On free subsets of $E_n$

By

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In an earlier paper 1) I have given an affirmative answer, for the cases $n = 2, 3$, to the following question of Mr. K. Borsuk 2): Is every free, locally contractible 3), compact continuum which cuts $E_n$ an $(n - 1)$-dimensional manifold? More precisely, in $E_n$ every free, locally connected compact continuum $M$ which cuts $E_n$ is a simple closed curve (1-manifold), and in $E_n$ every such continuum for which, moreover, the Betti number $p^1(M) = 0$, is a closed 2-dimensional manifold. In the present paper these results are extended to higher dimensions, using the notion of generalized closed $n$-manifold 4). In addition Theorem 7 of C. P. B. is extended in a similar way, and one or two special types of transformations are considered.

1) Concerning a problem of K. Borsuk, Fund. Math. 21 (1933), pp. 156--167 (to be referred to hereafter as C. P. B.).

2) Fund. Math. 30 (1933), p. 285, Prob. 56. Borsuk calls a subset $M$ of $E_n$ free in this space if for every $s > 0$ there exists an $s$-transformation of $M$ into a set $M'$ such that $M - M' = 0$; see K. Borsuk, Über die Fundamentalgruppe, Monatsh. f. Math. u. Phys. 41 (1934), pp. 64--77.

3) "lokale zusammenziehbar".

4) As introduced in my paper Generalized closed manifolds in $n$-space, Annals of Math. 55 (1934), pp. 876--908 (hereafter referred to as G. C. M.), a generalized closed $n$-manifold (e.g. c. n.m.) is a compact metric space $M$ of dimension $n$ such that: 1) $p^q(M) = 1$ and for any proper closed subset $F$ of $M$, $p_q(F) = 0$; 2) for $1 \leq i \leq n - 1$, there exists $s > 0$ such that if $\gamma$ is a cycle of $M$ of diameter $< s$, then $\gamma = 0$ in $M$; 3) if $p$ is a point of $M$ and $s$ a positive number, there exist $\epsilon > 0$ and $\eta > 0$, such that if $\gamma'$ is a cycle of $M$ of diameter $< \epsilon$, then $\gamma' = 0$ on $M - B(P, \epsilon)$; if $P$ is a point of $M$ of dimension $n - 1$, then $\gamma = 0$ on $M - S(P, \eta)$.

Theorem 1. In $E_n$ ($n > 2$), let $C$ be a free, compact continuum which cuts $E_n$, is locally i-connected 5) for $0 \leq i \leq n - 3$, and for which $p^1(C)$ is finite. Then $C$ is a g. c. ($n - 1$)-m.

Proof. As shown in Theorem 1 of C. P. B., $C$ is a common boundary of two domains $D$ and $H$ such that $D + H + C = E_n$. Since $C$ is connected, a continuous transformation of $C$ yields a connected point set; consequently, if $C'$ is a transform of $C$ such that $C - C' = 0$, then $C'$ must lie wholly in $D$ or in $H$. As $C$ is free, we can assume that for arbitrary $\epsilon > 0$, there is an $\epsilon$-transform of $C$ in $H$.

Given $\epsilon > 0$, let $\delta > 0$ be such that if $P$ is a point of $C$, then any i-cycle $(1 \leq i \leq n - 3)$ of $C - S(P, \delta)$ bounds on $C - S(P, \delta/2)$. Consider a cycle $\gamma'$ of $D - S(P, \delta)$, considering $P$ henceforth as a fixed point. By the Lemma of C. P. B., there exists $\eta > 0$ such that if $\gamma'$ links $C$ in $S(P, \epsilon)$ and $C'$ is any $\eta$-transform of $C$, then $\gamma'$ links $C'$ in $S(P, \delta/4)$. We suppose that $\gamma'$ links $C$ in $S(P, \epsilon)$.

Let $K^{i+1}$ be any $(i + 1)$-chain bounded by $\gamma'$ in $S(P, \delta)$. There is no loss of generality in supposing $|\gamma'|$ connected, and it will be convenient to do so. Let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be a monotonic decreasing sequence of positive numbers converging to zero, where $\gamma_n < \gamma'|S(P, \delta)|$, and let $K^{i+1}$ be subdivided so that its cells are all of diameter $< \epsilon_n$. The set of those closed $(i + 1)$-cells on $K^{i+1}$ that do not meet $C$ form a complex $L_i$. The component of $|L_i|$ which contains $|\gamma'|$ forms the basis for a chain $K^{i+1} \rightarrow \gamma' - \gamma_i$.

