On free subsets of E_n.

By

R. L. Wilder (Ann Arbor, U. S. A).

In an earlier paper [1]) I have given an affirmative answer, for the cases $n = 2, 3$, to the following question of Mr. K. Borsuk [2]): Is every free, locally contractible [3]), compact continuum which cuts E_n an $(n-1)$-dimensional manifold? More precisely, in E_2 every free, locally connected compact continuum M which cuts E_2 is a simple closed curve (1-manifold), and in E_3 every such continuum for which, moreover, the Betti number $p^1(M) = 0$, is a closed 2-dimensional manifold. In the present paper these results are extended to higher dimensions, using the notion of generalized closed n-manifold [4]). In addition Theorem 7 of C. P. B. is extended in a similar way, and one or two special types of transformations are considered.

[1]) *Concerning a problem of K. Borsuk*, Fund. Math. 21 (1933), pp. 156—167 (to be referred to hereafter as C. P. B.).

[2]) Fund. Math. 20 (1933), p. 285, Prob. 54. Borsuk calls a subset M of E_n *free* in this space if for every $\varepsilon > 0$ there exists an ε-transformation of M into a set M' such that $M \cdot M' = 0$; see K. Borsuk, *Über die Fundamentalgruppe...*, Monatsh. f. Math. u. Phys. 41 (1934), pp. 64—77.

[3]) „lokal zusammenziehbar".

[4]) As introduced in my paper *Generalized closed manifolds in n-space*, Annals of Math. 35 (1934), pp. 876—903 (hereafter referred to as G. C. M.), a generalized closed n-manifold (= g. c. n-m.) is a compact metric space M of dimension n such that 1) $p^n(M) = 1$ and for any proper closed subset F of M, $p_n(F) = 0$; 2) for $1 \leqq i \leqq n-1$, there exists $\eta > 0$ such that if γ^i is a cycle of M of diameter $< \eta$, then $\gamma^i \sim 0$ in M; 3) if P is a point of M and ε a positive number, there exist δ and η, $\varepsilon > \delta > \eta > 0$, such that if γ^i $(0 \leqq i \leqq n-2)$ is a cycle of $F(P, \delta)$, then $\gamma^i \sim 0$ on $S(P, \varepsilon) - S(P, \eta)$; and if γ^{n-1} is a cycle of $F(P, \delta)$, then $\gamma^{n-1} \sim 0$ on $M - S(P, \eta)$. In E_n, for $n = 2, 3$, the g. c. $(n-1)$-m. is the ordinary $(n-1)$-manifold.

Theorem 1. *In E_n $(n > 2)$, let C be a free, compact continuum which cuts E_n, is locally i-connected [5]) for $0 \leqq i \leqq n-3$, and for which $p^1(C)$ is finite. Then C is a g. c. $(n-1)$-m.*

Proof. As shown in Theorem 1 of C. P. B., C is a common boundary of two domains D and H such that $D + H + C = E_n$. Since C is connected, a continuous transformation of C yields a connected point set; consequently, if C' is a transform of C such that $C \cdot C' = 0$, then C' must lie wholly in D or in H. As C is free, we can assume that for arbitrary $\varepsilon > 0$, there is an ε-transform of C in H.

Given $\varepsilon > 0$, let $\delta > 0$ be such that if P is a point of C, then any i-cycle $(1 \leqq i \leqq n-3)$ of $C \cdot S(P, \delta)$ bounds on $C \cdot S(P, \varepsilon/2)$. Consider a cycle γ^i of $D \cdot S(P, \delta)$, considering P henceforth as a fixed point. By the Lemma of C. P. B., there exists $\eta > 0$ such that if γ^i links C in $S(P, \varepsilon)$ and C' is any η-transform of C, then γ^i links C' in $S(P, 3\varepsilon/4)$. We suppose that γ^i links C in $S(P, \varepsilon)$.

