Abel and (C, α, β) (α > 0, β > 0) means.

Corrigenda to the paper "On the differentiability of multiple integrals" by A. Zygmund (Fundamenta Mathematicae, vol. 29, p. 145–149).

Prof. Banach kindly called my attention to the fact that the proof of the lemma on p. 146 is incomplete, for the argument on p. 146, line 15, is valid in the case k = 1 only. The proof may be completed in various ways, and, in particular, as follows.

We have to show that, given any functions \(a = a(x, y), b = b(x, y)\), the expression
\[ \mu(u, v) = \int \int L_0(u - u) L_0(v - v) \, du \, dv \]
satisfies an inequality \(L_0[\mu] \leq A_\mu\), where \(A_\mu\) depends on \(g\) only. In the first place, we observe that, given any function \(a = a(x)\), the function \(\lambda(u) = \int L_0(u - u) \, du\) belongs to every \(\mathcal{E}_\mu\) and the integral of \(\lambda(u) \) over \(0 \leq u \leq 1\) does not exceed a constant \(B_\mu\). This is an analogue for the one-dimensional space, of the result which we have to prove; the proof follows by an argument similar to that of section 5 of the paper. Assuming this, let us consider any of the terms of the sum \(\frac{1}{g} \left( \sum_{i=1}^{g} \left( \sum_{j=1}^{g} \ldots \sum_{j=1}^{g} \right) \right)\) on p. 146, line 9. Suppose first that \(k + l\), e.g. \(k = 1, l = 2\). Integrating first with respect to \(x_1, x_2, y_1, y_2, \ldots, y_g\), and then with respect to \(x_1, x_2, y_1, y_2\), we obtain
\[ \int \int \left| \frac{1}{g} \right| \, dx_1 \, dy_1 \, \mu^{r-1}(x_1, y_1) \int L_2(b_1(x_1 - x_2)) \, dx_1 \int L_2(b_1(y_1 - y_2)) \, dy_1. \]

Applying Hölder's inequality with the three exponents \(i/2, i/2, i/2\), we see that the integral does not exceed \(B_\mu^2 \mu^{-1}(\mu)\). If \(k = 1, e.g. k = l = 1\), the integral is equal to \(\frac{1}{g} \int \mu^{r-1}(x_1, y_1) \, dx_1 \, dy_1 \leq \mu^{r-1}(\mu)\). Collecting the terms, we finally obtain
\[ B_\mu^2 \mu^{-1}(\mu) \leq C_\mu \left( B_\mu^{-1}(\mu) + B_\mu^{-2}(\mu) \right), \]
where \(C_\mu\) depends on \(g\) only. It is plain sufficient to consider the case when \(a(x, y)\) and \(b(x, y)\) have a positive minimum. Then \(L_0[\mu]\) is finite, and so does not exceed the largest root of the equation \(i^2 - C_\mu (i-1) + i-2 = 0\). This completes the proof.

On the strong derivatives of functions of intervals.

By S. Saks (Warszawa).

Introduction. Given a set of \(2p\) numbers \(x_1 \leq x_2 \leq \ldots \leq x_{2p}\) and \(x_2 \leq y_2 \leq \ldots \leq y_{2p}\), the set of points \((x_1, x_2, \ldots, x_{2p})\) such that \(a_i \leq x_i \leq b_i\) for \(i = 1, 2, \ldots, p\), will be denoted as the interval \([a_1, b_1; a_2, b_2; \ldots; a_p, b_p]\) of the \(p\)-dimensional space \(\mathbb{R}_p\). If \(F(I)\) is an additive function of intervals and \(L_0\) an interval in \(\mathbb{R}_p\), then \(V(F; L_0)\) will denote the total (absolute) variation of \(F\) over \(L_0\). If \(F^*(I)\) is a function of intervals of bounded variation then it may be extended as a completely additive function of sets to the family of all sets measurable \(^1\) \((B)\); accordingly, in this case, \(V(F; A)\) for any set \(A\) measurable \((B)\) will mean the total variation of \(F\) over \(A\).

If \((x_1, x_2, \ldots, x_p)\) is a point in the space \(\mathbb{R}_p\) and \(F^*(I)\) a function of intervals, then the lower and upper limits of the quotient \(F^*(I)/\text{meas } I\), where \(I\) is an arbitrary interval containing \((x_1, x_2, \ldots, x_p)\) and \(d(I) \to 0\), will be called the lower and upper strong derivatives of \(F^*(I)\) at the point \((x_1, x_2, \ldots, x_p)\), and denoted by \(F_+^*(x_1, x_2, \ldots, x_p)\) and \(F_-^*(x_1, x_2, \ldots, x_p)\) respectively. In the case when they are equal we shall write \(F^*_*(x_1, x_2, \ldots, x_p)\) for their common value, that will be called the strong derivative \(^7\) of \(F^*(I)\) at the point considered. \(F^*_*(x_1, x_2, \ldots, x_p)\) will as usually denote the derivative of \(F^*(I)\) in the ordinary sense.


\(^7\) Some problems concerning the strong derivative of additive functions of intervals have been recently discussed in a series of papers published in these Fundamenta; see the list at the end of this note.
of a bounded variation. More precisely, Theorem B may in that case be stated as follows:

B III. If a function \( \varphi(x_1, x_2, \ldots, x_p) \) is summable in the interval \( S_0 = [0, 1; 0, 1; \ldots; 0, 1] \) and \( F(I) \) denotes the indefinite integral of \( \varphi \), then, for every \( \eta > 0 \), there exists an additive continuous and singular \(^6\) function of intervals \( A(I) \), such that \( V(A; S_0) \leq \eta \) and that the function \( \Phi(I) = F(I) + A(I) \) satisfies the condition (a) of Theorem A.

