As we have already noted in Theorem 1 of C. P. B., a compact continuum which cuts E_n is the common boundary of two domains D and H. If for every $\varepsilon > 0$ there exist ε-transformations of C into subsets of both D and H that do not meet C, we shall say, for brevity, that C is *two-way free*. Concerning such continua we may state:

Theorem 4a. *In E_n, where n is odd and > 1, let C be a compact continuum which cuts E_n, is two-way free, and is locally i-connected for $0 \leqq i \leqq (n-3)/2$ $(=j)$. Then if $p^{j+1}(C)$ is finite, C is a g. c. $(n-1)$-m.*

Theorem 4b. *In E_n, where n is even and > 2, let C be a compact continuum which cuts E_n, is two-way free, and is locally i-connected for $0 \leqq i \leqq (n-2)/2$. Then C is a g. c. $(n-1)$-m.*

Theorems 4a and 4b are proved by proceeding, as in the proof of Theorem 1, to obtain local connectedness properties of *both* the complementary domains, and by applying the results stated in the reference given in footnote [11]).

In C. B. P. it was shown [17]) that for $n = 2, 3$, a compact continuum which cuts E_n and is continuously deformable without meeting itself is a closed $(n-1)$-manifold. This result is contained in the following theorem:

Theorem 5. *In E_n, a compact continuum which cuts E_n and is continuously deformable without meeting itself is a g. c. $(n-1)$-m.*

Proof. Denoting the continuum by C, and the complementary domains of which it is the common boundary (Theorem 1, C. P. B.) by D and H, we may suppose the deformation of C to take place in H. If D is not uniformly locally i-connected for an i such that $0 \leqq i \leqq n-2$, there exist an $\varepsilon > 0$, a point A of C, and a sequence of i-cycles γ_k^i of D whose diameters converge to zero and have the point A as topological limit, and each of which links C in $S(A, \varepsilon)$. By practically the same argument as given for the case $i = 0$ in the fourth and fifth paragraphs of the proof of Theorem 5 in C. P. B., we may show a contradiction. The theorem then follows from Principal Theorem C of G. C. M.

[17]) P. 165, Corollary, and Theorem 7.

On differentiation of Lebesgue double integrals.

By

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The theorem proved in this note gives an answer to a problem put forward by S. Saks [1]).

Denote by $r(u, v)$ a rectangle (and its area) with sides parallel to the coordinate axes and containing the point (u, v). Given a function $f(x, y)$ summable in a domain D denote by $\overline{D}_{u,v} \iint f(x, y)\,dx\,dy$, $\underline{D}_{u,v} \iint \ldots$, $D_{u,v} \iint \ldots$ the upper limit, the lower limit and the limit (if it exists) of the ratio

$$\frac{1}{r(u,v)} \iint\limits_{r(u,v)} f(x, y)\,dx\,dy, \quad \text{as} \quad d\,r(u, v) \to 0$$

where $d\,r(u, v)$ is the diameter of $r(u, v)$.

It is well known that for a bounded function $f(x, y)$

$$D_{u,v} \iint f(x, y)\,dx\,dy$$

exists and is equal to $f(u, v)$ at almost all points of the domain. In the case of an unbounded function this is true only with an additional condition that the ratio of the larger side of $r(u, v)$ to the smaller side remains bounded. If this restriction is not imposed then the inequality

$$\overline{D}_{u,v} \iint f(x, y)\,dx\,dy > \underline{D}_{u,v} \iint f(x, y)\,dx\,dy$$

may hold on a set of positive measure. Saks' problem is: *May both terms of this inequality be finite on a set of positive measure?*

[1]) S. Saks. *Remark on the differentiability of the Lebesgue indefinite integral.* Fund. Math. T. XXII, pp. 257—261.

To solve this problem I must first prove two lemmas.

Lemma 1. *A function $f(x, y)$, summable in a rectangle R (whose area is denoted also by R) with sides parallel to the coordinate axes, satisfies one of the two conditions,*

$$\textit{either}\quad f(x,y)=0, \quad \textit{or} \quad |f(x,y)|>A$$

and

$$\iint_R f(x,y)\,dx\,dy = BR$$

where $A>B>0$. Then there exists a set of non-overlapping rectangles with sides parallel to the coordinate axes, of total area $\geqq \frac{B}{A}R$, in each of which the mean value of $f(x,y)$ is equal to A.