Let K^{i+1} be any $(i+1)$-chain bounded by γ^i in $S(P, \delta)$. There is no loss of generality in supposing $|\gamma^i|$ [6]) connected, and it will be convenient to do so. Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k, \ldots$ be a monotonic decreasing sequence of positive numbers converging to zero, where $\varepsilon_1 < \varrho(|\gamma^i|, C)$, and let K^{i+1} be subdivided so that its cells are all of diameter $< \varepsilon_1$. The set of those closed $(i+1)$-cells on K^{i+1} that do not meet C form a complex L_1. The component of $|L_1|$ which contains $|\gamma^i|$ forms the basis for a chain $K_1^{i+1} \to \gamma^i - i_1$.

We now make a subdivision of K^{i+1} of mesh $< \varepsilon_2$ and $< \varrho(|K_1^{i+1}|, C)$. The set of those closed $(i+1)$-cells of this subdivision that do not meet C form a complex $L_2 \supset L_1$. The component of $|L_2|$ which contains $|\gamma^i|$ forms the basis for a chain $K_2^{i+1} \to \gamma^i - i_2$. We have

$$K_2^{i+1} - K_1^{i+1} \to i_1 - i_2.$$

Proceeding in this manner, we obtain for each positive integer k a chain K_k^{i+1} and a cycle i_k such that

$$|K_{k+1}^{i+1}| \supset |K_k^{i+1}|, \qquad K_k^{i+1} \to \gamma^i - i_k,$$
$$K_{k+1}^{i+1} - K_k^{i+1} \to i_k - i_{k+1}.$$

[5]) In the sense employed here (see P. Alexandroff, Annals of Math. 30, p. 181, footnote), a metric space is locally i-connected at a point P if for arbitrary $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that any cycle $\gamma_i \subset S(P, \delta_\varepsilon)$ bounds on $S(P, \varepsilon)$; the space is locally i-connected if it is locally i-connected at every point. For $i = 0$, this notion coincides with the local connectedness of point set theory.

[6]) If L denotes either a chain or a complex, we denote the set of points on L by $|L|$.

For each k, there can be assigned a positive number η_k such that all vertices (0-cells) of $K_{k+1}^{l+1} - K_k^{l+1}$ are within a distance η_k of C and such that $\lim_{k\to\infty} \eta_k = 0$ [7]). Consequently the cycles i_k and chains $K_{k+1}^{l+1} - K_k^{l+1}$ may be considered as forming „infinitesimal alterations" [8]) of cycles and chains on C; in particular, then, the cycles i_k form infinitesimal alterations of the elements of a convergent Vietoris cycle $\Gamma^l = (c_1, c_2, \dots, c_k, \dots)$ on $C \cdot S(P, \delta)$. By hypothesis, Γ^l bounds a chain Q^{l+1} on $C \cdot S(P, \varepsilon/2)$.

Let $\varrho(C, C') = \theta$, where C' is an η-transform of C as defined previously. Denoting the elements of Q^{l+1} by q_k, where $q_k \to c_k$, we choose an integer m such that the cells of both i_m and c_m are of diameter $< \theta/3$, the distance from any vertex of i_m to its correspond in c_m is $< \theta/3$, and the cells of q_m form a $\theta/3$-complex. Let $\bar{q}_m$ denote the complex consisting of the cells on q_m, with, however, the vertices of c_m replaced by their corresponds in i_m. Based on a geometrical realization of the complex $\bar{q}_m$, we obtain a chain $F^{l+1} \to i_m$. No cell of F^{l+1} meets C', and $|F^{l+1}| \subset S(P, 3\varepsilon/4)$. From the relations

$$K_m^{l+1} \to \gamma^l - i_m \qquad \text{in } S(P, 3\varepsilon/4) - C',$$

$$F^{l+1} \to i_m \qquad \text{in } S(P, 3\varepsilon/4) - C',$$

we get

$$K_m^{l+1} + F_m^{l+1} \to \gamma^l \qquad \text{in } S(P, 3\varepsilon/4) - C'.$$

But η was chosen so that γ^l links C' in $S(P, 3\varepsilon/4)$. Thus the supposition that γ^l links C in $S(P, \varepsilon)$ leads to a contradiction, and it follows that D is uniformly locally i-connected, for $1 \leqq i \leqq n-3$.