Indeed, let \( V_\alpha \) denote the integral of \( |\varphi| \) over \( S_0 \), and let \( \varepsilon = \eta / V_\alpha \) (we may obviously admit \( V_\alpha > 0 \)). Let \( \Phi(I) \) be a continuous additive function of intervals subject to the conditions (a) and (b) of Theorem A, and let \( A(I) = F(I) - F(I) \). As \( A' = A' = F' = \Phi \), \( \varphi = 0 \) almost everywhere in \( S_0 \), the function \( A \) is a singular one. Thus there exists a set of measure zero \( E \) in \( S_0 \) such that \( V(A; S_0) = V(A; E) \). Consequently

\[
V(\Phi; S_0) = V(\Phi; E) + V(\Phi; S_0 - E) = V(A; E) + V(F; S_0) = \\
\leq V(A; S_0) + V_\alpha.
\]

and so, in virtue of (b), we get \( V(A; S_0) \leq \varepsilon V_\alpha = \eta \), which proves the result.

For the case of one variable \( (p = 1) \), Theorem A was established by Luzin \(^7\). In that case the condition (b) is naturally omitted as a superfluous one, since in \( \mathbb{R}_1 \) the condition (a) is satisfied by the indefinite integral \( \sigma(x) \) of the given function \( \varphi(x) \) whenever the latter is summable in \([0, 1] \). The same remark applies to the case of a space \( \mathbb{R}_p \) for arbitrary \( p \) provided that the ordinary derivative is considered instead of the strong one.

For the sake of simplicity we shall consider the case of the plane. The reasoning are actually the same in the general case. Thus, in the sequel, by an interval \([a, b; c, d]\) we shall mean the rectangle whose sides are parallel to the coordinate axes and whose opposite corners, lower left hand and upper right-hand, are

---

\(^{6}\) An additive function of intervals is said to be singular if it is of bounded variation and has the derivative (in the ordinary sense) almost everywhere equal to zero.

\(^{7}\) Cf. e.g. de la Vallée-Poussin, L. c. 1), pp. 102—112; Saks, L. c. 2, pp. 256—257.

\(^{8}\) Luzin, Integral and Trigonometrical Series (in Russian), Moscow (1916), pp. 34—41; Sur la notion de l'intégrale, Annales d'Math. (3), t. 20 (1917), pp. 77—129.
(a, e) and (b, d) respectively. Given a function $F(x, y)$ of two variables, the corresponding function of intervals will be denoted by the same letter; i.e., we shall put $F(I) = F([a, b], [c, d]) = F(a, b) + F(a, d) + F(c, d) = F(c, b)$ for $I = [a, b; c, d].$

I.

1. Bohr construction. We shall recall some details of a Bohr construction upon which the proof of Theorem A actually rests.

Let $S = [a, b; c, d]$ be an interval, $a > 1$ an arbitrary number and $N = E_a.$ Consider a finite set of subintervals in $S$

$$I_j^0 = \left[ a + \frac{j}{N}, a + \frac{j}{N} \right] \quad \text{where } j = 1, 2, \ldots, N.$$  

Let us put

$$A^0 = \sum_{i=1}^{N} I_i^0, \quad \Theta^0 = \prod_{i=1}^{N} I_i^0.$$  

We have

$$\text{meas } I_j^0 = N \text{ meas } \Theta^0 \quad \text{for } j = 1, 2, \ldots, N,$$

$$\text{meas } A^0 = N \left(1 + \frac{1}{2} + \frac{1}{N} \right) \text{ meas } \Theta^0 \geq N \log N \geq \frac{N}{2} \text{ meas } \Theta^0.$$  

The remaining part of $S$, viz. $S - A^0$, will be divided into a finite system of non-overlapping intervals. To each of them we apply the same operation as above to $S.$ We carry out this process a sufficient number of times till the area of the remainder is less than $(\text{meas } S)/N^{p-1}.$ Then we divide the remainder into a finite number of arbitrary non-overlapping intervals, $J^0, J^1, \ldots, J^s,$ say. So we obtain a subdivision of $S$ into a set of non-overlapping intervals

$$I_1^0, I_2^0, \ldots, I_N^0; \quad I_1^1, I_2^1, \ldots, I_N^1; \quad \ldots$$

such that, upon putting

$$A^0 = \sum_{i=1}^{N} I_i^0, \quad \Theta^0 = \prod_{i=1}^{N} I_i^0 \quad \text{for } i = 1, 2, \ldots, s,$$

we have

$$\text{meas } I_j^i = N \text{ meas } \Theta^0 \quad \text{for } i = 1, 2, \ldots, s; \quad j = 1, 2, \ldots, N.$$  


Derivatives of functions of intervals 239

(1.3) \quad \text{meas } A^0 \geq \frac{1}{2} N \log N \text{ meas } \Theta^0 \quad \text{for } i = 1, \ldots, s;

(1.4) \quad \text{meas } (J^0 + J^1 + \ldots + J^s) \leq \text{meas } S \left(\frac{N}{N+1}\right)^p.$$