Introduce the operation $O=O(R)$ on the rectangle R, in the following way. Divide the larger side of R (or any side in the case when R is a square) into equal parts of length $\geqq \frac{1}{4}$ and $<\frac{1}{2}$ of the smaller side and through the points of division draw lines parallel to the smaller side. If the mean value of $f(x,y)$ on each of the partial rectangles is $<A$ then the operation O is finished. If, on the other hand, the mean value of $f(x,y)$ on some of the partial rectangles is $\geqq A$, then we spread the sides of such rectangles parallel to the smaller side of R until all these rectangles are included in rectangles R_1 on each of which the mean value of $f(x,y)$ is equal to A. Rectangles R_1 are defined so that none of them can be enlarged. Denote by R_1' the aggregate of the partial rectangles of R or of the parts of these rectangles which do not belong to R_1 so that $R=R_1+R_1'$. The formation of the sets R_1 and R_1' is the operation O in the general case. Obviously the mean value of $f(x,y)$ on each of the rectangles of R_1' is $<A$.

Applying now the operation O to each rectangle of R_1' we shall again arrive at two aggregates R_2, R_2' of rectangles such that $R_1'=R_2+R_2'$ and that mean value of $f(x,y)$ on each rectangle of R_2 is equal to A and on each rectangle of R_2' is $<A$. Applying the operation O on each rectangle of R_2' and so on, we shall arrive at the set $U=R_1+R_2+R_3+\dots$ of non-overlapping rectangles on each of which the mean value of $f(x,y)$ is equal to A. Consider

now rectangles $r(u,v)$ for which the ratio of the smaller side to the larger is $\geqq \frac{1}{4}$. At almost all points of R

$$(1)\qquad \lim \frac{1}{r(u,v)}\iint_{r(u,v)} f(x,y)\,dx\,dy = f(u,v), \quad \text{as} \quad d\,r(u,v)\to 0.$$

Any such point for which $f(x,y)>0$, and *a fortiori* $>A$, belongs to U. For otherwise it must belong to each of the sets $R_1', R_2', \dots$ and consequently it must belong to a sequence of rectangles $r(u,v)$, decreasing to zero, on each of which the mean value of $f(x,y)$ is $<A$, i. e.

$$\frac{1}{r(u,v)}\iint_{r(u,v)} f(x,y)\,dx\,dy < A,$$

which is impossible by (1).

Thus at almost all points of $R-U$

$$f(x,y)\leqq 0,$$

and consequently

$$\iint_U f(x,y)\,dx\,dy \geqq BR$$

On the other hand, the mean value of $f(x,y)$ on any rectangle of U is equal to A. Consequently

$$\iint_U f(x,y)\,dx\,dy = A\cdot m\,U$$

i. e.

$$m\,U \geqq \frac{B}{A} R,$$

which proves the lemma.

Lemma 2. *For a plane set U and a set $G=\Sigma r$ of rectangles r with sides parallel to the coordinate axes the following condition is satisfied:*

the mean density of U on every rectangle r is greater than α, i. e.

$$m(U\times r)>\alpha\cdot m r;$$

then

$$(2)\qquad m\,U > \frac{1}{416}\alpha^2\cdot m\,G.$$

We write $G = G_1 + G_1'$, where G_1 is the set of those rectangles of G, in which the vertical side is greater than, or equal to, the horizontal side. One of the two inequalities

$$m\, G_1 \geqq \frac{1}{2} m\, G, \qquad m\, G_1' \geqq \frac{1}{2} m\, G$$

is true. Suppose it is the first one

(3) $$m\, G_1 \geqq \frac{1}{2} m\, G.$$

For every rectangle of G_1 define the integer n by the condition

(4) $$\frac{d}{n} \geqq c, \qquad \frac{d}{n+1} < c$$

where c and d $(c \leqq d)$ are the sides of the rectangle; divide the side d into n equal parts and correspondingly divide the rectangle into n equal rectangles by lines parallel to the side c through the points of division of d. It is easy to see that the ratio of the larger side to the smaller side in every partial rectangle is < 2. Take a positive number $\beta < \alpha$. Define a lower bound for the number k of partial rectangles on each of which the mean density of U is $> \beta$. We have obviously

$$k + (n-k)\,\beta \geqq n\,\alpha$$

i. e.

$$k \geqq \frac{\alpha - \beta}{1 - \beta}\, n$$

Putting $\beta = \frac{1}{2}\alpha$ we shall have

(5) $$k \geqq \frac{\frac{1}{2}\alpha}{1 - \frac{1}{2}\alpha}\, n > \frac{1}{2}\alpha\, n.$$

Thus the total sum of vertical sides of the partial rectangles on which the mean density of U is $> \frac{1}{2}\alpha$, is $> \frac{1}{2}\alpha\, d$.