By Theorem 2 of C. P. B., both domains D and H are uniformly locally 0-connected. Since $p^1(C)$ is finite, we know from well-known duality relations that $p^{n-2}(E_n - C)$ is finite. That C is therefore a g. c. $(n-1)$-m follows at once from Theorem 15 of G. C. M.

Remarks. As we shall note presently (see the Lemma below), when we are dealing with chains and cycles mod $m \geqq 2$, the assumption in Theorem 1 that $p^1(C)$ is finite becomes unnecessary in case $n \geqq 4$, since in this case the finiteness of $p^1(C)$ is a consequence of the local 1-connectedness of C. However, it is clear from the above proof that we can replace the assumption concerning

[7]) This follows easily, for instance, from the fact that the set of points on any complex is locally 0-connected.

[8]) See P. Alexandroff, Annals of Math. 30, p. 181.

$p^1(C)$ by the assumption that C is locally $(n-2)$-connected, in which case the conclusion of the theorem follows from Principal Theorem C of G. C. M. Since a locally contractible continuum is locally i-connected for all dimensions i [9]) we have the

Corollary. *A free, locally contractible continuum which cuts E_n is a g. c. $(n-1)$-manifold.*

Since every absolute neighborhood retract is locally contractible [10]), a similar corollary may be stated for the class of such sets.

If we assume, in the hypothesis of Theorem 1, that $p^{n-2}(C)$ is finite, and delete the condition that $p^1(C)$ be finite, we shall then have as a consequence that $p^1(E_n - C)$ and hence $p^1(H)$ is finite. This fact, together with the other local connectedness properties of D and H proved above, are sufficient to make C a g. c. $(n-1)$-m. [11]). Thus we can state

Theorem 1a. *The result of Theorem 1 remains true if the condition that $p^1(C)$ be finite is replaced by the condition that $p^{n-2}(C)$ be finite.*

For the case where $n \geqq 4$, and only chains mod $m \geqq 2$ are employed, we may obtain a much stronger theorem that Theorem 1. In order to establish this, we need the following Lemma which I have established in another (as yet unpublished) paper:

Lemma. *Let M be a locally compact metric space which, in terms of chains mod $m \geqq 2$, is locally i-connected for $0 \leqq i \leqq k$ (where k is a non-negative integer). Then, if F is a self-compact subset of M and U is an open subset of M containing F, at most a finite number of i-cycles of F are independent with respect to homologies in U* [12]).

Theorem 2. *In E_n $(n > 2)$, let C be a free, compact continuum which cuts E_n and which, in terms of chains mod $m \geqq 2$, is locally i-connected for $0 \leqq i \leqq (n-1)/2$. Then M is a g. c. $(n-1)$-m.*

Proof. By virtue of Principal Theorem C of G. C. M., we need only show that if P is a point of C and ε a positive number, there

[9]) See K. Borsuk, *Zur kombinatorischen Eigenschaften der Retrakte*, Fund. Math. 21 (1933), pp. 91—98, Hilfssatz.

[10]) K. Borsuk, *Über eine Klasse von lokal zusammenhängenden Räumen*, Fund. Math. 19 (1932), pp. 220—242, § 27.

[11]) See Bull. Amer. Math. Soc. 40 (1934), p. 805, abstract no. 354.

[12]) When $F \equiv M$, the finiteness of the numbers $p^i(M)$ follows, a fact that we referred to in connection with $p^1(C)$ in the above Remarks.

exists $\delta > 0$ such that any γ^i in $D \cdot S(P, \delta)$ bounds in $D \cdot S(P, \varepsilon)$, where $0 \leqq i \leqq n-2$ and D is as defined in the proof of Theorem 1. That such a δ exists for values of i such that $0 \leqq i \leqq (n-1)/2$ follows from the proof of Theorem 1. Hence in what follows we consider a fixed value of i such that $(n-1)/2 < i \leqq n-2$.