In the above subdivision of $S$ we have started from a set (1.1) of intervals with a common corner at $(a, e)$ and their opposite corners on the curve $(x-a; y-e) = (b-a; d-c) = (a, b; c, d)$ $N = (\text{meas } S)/N.$ So the total area $A^0$ of these intervals was approximately equal to

$$\frac{\text{meas } S}{N} \int_{j=1}^{N} \frac{dx}{x} = \log N = \frac{\text{meas } S}{N}.$$  

If, however, instead of an interval on the plane we considered an interval $S = [a_1, b_1; a_2, b_2; \ldots, a_p, b_p]$ in a space $R^p$, then we should have to start Bohr's construction from a set of $p$-dimensional intervals with the common corner $(a_1, a_2, \ldots, a_p)$ and the opposite corners belonging to a $(p-1)$-dimensional variety $(a_1 - a_1; a_2 - a_2; \ldots; a_p - a_p) = (\text{meas } S)/N^{p-1}.$ The total volume of these intervals, as we easily compute, is approximately equal to $\frac{(\log N)^{p-1}}{N} \cdot \text{meas } S,$ while their product is an interval of volume $(\text{meas } S)/N^{p-1}.$ Accordingly, passing to a space $R^p$ we merely have to replace $N \log N$ in (1.3) by $N \log^{p-1} N$ (with a suitable constant coefficient) and $(N+1)$ in (1.4) by $(N+1)^p.$ This explains the role of the $(p-1)$-st power of $\log |p|$ in the general extension of Theorem A.

Now let $\psi(x, y)$ be the function equal to $\alpha$ over the set $\sum_{i=1}^{s} \Theta_i^0 + \sum_{i=1}^{s} \Theta_i^1,$ and to 0 elsewhere. In virtue of (1.3)

$$\sum_{i=1}^{s} \text{meas } \Theta_i^0 \leq (N+1) \log (N+1) \sum_{i=1}^{s} \text{meas } A_i^0 \leq 8 \text{meas } S \alpha \log \alpha$$

Hence, by (1.4)

$$\int \int \psi \log \psi \ dx \ dy = \alpha \log \alpha \left( \sum_{i=1}^{s} \text{meas } \Theta_i^0 + \sum_{i=1}^{s} \text{meas } J_i^0 \right) \leq 9 \text{meas } S.$$  

On the other hand, it results from (1.2) and the definition of $\psi(x, y)$ that

$$\int \int \psi \ dx \ dy = \alpha \text{ meas } \Theta^0 \geq \text{meas } I_j^0 \quad \text{where } i = 1, 2, \ldots, s; \quad j = 1, 2, \ldots, N;$$

$$\int \int \psi \ dx \ dy = \alpha \text{ meas } J_j^0 \quad \text{where } i = 1, 2, \ldots, r.$$
Derivatives of functions of intervals

For suppose that \( \psi_n \) are defined and satisfy the above conditions for \( k \leq h - 1 \). Divide \( S_n \) into a finite number of intervals \( I_k \), \( \bar{I}_{k-1} \), \( \bar{I}_h \) so that the sum \( \sum I_k \psi_n(x, y) / [e(\alpha^h)]^n \) has a constant value over each of them, say \( C_k \) over \( I_k \) for \( n = 1, 2, \ldots, n_k \). This may obviously be done in virtue of the condition \( (C_k) \) which by assumption is satisfied for \( k = 1, 2, \ldots, h - 1 \). We can admit that \( I_k \) are of diameters less than \( 1/k \).

Now a number \( \alpha^h \) will be chosen sufficiently large so as to satisfy the condition \( (C_k) \) for \( k = h \). Since \( t / [e(\beta^h)]^n \) is continuous and tends with \( t \to +\infty \) the value of \( \alpha^h \) may be determined in such manner that \( \beta_n + \alpha^h / [e(\alpha^h)]^n \) belong to the set \( m \). Next the numbers \( \alpha^h \) for \( n = 1, \ldots, n_0 \) will be defined so that \( \alpha^h < \alpha^h < \ldots < \alpha^h \) and that the values \( \beta_n + \alpha^h / [e(\alpha^h)]^n \) all belong to the set \( m \). Thus upon putting \( \psi_n(x, y) = ||\psi_n (x, y) \) on \( I_k \), for \( n = 1, 2, \ldots, n_k \), \( \psi_n (x, y) \) is a function satisfying the conditions of the lemma of § 1 with \( S = I_k, \) and \( m = m \), it is readily seen that \( \psi_n (x, y) \) and \( \alpha^h, \alpha^h, \ldots, \alpha^h \) verify all the conditions \( (C_k) \), \( (C_k) \), \( (C_k) \), \( (C_k) \) and \( (C_k) \) for \( k = h \).