Denote by G_2 the set of all the partial rectangles with the mean density of U greater than $\frac{1}{2}\alpha$ corresponding to all rectangles of G_1, and define $m\, G_2$. Denote by l_ξ, λ_ξ the intersection of the line $x = \xi$

respectively with the sets G_1 and G_2 and let a, b be the extreme values of x for the set G_1 so that

(6) $$m\, G_1 = \int_a^b l_x\, dx, \qquad m\, G_2 = \int_a^b \lambda_x\, dx.$$

Obviously there exists a set of rectangles of G_1 covering the set l_x and such that no three of them have points in common [1]). Then we can choose a subset of non-overlapping ones whose intersection with l_x is $\geqq \frac{1}{2} l_x$, i. e. the sum of vertical sides of the rectangles of the subset is $\geqq \frac{1}{2} l_x$, and consequently we conclude from (5) that the sum of the vertical sides of the rectangles of G_2 corresponding to the rectangles of the subset is $> \frac{1}{4}\alpha\, l_x$, i. e.

$$\lambda_x > \frac{1}{4}\alpha\, l_x;$$

and consequently by (6) and (3)

(7) $$m\, G_2 > \frac{1}{4}\alpha\, m\, G_1 \geqslant \frac{1}{8}\alpha\, m\, G.$$

Take now any rectangle $ABCD$ for which the ratio of the larger side to the smaller is < 2 and let $A'B'C'D'$ be a similar rectangle with the same centre, and with sides parallel to those of $ABCD$ and 5 times greater than these. It is easy to see that no rectangle with sides parallel to those of $ABCD$ and of area less than, or equal to, that of $ABCD$ and with the same condition about the ratio of its sides, can have points in common with $ABCD$ and at the same time with the exterior of $A'B'C'D'$. From this remark it follows by the well known Vitali argument that the set G_2 has a subset G_3 of non-overlapping rectangles such that

(8) $$m\, G_3 > \frac{1}{26} m\, G_2.$$

[1]) We can clearly suppose that the set G of rectangles is finite.

We write now

$$m\,U \geqq m(U \times G_8) > \frac{\alpha}{2}\, m\, G_8$$

and by (8), (7)

$$m\,U > \frac{1}{416}\,\alpha^2\, m\, G,$$

which proves the lemma.

Theorem. *For any summable function $f(x, y)$ in a domain Δ there are at almost all points of the domain four possibilities with respect to the differentiation of $\iint f(x, y)\,dx\,dy$*

(i) $D_{u,v}\iint f(x, y)\,dx\,dy$ *exists*

(ii) $\overline{D}_{u,v}\iint f(x,y)\,dx\,dy = +\infty, \quad \underline{D}_{u,v}\iint f(x,y)\,dx\,dy = f(u,v)$

(iii) $\overline{D}_{u,v}\iint f(x,y)\,dx\,dy = f(u,v), \quad \underline{D}_{u,v}\iint f(x,y)\,dx\,dy = -\infty$

(iv) $\overline{D}_{u,v}\iint f(x,y)\,dx\,dy = +\infty, \quad \underline{D}_{u,v}\iint f(x,y)\,dx\,dy = -\infty.$

From the fact that at almost all points

$$\lim \frac{1}{r(u,v)} \iint_{r(u,v)} f(x, y)\,dx\,dy = f(u, v)$$

as $dr(u, v) \to 0$ under the condition that the ratio of the larger side to the smaller one remains bounded, it follows that at almost all points of Δ

$$\overline{D}_{u,v}\iint f\,dx\,dy \geqq f(u,v) \geqq \underline{D}_{u,v}\iint f\,dx\,dy,$$

and thus the theorem will be proved if we prove that each of the two sets

(9) $\quad \underset{u,v}{E}\{f(u,v) < \overline{D}_{u,v} < +\infty\}, \quad \underset{u,v}{E}\{-\infty < \underline{D}_{u,v} < f(u,v)\}$

is of measure zero. Suppose that this is not true and let the first of these sets have a positive measure. Then there exist positive numbers c, M and $N > 2M$ such that the set

$$E_1 = \underset{u,v}{E}\left\{\begin{matrix} f(u,v) + c < \overline{D}_{u,v} < f(u,v) + M \\ -N < f(u,v) < +N \end{matrix}\right\}$$

also has a positive measure. Write now

$$f(x, y) = f_1(x, y) + f_2(x, y),$$

where

$$\begin{aligned} f_1(x,y) &= f(x,y) \qquad \text{if } |f(x,y)| > N \\ &= 0 \qquad\qquad \text{otherwise.}\end{aligned}$$

Thus $f_2(x, y) = 0$ at all points of E_1.