Let η be any positive number $< \varepsilon$ and let $\eta_1, \eta_2, \eta_3, \ldots$ be a sequence of positive numbers such that $\varepsilon > \eta_1 > \eta_2 > \eta_3 > \ldots > \eta$. Let $\theta > 0$ be such that any $(n-i-1)$-cycle of $C \cdot S(P, 2\theta)$ bounds on $C \cdot S(P, \eta)$. Let $\delta_1 < \theta$. Suppose that no matter what $\delta > 0$ is selected, there is in $D \cdot S(P, \delta)$ a γ^i that fails to bound in $D \cdot S(P, \varepsilon)$. Then there exists such a γ^i in $D \cdot S(P, \delta_1)$ that we shall denote by γ_1^i. As γ_1^i links C in $S(P, \varepsilon)$, there exists $\beta_1 > 0$ and a β_1-transformation T_1 such that $T_1(C) \subset H$, and γ_1^i links $T_1(C)$ in $S(P, \eta_1)$. Let δ_2 be a positive number such that $T_1(C) \cdot S(P, \delta_2) = 0$. In $D \cdot S(P, \delta_2)$ there is a γ_2^i that links C in $S(P, \varepsilon)$. Then γ_1^i and γ_2^i are independent with respect to homologies in $S(P, \eta_1) - C \cdot S(P, \eta_1)$. For suppose we had

$$K_1^{i+1} \to c_1 \gamma_1^i + c_2 \gamma_2^i \quad \text{in} \quad S(P, \eta_1) - C \cdot S(P, \eta_1).$$

If K_2^{i+1} is the maximal portion of K_1^{i+1} forming a chain of H, then clearly $K_2^{i+1} \to 0$. Then

$$1) \; K^{i+1} = K_1^{i+1} - K_2^{i+1} \to c_1 \gamma_1^i + c_2 \gamma_2^i \quad \text{in} \quad D \cdot S(P, \eta_1) \subset S(P, \eta_1) - T_1(C) \cdot S(P, \eta_1).$$

However, there exists a chain L^{i+1} such that

$$2) \; L^{i+1} \to c_2 \gamma_2^i \qquad \text{in} \quad S(P, \delta_2) \subset S(P, \eta_1) - T_1(C) \cdot S(P, \eta_1).$$

From relations 1) and 2) we get

$$K^{i+1} - L^{i+1} \to c_1 \gamma_1^i \quad \text{in} \quad S(P, \eta_1) - T_1(C) \cdot S(P, \eta_1),$$

which contradicts the fact that γ_1^i links $T_1(C)$ in $S(P, \eta_1)$.

Having determined, by the process just indicated, cycles $\gamma_1^i, \ldots, \gamma_k^i$ that are linearly independent with respect to homologies in $D \cdot S(P, \eta_{k-1})$, we observe that since there are only a finite number of possible linear combinations of the γ's (mod m), there is a $\beta_k > 0$ such that if $T_k(C)$ is a β_k-transformation of C into a subset of H, every such linear combination links $T_k(C)$ in $S(P, \eta_k)$. Let $\delta_{k+1} > 0$ be such that $T_k(C) \cdot S(P, \delta_{k+1}) = 0$, and in $D \cdot S(P, \delta_{k+1})$ let γ_{k+1}^i be a cycle linking C in $S(P, \varepsilon)$. Then $\gamma_1^i, \ldots, \gamma_k^i, \gamma_{k+1}^i$ are linearly independent

with respect to homologies in $S(P, \eta_k) - C \cdot S(P, \eta_k)$. For if not, we would have by reasoning as above, a chain K^{i+1} such that

$$3) \; K^{i+1} \to c_1 \gamma_1^i + \ldots + c_k \gamma_k^i + c_{k+1} \gamma_{k+1}^i \text{ in } S(P, \eta_k) - T_k(C) \cdot S(P, \eta_k).$$

However, we have

$$4) \; L^{i+1} \to c_{k+1} \gamma_{k+1}^i \quad \text{in} \quad S(P, \delta_{k+1}) \subset S(P, \eta_k) - T_k(C) \cdot S(P, \eta_k),$$

and from relations 3) and 4) we get

$$K^{i+1} - L^{i+1} \to c_1 \gamma_1^i + \ldots + c_k \gamma_k^i \quad \text{in} \quad S(P, \eta_k) - T_k(C) \cdot S(P, \eta_k),$$

contradicting the fact that every linear combination of the cycles $\gamma_1^i, \ldots, \gamma_k^i$ links $T_k(C)$ in $S(P, \eta_k)$.