Now make a subdivision of $K^{i+1}$ of mesh $< \epsilon_i$ and $< p(|K^{i+1}|, C)$, and let $K^{i+1}$ be subdivided so that its cells are all of diameter $< \epsilon_i$. The set of those closed $(i + 1)$-cells on $K^{i+1}$ that do not meet $C$ form a complex $L_i$. The component of $|L_i|$ which contains $|\gamma'|$ forms the basis for a chain $K^{i+1} \rightarrow \gamma' - \gamma_i - \gamma_n$.

Proceeding in this manner, we obtain for each positive integer $k$ a chain $K^{i+1}$ and a cycle $\gamma_k$ such that

$$|K^{i+1}| \supset |K^{i+1}|, \quad K^{i+1} \rightarrow \gamma' - \gamma_k,$$

$$K^{i+1} - K^{i+1} \rightarrow \gamma_k - \gamma_n.$$

5) In the sense employed here (see P. Alexandroff, Annals of Math. 30, p. 181, footnote), a metric space is locally i-connected at a point $P$ if for arbitrary $\epsilon > 0$ there exists $d_y > 0$ such that any cycle $\gamma|S(P, d_y)$ bounds on $S(P, \delta)$; the space is locally i-connected if it is locally i-connected at every point.

For $i = 0$, this notion coincides with the local connectedness of point set theory.

6) If $L$ denotes either a chain or a complex, we denote the set of points on $L$ by $|L|$.
For each \( k \), there can be assigned a positive number \( \eta_k \) such that all vertices (0-cells) of \( K^{4k+1}_m - K^{4k+4}_m \) are within a distance \( \eta_k \) of \( C \) and such that \( \lim_{k \to \infty} \eta_k = 0 \). Consequently the cycles \( i_k \) and chains \( K^{4k+1}_m - K^{4k+4}_m \) may be considered as forming "infinitesimal alterations" of cycles and chains on \( C \); in particular, then, the cycles \( i_k \) form infinitesimal alterations of the elements of a convergent Vastoris cycle \( I^0 = (c_0, c_1, \ldots, c_n) \) on \( C \cdot S(P, \delta) \). By hypothesis, \( I^0 \) bounds a chain \( Q^{4k+1}_m \) on \( C \cdot S(P, \varepsilon/2) \).

Let \( q(C, C') = \theta \), where \( C' \) is an \( \eta \)-transform of \( C \) as defined previously. Denoting the elements of \( Q^{4k+1}_m \) by \( q_k \), where \( q_k \to c_k \), we choose an integer \( m \) such that the cells of both \( i_m \) and \( c_m \) are of diameter \( < \theta/3 \), the distance from any vertex of \( i_m \) to its corresponding \( c_m \) is \( < \theta/3 \), and the cells of \( g_m \) form a \( \theta/3 \)-complex. Let \( g_m \) denote the complex consisting of the cells of \( g_m \) with, however, the vertices of \( c_m \) replaced by their correspondents in \( i_m \). Based on a geometrical realization of the complex \( g_m \), we obtain a chain \( F^{4k+1}_m \to i_m \). No cell of \( F^{4k+2}_m \) meets \( C' \), and \( F^{4k+2}_m \subset S(P, 3\varepsilon/4). \) From the relations

\[
K^{4k+1}_m \to \gamma - i_m \quad \text{in } S(P, 3\varepsilon/4) \to C',
\]

\[
F^{4k+1}_m \to i_m \quad \text{in } S(P, 3\varepsilon/4) \to C',
\]

we get

\[
K^{4k+1}_m + F^{4k+1}_m \to \gamma \quad \text{in } S(P, 3\varepsilon/4) \to C'.
\]

But \( \eta \) was chosen so that \( \gamma' \) links \( C' \) in \( S(P, 3\varepsilon/4) \). Thus the supposition that \( \gamma' \) links \( C \) in \( S(P, \varepsilon) \) leads to a contradiction, and it follows that \( D \) is uniformly locally connected, for \( 1 \leq i \leq n - 3 \).