Now at almost all points

$$\overline{D}_{u,v}\iint f(x,y)\,dx\,dy = \overline{D}_{u,v}\iint f_1(x,y)\,dx\,dy + f_2(u,v).$$

Consequently at almost all points of E_1

$$\overline{D}_{u,v}\iint f(x,y)\,dx\,dy = \overline{D}_{u,v}\iint f_1(x,y)\,dx\,dy + f(u,v).$$

Denote the set of these points by E_2. We have

$$m\,E_2 = m\,E_1, \quad E_2 = \underset{u,v}{E}\left\{\begin{matrix} c < \overline{D}_{u,v}\iint f_1\,dx\,dy < M \\ f_1 = 0 \end{matrix}\right\}.$$

Given a positive number δ denote by E_3 the subset of the points (u, v) of E_2 at which

$$\iint_{r(u,v)} f_1(x,y)\,dx\,dy < M \cdot m\,r(u,v)$$

if only $dr(u,v) < \delta$. Taking δ small enough we shall have $m E_3 > 0$. Include now the set E_3 in an open set H such that

(10) $$m\,H - m\,E_3 < \frac{1}{1664}\,\frac{c^2}{M^2} \cdot m\,E_3.$$

Each point (u, v) of E_3 can be included in a rectangle $r(u, v) \subset H$ of diameter $< \delta$ and on which the mean value of $f_1(x, y)$ is $> c$. (It will, of course, be $< M$). Denote the set of points formed by all these rectangles by G. By Lemma 1 on each rectangle of G we can construct a set of non-overlapping rectangles with mean value of $f_1(x, y)$ equal to $2M$ and of total area $> \frac{c}{2M}$ times the area of the rectangle. Denote by U the set formed by all such rectangles corresponding to all rectangles of G. We have

the mean density of U on every rectangle of G is $> \frac{c}{2M}$.

Therefore by Lemma 2

$$m\,U > \frac{1}{416}\;\frac{c^2}{4\,M^2}\,m\,G$$
$$> \frac{1}{1664}\,\frac{c^2}{M^2}\,m\,E_3.$$

As U is included in H we conclude from (10) that

$$m\,(E_3 \times U) > 0$$

which is impossible since by the definition of E_3 no point of E_3 can belong to a rectangle of diameter $<\delta$ on which mean value of $f_1(x, y)$ is $\geqq M$. Thus the first of the sets (9) cannot have a positive measure. Similarly it can be proved that the second of them is also of measure zero, and in this way the proof of the theorem is completed.

Note on the differentiability of multiple integrals.

By

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§ 1.

Let $f(x_1, x_2, \ldots, x_k) = f(P)$ be an L-integrable function defined in the cell

(S) $$0 \leqslant x_i \leqslant 1 \qquad (1 \leqslant i \leqslant k).$$

We shall say that the integral of the function f is *strongly differentiable* at the point P_0, if

(1) $$\lim_{\delta(I)\to 0} \frac{1}{|I|} \int_I f(P)\,dP$$

exists and is finite; here I denotes any cell with sides parallel to the axes, contained in S, and containing P_0; $|I|$ denotes the measure, and $\delta(I)$ the diameter of I. The limit (1) will be called the *strong derivative* of the integral of f at the point P_0.

The following results have recently been established [1]).

Theorem A. *There is a function $f(P) \in L$ such that its integral is nowhere strongly differentiable.*

[1]) Theorem A was proved by S. Saks, *Théorie de l'intégrale*, Warszawa, 1933, pp. 1—288, esp. p. 232, *Fund. Math.* 22 (1934), 257—261, and independently by Busemann and Feller, *Fund. Math.* 22 (1934), 226—256. Theorem B, for bounded functions, was proved by Saks, *Théorie de l'intégrale,* p. 232, Busemann and Feller, *loc. cit.* F. Riesz, *Fund. Math.* 22, p. 221—226, and, for functions of the class L^p, $p>1$, by A. Zygmund, *Fund. Math.* 23 (1934), 143—149 (see also Corrigenda at the end of this paper).