We thus show the existence of an infinite set of such cycles γ_k^i every linear combination of which links C is $S(P, \eta)$. As the γ's are therefore linearly independent with respect to homologies in $E_n - [C \cdot S(P, \eta) + F(P, \eta)]$, they are uniquely linked with a set of cycles z_k^{n-i-1} of $C \cdot S(P, \eta) + F(P, \eta)$.

As a Vietoris cycle, let $z_k^{n-i-1} = (z_{k,1}, z_{k,2}, \ldots, z_{k,m}, \ldots)$, where $z_{k,m}$ is an ε_m-cycle, $\lim_{m \to \infty} \varepsilon_m = 0$. Let $h_{k,m}$ denote the portion of $z_{k,m}$ whose abstract $(n-i-1)$-cells lie wholly in $\overline{S(P, \theta)}$. Then $K_{k,m}$, where $h_{k,m} \to K_{k,m}$, may, without loss of generality, be assumed to lie on $C \cdot F(P, \theta)$ [13]), and since the sequence $K_{k,1}, K_{k,2}, \ldots, K_{k,m}, \ldots$ contains a convergent subsequence forming a Vietoris cycle Γ_k^{n-i-2} of $C \cdot F(P, \theta)$, we can assume for the sake of brevity, that Γ_k^{n-i-2} is the former sequence itself [14]).

Since there are infinitely many cycles Γ_k^{n-i-2}, there exists, by virtue of the above Lemma, a relation

$$B^{n-i-1} \to c_1 \Gamma_{s_1}^{n-i-2} + \ldots + c_r \Gamma_{s_r}^{n-i-2} \quad \text{in} \quad C \cdot [S(P, 2\theta) - S(P, \delta_1)],$$

where the elements of the chain B^{n-i-1} are $b_1, b_2, \ldots, b_m, \ldots$ and $b_m \to c_1 K_{s_1,m} + \ldots + c_r K_{s_r,m}$. We assume that from the sequence of cycles $c_1 h_{s_1,m} + \ldots + c_r h_{s_r,m} - b_m$ there has been chosen a convergent

[13]) If z_k^{n-i-1} lay wholly in $\overline{S(P, \theta)}$, it would bound on $C \cdot S(P, \eta)$.

[14]) Actually, we should choose the convergent subsequence $\Gamma_k^{n-i-2} = (K_{k,n_1}, K_{k,n_2}, \ldots, K_{k,n_m}, \ldots,)$, and henceforth replace z_k^{n-i-1} by the homologous cycle formed by the sequence $(z_{k,n_1}, z_{k,n_2}, \ldots, z_{k,n_m}, \ldots)$. And when other convergent sequences are selected hereafter, it will usually mean that the cycles just treated will have to be replaced again by subsequences to harmonize with the new cycles.

subsequence forming a Vietoris cycle v^{n-i-1} and that the subscripts here and in preceding sequences have been reassigned to accord with our notation. Similarly, if $z_{k,m} - h_{k,m} = g_{k,m}$, we obtain a Vietoris cycle w^{n-i-1} from the sequence $c_1 g_{s_1,m} + \ldots + c_r g_{s_r,m} + b_m$, with the necessary reassignments of subscripts as before.

Since

$$c_1 z_{s_1}^{n-i-1} + \ldots + c_r z_{s_r}^{n-i-1} = v^{n-i-1} + w^{n-i-1}\ {}^{15}),$$

and since every linear combination of the z's is linked with a linear combination of the γ's, it follows that $v^{n-i-1} + w^{n-i-1}$ must be linked with a linear combination of the γ's. But $|v^{n-i-1}| \subset S(P, 2\theta)$, and therefore $v^{n-i-1} \sim 0$ on $C \cdot S(P, \eta)$. Thus w^{n-i-1} must be linked with a linear combination of the γ's. However, $|w^{n-i-1}| \subset E_n - S(P, \delta_1)$, whereas the γ's all lie in $S(P, \delta_1)$; consequently w^{n-i-1} cannot be linked with a linear combination of the γ's. This contradiction completes the proof.