We now turn to the proof of Theorem A. Let us set

\[
\psi(x, y) = \sum_{h=0}^{\infty} \psi_n(x, y) / [e(\alpha^h)]^n
\]

and let \( \psi_n (x, y) \) denote the \( n \)th partial sum of \( \psi (x, y) \). Let

\[
R_n = E \{ \psi_n (x, y) \} = E \{ \psi_k (x, y) \} \geq \alpha^h \}
\]

It results from \((C_k)\) and \((C_k)\) that over \( R_n \) we have

\[
\psi_n (x, y) \leq \sum_{h=0}^{\infty} \alpha^h / [e(\alpha^h)]^n \leq \alpha^h / [e(\alpha^h)]^n
\]

whence \( \psi_n (x, y) \leq 2 \psi_n (x, y) / [e(\alpha^h)]^n \) for \( (x, y) \in R_n \), and in virtue of \((C_k)\)

\[
\int S_1 \int \psi_1 \log \psi_1 \, dx \, dy \leq \int S_1 \int \psi_1 \log \psi_1 \, dx \, dy +
\]

\[
\int S_1 \int \psi_1 \log \psi_1 \, dx \, dy \leq \int S_1 \int \psi_1 \log \psi_1 \, dx \, dy +
\]

\[
\int S_1 \int \psi_1 \log \psi_1 \, dx \, dy \leq \frac{18}{[e(\alpha^h)]^n} \left( 1 + \log \left( \frac{2}{[e(\alpha^h)]^n} \right) \right).
\]

Fundaemtla Mathematische T. XXV. 16
Derivatives of functions of intervals

II.

1. Lemmas. The proof of Theorem B will rest on a few elementary lemmas.

Lemma 1. Given a linear function \( F(x) = px + q \) in an interval \([a, b]\) and positive numbers \( \epsilon, \eta < 1 \), there exist a monotonic continuous function \( \Phi(x) \) in \([a, b]\) and a set \( P \) of measure \( \eta \cdot (b - a) \) such that

\[
(1.1) \quad |\Phi(x) - F(x)| < \epsilon \quad \text{in} \quad [a, b], \quad \Phi(a) = F(a), \quad \Phi(b) = F(b);
\]

\[
(1.2) \quad \Phi'(x) = 0 \quad \text{almost everywhere in} \quad [a, b];
\]

\[
(1.3) \quad |\Phi'(x) - F'(x)| \leq \frac{4|p|}{\eta} |x' - x| \quad \text{for} \quad x' \in [a, b], \quad x \in [a, b] - P.
\]

Proof: Divide \([a, b]\) into \( n \) intervals \( I_1, I_2, \ldots, I_n \) of equal length \( \delta = (b - a)/n < \epsilon/p \). Next, divide each interval \( I_k = [a + (k - 1) \delta, a + k \delta] \) into five sub-intervals, viz.

\[
J_{k0}^0 = [a + (k - 1) \delta, a + (k - 1 + \eta/4) \delta],
\]

\[
J_{k1}^0 = [a + (k - 1 + \eta/4) \delta, a + (k - 1 + \eta/2) \delta],
\]

\[
J_{k2}^0 = [a + (k - 1 + \eta/2) \delta, a + (k - \eta/2) \delta],
\]

\[
J_{k3}^0 = [a + (k - \eta/2) \delta, a + (k - \eta/4) \delta],
\]

\[
J_{k4}^0 = [a + (k - \eta/4) \delta, a + k \delta].
\]

The intervals \( J_{kj}^0 \) (where \( k = 1, 2, \ldots, n; j = 1, 2, 3, 4 \)) are of equal length \( \eta \delta/4 \), and so upon denoting their sum by \( P \), we have \( \text{meas} \ P = 4n \eta \delta/4 = \eta \cdot (b - a) \).

Now let \( \Phi(x) \) denote a continuous and monotonic function in \([a, b]\) which has the derivative \( \Phi'(x) \) almost everywhere equal to 0, coincides with \( F(x) \) at the middle- and end-points of the intervals \( I_k \) and is constant in the intervals \( J_{k0}^0 + J_{k1}^0 + J_{k2}^0 \). Since the oscillation of \( \Phi(x) - F(x) \) over each interval \( I_k \) is less than, or equal to, \( |p| \delta < \epsilon \) we see at once that the condition (1.1) is satisfied. In order to
prove (1.3) we may evidently suppose that both points \( x' \) and \( x \) belong to the same interval \( I_a \), i.e., that \( x \in I_a \) and \( x' \in I_a \). Then if \( x' \in \mathbb{J}^p + I_a \), we have \( \Phi(x') = \Phi(x) \) and the inequality in (1.3) is obvious. If, however, \( x' \neq \mathbb{J}^p + I_a \) then \( |x' - x| \geq \eta \delta/4 \) and, since \( |\Phi(x') - \Phi(x)| \leq |p| \delta \), we again obtain the inequality in (1.3).

**Lemma 2.** Let \( \psi(x) \) be a measurable function in \([a, b]\) such that \( |\psi(x)| \leq M < \infty \). Then for any pair of positive numbers \( \epsilon, \eta < 1 \) there exists a continuous function \( \tilde{\psi}(x) \) on \([a, b]\) and a set \( P \) of measure \( \eta \cdot (b - a) \) such that

\[
|\tilde{\psi}(x)| \leq \epsilon \quad \text{in} \quad [a, b],
\]

\[
\tilde{\psi}'(x) = \psi(x) \quad \text{almost everywhere in} \quad [a, b],
\]

\[
|\tilde{\psi}(x') - \tilde{\psi}(x)| \leq \frac{5M}{\eta} |x' - x| \quad \text{whenever} \quad x' \in [a, b] \quad \text{and} \quad x \in [a, b] - P,
\]

\[
\nu(\tilde{\psi}; a, b) \leq 2 \int \psi(x) \, dx.
\]