We turn now to the case where the set C under consideration is what we might call *free in the strong sense;* i. e., the case where for every $\varepsilon > 0$ there exists an ε-transformation of C into a set C' that is *homeomorphic* with C and such that $C \cdot C' = 0$. Concerning such sets we prove the following theorem, reserving until after the proof such remarks as we wish to make concerning relationships with preceding theorems.

Theorem 3. *In E_n $(n > 2)$, let C be a compact continuum which cuts E_n, is free in the strong sense, and is locally i-connected for $0 \leqq i \leqq j$, where $j = (n-3)/2$ or $(n-2)/2$ according as n is odd or even. Then if $p^{j+1}(C)$ is finite, C is a g. c. $(n-1)$-m.*

Proof. As in the proof of Theorem 1. we have C a common boundary of uniformly locally 0-connected domains D and H where for every $\varepsilon > 0$ there exists an ε-transformation of C into a subset of H that we can now assume is a homeomorph of C. That D is uniformly locally i-connected for $1 \leqq i \leqq j$ is proved as in Theorem 1. We can prove that H is uniformly locally i-connected for the same values of i by proceeding as follows: Given an arbitrary $\varepsilon > 0$, let $\delta > 0$ be such that an i-cycle of C of diameter $< \delta$ bounds a chain of C of diameter $< \varepsilon/2$. There exists $\eta > 0$ such that if C' is a homeomorph of C (in H) which is the result of an η-transformation of C, then any i-cycle of C' of diameter $< \delta$ bounds a chain of C' of diameter $< \varepsilon$ (by elementary continuity considerations). Now let P be a point of C and γ^i a cycle of $H \cdot S(P, \delta)$. Without loss of generality we may assume that $|\gamma^i|$ is a connected point set, and it is easily shown [16]) that there exists a positive number θ such that a θ-transformation of C into a subset of H yields a set which separates $|\gamma^i|$ from C; let C' denote the result of such a transformation. By methods similar to those used in the proof of Theorem 1, we may show the existence of chains K_m^{i+1}, F_m^{i+1} (with, however, the roles of C and C' reversed) such that $K_m^{i+1} + F_m^{i+1} \to \gamma_i$ in $S(P, \varepsilon) - C \cdot S(P, \varepsilon)$, for m great enough. Accordingly, H is uniformly locally i-connected as stated above, and with the assumption that $p^{j+1}(C)$ is finite the theorem follows [11]).

Remarks. On comparing Theorems 2 and 3 we note that in an *even*-dimensional E_n, Theorem 2, except for its restriction on the type of chains, is the stronger theorem, since Theorem 2 in this case requires the local i-connectedness of C only for $0 \leqq i \leqq (n-2)/2$, whereas Theorem 3 requires the finiteness of $p^{\frac{n}{2}}(C)$ as well as the stronger type of transformation. However, aside from its freedom from restriction on the type of chains employed, Theorem 3 yields additional information in the case of an *odd*-dimensional E_n, since it assumes the local i-connectedness of C only for $0 \leqq i \leqq (n-3)/2$, whereas Theorem 2 requires local i-connectedness of C for $i = (n-1)/2$ as well.

We call attention also to the fact that in the case of E_3, Theorem 1 is a stronger theorem than Theorem 2, being in this case the same as Theorem 3 of C. P. B.

In connection with the distinction between „free" and „free in the strong sense", it would be interesting to know under what conditions the first implies the second. A related question is the following: If C is a compact, closed subset of E_n, and $C' = f(C)$ the result of a continuous transformation f, under what conditions can f be ε-extended, for arbitrary $\varepsilon > 0$, into a homeomorphism of C? For instance, when will there exists a $\delta > 0$ such that if f is a δ-transformation of C, then f can be ε-extended, for arbitrary $\varepsilon > 0$, into a homeomorphism of C?