**Proof.** Let \( G(x) \) be an indefinite integral of \( \psi(x) \) in \([a, b]\). Divide \([a, b]\) into a finite number of intervals \( I_1, I_2, \ldots, I_n \) so that the oscillation of \( G(x) \) be less than \( \epsilon/2 \) on each of them. Let \( F(x) \) denote the function which is linear in each of these intervals and coincides with \( G(x) \) at their end-points. The angular coefficient of \( F(x) \) in every interval \( I_a \) is less than, or equal to \( M \). Hence, by applying the preceding lemma to the function \( F(x) \) in each of the intervals \( I_a \), we obtain a set \( P \) of measure \( \eta \cdot (b - a) \) and a function \( \Phi(x) \), continuous in \([a, b]\) and monotonic in each \( I_a \), such that

\[
|\Phi(x) - F(x)| < \epsilon/2 \quad \text{in} \quad [a, b],
\]

\[
\Phi'(x) = 0 \quad \text{almost everywhere in} \quad [a, b],
\]

\[
|\Phi(x') - \Phi(x)| \leq \frac{4M}{\eta} |x' - x| \quad \text{whenever} \quad x' \in [a, b] \quad \text{and} \quad x \in [a, b] - P.
\]

Let \( \tilde{\psi}(x) = G(x) - \Phi(x) \). Since \( |G(x) - F(x)| \leq \epsilon/2 \) throughout the interval \([a, b]\), and \( |\Phi(x') - \Phi(x')| \leq \frac{M}{\eta} |x' - x'| \) for any pair of points \( x', x'' \) in \([a, b]\), we conclude at once from (1.4), (1.5) and (1.6) that the function \( \tilde{\psi}(x) \) together with the set \( P \) satisfy the required conditions of the lemma.

**Derivatives of functions of intervals**

**Lemma 3.** Let \( f(x, y) \) be a measurable function, \( E \) a closed set in the square \( S_0 = [0, 1; 0, 1] \) and \( \sigma < 1 \) a positive number. Then there exist in \( S_0 \) an additive continuous function of intervals \( F(I) \) and a measurable set \( Q \) such that

(i) \[ \text{meas } Q \leq \sigma, \]

(ii) \[ E^*(x, y) = 0 \quad \text{at every point} \quad (x, y) \in E, \]

(iii) \[ \int_{S_0 \setminus E} |F*(x, y) - f(x, y)| \, dx \, dy < \sigma^3, \]

(iv) \[ |F(I)| \leq \sigma \cdot \text{meas } I \quad \text{whenever} \quad I \cap \{E - Q\} = \emptyset, \]

(v) \[ \nu(F; S_0) \leq 4 \int_{S_0 \setminus E} |f(x, y)| \, dx \, dy, \]

(vi) \[ |F(I)| < \sigma \quad \text{for any interval} \quad I \subset S_0. \]

**Proof.** We can approximate \( f(x, y) \) by a function

\[
f_0(x, y) = \sum_{k=1}^{n} g_k(x) h_k(y)
\]

where \( h_k(x) \) and \( g_k(x) \) are continuous functions in \([0, 1]\), so that

\[
\int_{S_0} \int_{S_0} |f(x, y) - f_0(x, y)| \, dx \, dy < \sigma^3.
\]

\[
\sum_{k=1}^{n} \int_{S_0} |g_k(x) h_k(y)| \, dx \, dy < \int_{S_0} \int_{S_0} |f(x, y)| \, dx \, dy.
\]

On the other hand, an aggregate of non-overlapping intervals \( I_1, I_2, \ldots, I_n \) may be determined in \( S_0 \setminus E \) so that

\[
\int_{S_0 \setminus E} |f(x, y)| \, dx \, dy < \frac{\sigma^2}{2}
\]

where \( E = \sum_{k=1}^{n} I_k \).

10) In order to satisfy the condition (1.6) it is convenient to set the function \( f(x, y) \) in the form \( f = f_1 - f_4 \), where \( f_3 \geq 0, f_4 \geq 0 \) and \( f_2 f_4 = 0 \), and then to apply the well-known methods of approximation to \( f_1 \) and \( f_4 \) separately.
Let $M$ denote the upper bound of the values of $|g_k(x)|$ and $|h_k(x)|$ in $[0, 1]$ and let $l = d(E, R) > 0$ be the distance of the sets $E$ and $R$. Then, in virtue of Lemma 2 a system of functions $G_k(x)$, $H_k(x)$, where $k = 1, \ldots, n$, and a set $P$ of linear measure less than $\sigma/2$, may be defined in $[0, 1]$ so as to satisfy the following conditions:

\begin{align}
(1.10) \quad |G_k(x)| & \leq \frac{l \sigma}{160 Mn^2 m} \quad |H_k(x)| \leq \frac{l \sigma}{160 Mn^2 m} \quad \text{in} \quad [0, 1]; \\
(1.11) \quad G_k(x) = g_k(x), \quad H_k(x) = h_k(x) \quad \text{almost everywhere in} \quad [0, 1]; \\
(1.12) \quad |G_k(x') - G_k(x)| \leq \frac{20 Mn \|x' - x\|}{\sigma}, \\
|H_k(x') - H_k(x)| \leq \frac{20 Mn \|x' - x\|}{\sigma}, \\
\text{whenever} \quad x' \in [0, 1] \quad \text{and} \quad x \in [0, 1] - P; \\
(1.13) \quad V(G_k; 0, 1) \leq \frac{1}{2} \int_0^1 |g_k(x)| \, dx, \quad V(H_k; 0, 1) \leq \frac{1}{2} \int_0^1 |h_k(x)| \, dx.
\end{align}

Let us now put

\begin{equation}
\mathcal{F}(x, y) = \sum_{k=1}^n G_k(x) H_k(y), \quad \mathcal{F}(I) = \sum_{k=1}^n \mathcal{F}(I \cdot I_k),
\end{equation}

and let $Q$ denote the plane set of points $(x, y)$ whose at least one coordinate belongs to $P$. It will be shown that the function $\mathcal{F}(I)$ and the set $Q$ satisfy the conditions (i–vi).

Indeed, we have $\text{mes} Q \leq 2 \text{lin mes} P < \sigma$ i.e. the condition (i). Next, by (1.14), the function $\mathcal{F}(I)$ vanishes for any interval outside of $R$, whence $\mathcal{F}^*(x, y) = 0$ identically over $S_1 - R$, and in particular over the set $E$. Further, by (1.14) and (1.11)

\begin{align}
\mathcal{F}^*(x, y) &= \sum_{k=1}^n G_k(x) H_k^*(y) = \sum_{k=1}^n g_k(x) h_k^*(y) = f_k(x, y) \quad \text{almost everywhere;}
\end{align}

and therefore, on account of (1.14), (1.9) and (1.7), we have

\begin{align}
\int_{s_0 - c}^s \int_{s_0 - c}^s |\mathcal{F}^*(x, y) - f(x, y)| \, dy \, dx & \leq \int_{s_0 - c}^s \int_{s_0 - c}^s |\mathcal{F}^*(x, y) - f_k(x, y)| \, dy \, dx + \\
+ \int_{s_0 - c}^s \int_{s_0 - c}^s |f_k(x, y) - f(x, y)| \, dy \, dx & \leq \int_{s_0 - c}^s \int_{s_0 - c}^s |\mathcal{F}^*(x, y) - f_k(x, y)| \, dy \, dx + \\
+ \int_{s_0 - c}^s \int_{s_0 - c}^s |f_k(x, y) - f(x, y)| \, dy \, dx & < \sigma^2,
\end{align}

which means the condition (iii).

In order to prove the condition (iv) consider an arbitrary interval $I = [a, b; c, d]$ such that $I \cdot (E - Q) \neq 0$. The inequality $\mathcal{F}(I) \leq \sigma \text{meas} I$ is obvious in case $I \cdot R = 0$, for then $\mathcal{F}(I) = 0$. Hence suppose that $I \cdot R \neq 0$. Then $\text{d}(I) \geq d(E, R) = l$, and therefore one at least of the numbers $b - a, c - d$ is greater than $l/2$. Suppose that $b - a > l/2$.

We may also suppose that one of the corners of $I$, say the point $(a, c)$, belongs to $E - Q$; for, otherwise, we might divide $I$ into four, or two, intervals so that their common corner should belong to $E - Q$.

Now let $J = [a, \beta; \gamma, \delta]$ be an arbitrary sub-interval of $I$. As by assumption the point $(a, c)$ does not belong to $Q$, its ordinate $c$ certainly does not belong to the set $P$ and, in view of (1.13), we have

\begin{align}
|H_k(\delta) - H_k(\gamma)| & \leq |H_k(\delta) - H_k(c)| + |H_k(\gamma) - H_k(c)| \leq \frac{40 Mn (d - c)}{\sigma} \\
& \text{for} \quad k = 1, 2, \ldots, n.
\end{align}

On the other hand, in virtue of (1.10)

\begin{align}
|G_k(\beta) - G_k(a)| & \leq \frac{l \sigma}{80 Mn^2 m} \leq \frac{\sigma \cdot (b - a)}{40 Mn^2 m},
\end{align}

Hence, for every interval $J = [a, \beta; \gamma, \delta] \subset I$

\begin{align}
\mathcal{F}(J) & \leq \sum_{k=1}^n |G_k(\beta) - G_k(a)| \cdot |H_k(\delta) - H_k(\gamma)| \\
& \leq n \cdot \frac{\sigma \cdot (b - a)}{40 Mn^2 m} \cdot \frac{40 Mn (d - c)}{\sigma} \leq \frac{\sigma \text{meas} I}{m}.
\end{align}

Thus $|\mathcal{F}(I)| \leq \sum |\mathcal{F}(I \cdot I_k)| \leq \sigma \text{meas} I$, and so the condition (iv) is established. Further from (1.14), (1.3) and (1.8) we derive

\begin{align}
\mathcal{V}(\mathcal{F}; S_0) &= \mathcal{V}(\mathcal{F}; R) \leq \mathcal{V}(\mathcal{F}; S_0) = \sum_{k=1}^n \mathcal{V}(G_k; 0, 1) \cdot \mathcal{V}(H_k; 0, 1) \\
& \leq 4 \sum_{k=1}^n \int_{s_0 - c}^s |g_k(x)| \, dx \cdot \int_{s_0 - c}^s |h_k(x)| \, dx \leq 4 \int_{s_0 - c}^s f(x, y) \, dy \, dx
\end{align}
which proves the condition (v). Finally, (vi) apparently follows from (1.14) and (1.10).