[15]) By definition, the general element of the cycle in the left-hand member of this relation is $c_1 z_{s_1,m} + \ldots + c_r z_{s_r,m}$.

[16]) For instance, see the first seven lines of the proof of theorem 1, C. P. B., with P as a point of $|\gamma^i|$ instead of C.

As we have already noted in Theorem 1 of C. P. B., a compact continuum which cuts E_n is the common boundary of two domains D and H. If for every $\varepsilon > 0$ there exist ε-transformations of C into subsets of both D and H that do not meet C, we shall say, for brevity, that C is *two-way free*. Concerning such continua we may state:

Theorem 4a. *In E_n, where n is odd and > 1, let C be a compact continuum which cuts E_n, is two-way free, and is locally i-connected for $0 \leqq i \leqq (n-3)/2$ $(=j)$. Then if $p^{j+1}(C)$ is finite, C is a g. c. $(n-1)$-m.*

Theorem 4b. *In E_n, where n is even and > 2, let C be a compact continuum which cuts E_n, is two-way free, and is locally i-connected for $0 \leqq i \leqq (n-2)/2$. Then C is a g. c. $(n-1)$-m.*

Theorems 4a and 4b are proved by proceeding, as in the proof of Theorem 1, to obtain local connectedness properties of *both* the complementary domains, and by applying the results stated in the reference given in footnote [11]).

In C. B. P. it was shown [17]) that for $n = 2, 3$, a compact continuum which cuts E_n and is continuously deformable without meeting itself is a closed $(n-1)$-manifold. This result is contained in the following theorem:

Theorem 5. *In E_n, a compact continuum which cuts E_n and is continuously deformable without meeting itself is a g. c. $(n-1)$-m.*

Proof. Denoting the continuum by C, and the complementary domains of which it is the common boundary (Theorem 1, C. P. B.) by D and H, we may suppose the deformation of C to take place in H. If D is not uniformly locally i-connected for an i such that $0 \leqq i \leqq n-2$, there exist an $\varepsilon > 0$, a point A of C, and a sequence of i-cycles γ_k^i of D whose diameters converge to zero and have the point A as topological limit, and each of which links C in $S(A, \varepsilon)$. By practically the same argument as given for the case $i = 0$ in the fourth and fifth paragraphs of the proof of Theorem 5 in C. P. B., we may show a contradiction. The theorem then follows from Principal Theorem C of G. C. M.

[17]) P. 165, Corollary, and Theorem 7.

On differentiation of Lebesgue double integrals.

By

A. S. Besicovitch (Cambridge, England).

The theorem proved in this note gives an answer to a problem put forward by S. Saks [1]).

Denote by $r(u, v)$ a rectangle (and its area) with sides parallel to the coordinate axes and containing the point (u, v). Given a function $f(x, y)$ summable in a domain D denote by $\overline{D}_{u,v} \int\int f(x, y)\,dx\,dy$, $\underline{D}_{u,v} \int\int \dots$, $D_{u,v} \int\int \dots$ the upper limit, the lower limit and the limit (if it exists) of the ratio

$$\frac{1}{r(u, v)} \int\int_{r(u,v)} f(x, y)\,dx\,dy, \quad \text{as} \quad d\,r(u, v) \to 0$$

where $d\,r(u, v)$ is the diameter of $r(u, v)$.

It is well known that for a bounded function $f(x, y)$

$$D_{u,v} \int\int f(x, y)\,dx\,dy$$

exists and is equal to $f(u, v)$ at almost all points of the domain. In the case of an unbounded function this is true only with an additional condition that the ratio of the larger side of $r(u, v)$ to the smaller side remains bounded. If this restriction is not imposed then the inequality

$$\overline{D}_{u,v} \int\int f(x, y)\,dx\,dy > \underline{D}_{u,v} \int\int f(x, y)\,dx\,dy$$

may hold on a set of positive measure. Saks' problem is: *May both terms of this inequality be finite on a set of positive measure?*

[1]) S. Saks. *Remark on the differentiability of the Lebesgue indefinite integral.* Fund. Math. T. XXII, pp. 257—261.