2. Proof of Theorem B. Let \( \varphi(x, y) \) be a measurable function in the square \( S_0 = [0, 1] \times [0, 1] \) and \( \varepsilon < 1 \) a positive number. Let \( V_0 = \int \int_{S_0} |\varphi(x, y)| \, dx \, dy \). For the sequel we may clearly suppose that \( 32 \varepsilon < V_0 \). We shall define a non-decreasing sequence of closed sets \( \{ E_n \} \), a sequence of measurable sets \( \{ Q_n \} \) and a sequence of continuous additive functions of intervals \( \{ F_n(I) \} \) so as to satisfy the following conditions:

\[
(P_1) \quad \text{meas } (S_0 - E_n) < \frac{1}{2^n},
\]

\[
(P_2) \quad \text{meas } Q_n < \frac{1}{2^n},
\]

\[
(P_3) \quad |F_n(I)| < \frac{1}{2^n} \quad \text{for } n > 1 \text{ and every interval } I \subset S_0;
\]

\[
(P_4) \quad F_n(I) \leq \frac{\text{meas } I}{2^n} \quad \text{whenever } n > 1 \text{ and } I \cap (E_{n-1} - Q_n) \neq \emptyset;
\]

\[
(P_5) \quad V(F_s; S_0) \leq \frac{e V_0}{2^n} \quad \text{for } n > 1; \quad V(F_s; S_0) \leq V_0;
\]

\[
(P_6) \quad \sum_{k=1}^{n} |F_k^*(x, y)| - |\varphi(x, y)| < e^2 \text{ on } E_n; \quad F_n^*(x, y) = 0 \text{ on } E_{n-1} \text{ for } n > 1;
\]

\[
(P_7) \quad \int_{S_0} \int_{S_0} \left| \varphi(x, y) \right| + \sum_{k=1}^{n} |F_k^*(x, y)| \, dx \, dy \leq \frac{e V_0}{2^{n+5}}.
\]

We proceed by induction. First, let \( E_0 \) be a closed set in \( S_0 \) such that \( \varphi(x, y) \) is continuous over \( E_0 \) and that \( \int \int_{S_0} |\varphi(x, y)| \, dx \, dy \leq e V_0 / 32 \); let \( F_1(I) = \int \int_{I} |\varphi(x, y)| \, dx \, dy \). Since \( \varphi \) is bounded over \( E_0 \) we have \( F_1^*(x, y) = |\varphi(x, y)| \) at almost every point \(^{11)} \) \((x, y) \in E_0 \). Let \( E_1 \) be a closed subset of \( E_0 \) such that \( F_1^*(x, y) = \varphi(x, y) \) everywhere on \( E_1 \) and that \( \int \int_{S_0} |\varphi(x, y)| \, dx \, dy \leq e V_0 / 32 \). On putting \( Q_1 = 0 \), we see at once that \( E_1, F_1(I) \) and \( Q_1 \), as defined above, satisfy the conditions \( (P_1 - P_7) \) for \( n = 1 \).

\(^{11)} \) See for instance Risses, F., 5; Busemann, F., Feller, 2, pp. 247—260; Saks, I., 1, p. 282. For more general results see Besicovitch 1; Ward, 7.

**Derivatives of functions of intervals**

Now let us suppose that \( E_n, F_n(I) \) and \( Q_n \) are defined and satisfy these conditions for \( n \leq p - 1 \). Put

\[
f_p(x, y) = \varphi(x, y) - \sum_{k=1}^{p-1} F_k^*(x, y).
\]

In virtue of \((P_7)\) for \( n = p - 1 \) we have

\[
\int \int_{S_0} |f_p(x, y)| \, dx \, dy \leq \frac{e V_0}{2^{p+5}}.
\]

Next, let \( \sigma_p < e^2 / 2^n \) be a positive number such that, for any measurable set \( A \) in \( S_0 \),

\[
\int \int_{A} \left| \varphi(x, y) \right| + \sum_{k=1}^{p-1} |F_k^*(x, y)| \, dx \, dy \leq \frac{e V_0}{2^{p+6}} \text{ if } \text{meas } A \leq \sigma_p.
\]

Let \( Q_p = Q \) be a set and \( F_p = F \) an additive continuous function, satisfying the conditions (i—vi) of Lemma 3 with \( \sigma = \sigma_p \), \( E = E_{p-1} \) and \( f(x, y) = f_p(x, y) \). So we have

\[
\int \int_{S_0} |F_p^*(x, y) - f_p(x, y)| \, dx \, dy < \sigma_p,
\]

and therefore the inequality

\[
\left| \sum_{k=1}^{p} F_k^*(x, y) - \varphi(x, y) \right| = |F_p^*(x, y) - f_p(x, y)| < \sigma_p < e^2
\]

holds everywhere on \( S_0 - E_{p-1} \), except at most on a subset of measure less than \( \sigma_p \). Let \( H_p \) denote a closed subset of \( S_0 - E_{p-1} \) such that (2.4) holds on it everywhere and that

\[
\text{meas } (S_0 - E_{p-1} - H_p) \leq \sigma_p.
\]

Hence on putting \( E_p = E_{p-1} + H_p \) we have

\[
\text{meas } (S_0 - E_p) \leq \sigma_p.
\]

We shall show that \( E_{p+1}, Q_p, F_p \) satisfy the required conditions \( (P_1 - P_7) \) for \( n = p \). Indeed, the condition \( (P_1) \) is already contained in (2.5). Next, for \( \sigma = \sigma_p \), \( Q = Q_p, f = f_p \) the conditions (i),
Derivatives of functions of intervals

Thus

\[(2.10) \quad \mathcal{V}(I) \leq \varepsilon \cdot \text{meas } I \quad \text{whence} \quad \mathcal{V}(\mathcal{S}_n; S_0) \leq \varepsilon \cdot \frac{\varepsilon}{2}, \quad V_0, \]

and in view of (2.9) we have \(\mathcal{V} (\mathcal{S}_n; S_0) \leq (1 - \varepsilon) \cdot V_0\), which is the relation (b) of Theorem B. Also from the uniform convergence of the series (2.8) and from the first relation (2.10), it follows that \(\mathcal{D}(I)\) is a continuous additive function of intervals.

It remains to prove that

\[(2.11) \quad \mathcal{D}(x, y) = \varphi(x, y) \quad \text{almost everywhere in } S_0.\]

For this purpose, let \((x, y)\) be an arbitrary point in \(A\). Since \(\mathcal{E}_n\) is a non-decreasing sequence, there exists a positive integer \(k_0\) such that for \(k \geq k_0\) the point \((x, y)\) belongs to any set \(\mathcal{E}_k\) while to no set \(\mathcal{Q}_k\). Therefore, in view of (P_4) for every \(k \geq k_0\) and every interval \(I\) which contains \((x, y)\)

\[
\left| \sum_{k=1}^{\infty} \mathcal{F}_k(I) \right| \leq \sum_{k=1}^{\infty} \frac{\mathcal{F}_k(I)}{\text{meas } I} \leq \frac{1}{2^{k_0}}.
\]

passing to the limit as \(d(I) \to 0\) and \(k \to \infty\), we receive thus

\[(2.12) \quad \mathcal{F}(x, y) = \sum_{k=1}^{\infty} \mathcal{F}_k(x, y) = \varphi(x, y) - \psi(x, y)\]

for any point \((x, y)\) in \(A\), and \(s_0\), on account of (P_4) and (P_3), almost everywhere in \(S_0\). On the other hand, as \(\psi(x, y)\) is bounded in \(S_0\) (except perhaps on a set of measure zero), we have \(\mathcal{D}(x, y) = \varphi(x, y)\) almost everywhere \(1)\); which together with (2.2) establishes the relation (2.11) and completes the proof of Theorem B.

3. Remarks. To the theorem established in the preceding section a few remarks may be added. It is easily seen that the function \(\mathcal{D}(x, y)\) subject to the conditions (a) and (b) of Theorem B may be defined so as to satisfy some more conditions of regularity; e. g. it may be supposed to be a sum of two functions \(\mathcal{D}_1(x, y)\) and \(\mathcal{D}_2(x, y)\) of which \(\mathcal{D}_1(x, y)\) is an indefinite integral of a bounded function, and \(\mathcal{D}_2(x, y)\) has continuous partial derivatives of the first order \(1)\). \(\partial \mathcal{D}_1/\partial x, \partial \mathcal{D}_2/\partial y\) everywhere in \(S_0\) and those of the

\(^1\) It may be even supposed that \(\mathcal{D}_2(x, y)\) has continuous partial derivatives \(\partial \mathcal{D}_1/\partial x, \partial \mathcal{D}_2/\partial y\) of all orders.
Über die stetigen Abbildungen der Strecke.
Von
Stefan Mazurkiewicz (Warszawa).

Bezeichnungen. Sind $a, b$ reelle Zahlen, so bezeichnen wir mit $[a, b]$ das offene, mit $[a, b]$ das abgeschlossene Intervall mit den Endpunkten $a, b$. $R^k$ bezeichnet den $k$-dimensionalen kartesischen Raum, $\sigma(P, Q)$ die Entfernung zweier Punkte $P, Q$ aus $R^k$. Sind $A$ und $B$ zwei Punkmenge, so ist $\sigma(A, B) = \inf \sigma(P, Q)$, wo $P \in A$ und $Q \in B$. $d(A)$ ist der Diameter der Punkmenge $A \subseteq R^k$; ist $A \subseteq R^k$, $\lambda > 0$, so bezeichnet $U(A, \lambda)$ die Menge aller Punkte $P \in R^k$, die der Ungleichung $\sigma(P, A) < \lambda$ genügen. $C^k$ bezeichnet die Menge der in $[0, 1]$ definierten und stetigen Funktionen $f$, die der Bedingung $f(0) = f(1)$ genügen. $C^k$ wird durch die Formel: $\max |f(t)|$, $t \in [0, 1]$, so ist $f(\alpha, \beta)$ die Menge der Punkte $f(t)$ mit $t \in [\alpha, \beta]$. Wenn $P \in [0, 1]$, $Q \in [0, 1]$, so bezeichnet $\sigma(P, Q)$ die relative Entfernung von $P$ und $Q$ in Bezug auf $[0, 1]$.

H. Jarnik hat folgenden Satz bewiesen:

Es gibt in $C^k$ eine Residualmenge $W^*$, so dass jedes $f \in W^*$ folgende Eigenschaft besitzt: ist $0 \leq \alpha < \beta \leq 1$, so ist jeder Punkt der Kurve $f(\alpha, \beta)$ ein irregulärer Punkt dieser Kurve.

H. Knaster hat mir die Vermutung mitgeteilt, dass folgender Satz richtig ist.

Satz. Es gibt in $C^k$ eine Residualmenge $W^*$, so dass für $f \in W^*$ die Kurve $f(0, 1)$ mit der Universalkurve von Sierpiński homöomorph ist.


References.