By Theorem 2 of C. P. B., both domains \( D \) and \( H \) are uniformly locally connected. Since \( p^1(C) \) is finite, we know from well-known duality relations that \( p^{n-2}(E_n - C) = \) finite. That \( C \) is therefore a g. c. \((n-1)-m \) follows at once from Theorem 15 of G. C. M.

**Remarks.** As we shall note presently (see the Lemma below), when we are dealing with chains and cycles mod \( m \geq 2 \), the assumption in Theorem 1 that \( p^1(C) \) is finite becomes unnecessary in case \( n \geq 4 \), since in this case the finiteness of \( p^1(C) \) is a consequence of the local 1-connectedness of \( C \). However, it is clear from the above proof that we can replace the assumption concerning

\[
p^1(C)
\]

by the assumption that \( C \) is locally \((n-2)-connected \), in which case the conclusion of the theorem follows from Principal Theorem C of G. C. M. Since a locally contractible continuum is locally \( i \)-connected for all dimensions \( i \), we have the

**Corollary.** A free, locally contractible continuum which cuts \( E_n \) is a g. c. \((n-1)-m \)-manifold.

Since every absolute neighborhood retract is locally contractible, a similar corollary may be stated for the class of such sets.

If we assume, in the hypothesis of Theorem 1, that \( p^{n-2}(C) \) is finite, and delete the condition that \( p^1(C) \) be finite, we shall then have as a consequence that \( p^1(E_n - C) \) and hence \( p^1(H) \) is finite. This fact, together with the other local connectedness properties of \( D \) and \( H \) proved above, are sufficient to make \( C \) a g. c. \((n-1)-m \). Thus we can state

**Theorem 1a.** The result of Theorem 1 remains true if the condition that \( p^1(C) \) be finite is replaced by the condition that \( p^{n-2}(C) \) be finite.

For the case where \( n \geq 4 \), and only chains mod \( m \geq 2 \) are employed, we may obtain a much stronger theorem that Theorem 1.

In order to establish this, we need the following Lemma which I have established in another (as yet unpublished) paper:

**Lemma.** Let \( M \) be a locally compact metric space which, in terms of chains mod \( m \geq 2 \), is locally \( i \)-connected for \( 0 \leq i \leq k \) (where \( k \) is a non-negative integer). Then, if \( F \) is a self-compact subset of \( M \) and \( U \) is an open subset of \( M \) containing \( F \), at most a finite number of \( i \)-cycles of \( F \) are independent with respect to homologies in \( U \).

**Theorem 2.** In \( E_n \) \((n > 2) \), let \( C \) be a free, compact continuum which cuts \( E_n \) and which, in terms of chains mod \( m \geq 2 \), is locally \( i \)-connected for \( 0 \leq i \leq (n-1)/2 \). Then \( M \) is a g. c. \((n-1)-m \).

**Proof.** By virtue of Principal Theorem C of G. C. M., we need only show that if \( F \) is a point of \( C \) and \( \varepsilon \) a positive number, there

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1) This follows easily, for instance, from the fact that the set of points on any complex is locally 0-connected.

exists \( \delta > 0 \) such that any \( \gamma' \) in \( D \cdot S(P, \delta) \) bounds in \( D \cdot S(P, \varepsilon) \), where \( 0 \leq i \leq n - 2 \) and \( D \) is as defined in the proof of Theorem 1. That such a \( \delta \) exists for values of \( i \) such that \( 0 \leq i \leq (n - 1)/2 \) follows from the proof of Theorem 1. Hence in what follows we consider a fixed value of \( i \) such that \( (n - 1)/2 < i \leq n - 2 \).

Let \( \eta \) be any positive number \( < \varepsilon \) and let \( \eta_1, \eta_2, \eta_3, \ldots \) be a sequence of positive numbers such that \( \varepsilon > \eta_1 > \eta_2 > \eta_3 > \ldots > \eta_n \). Let \( \theta > 0 \) be such that any \( (n - i - 1) \)-cycle of \( C \cdot S(P, 2 \theta) \) bounds on \( C \cdot S(P, \eta) \). Let \( \delta_1 < \theta \). Suppose that no matter what \( \delta > 0 \) is selected, there is in \( D \cdot S(P, \delta) \) a \( \gamma' \) that fails to bound in \( D \cdot S(P, \varepsilon) \). Then there exists such a \( \gamma' \) in \( D \cdot S(P, \delta_1) \) that we shall denote by \( \gamma_1 \).

As \( \gamma_1 \) links \( C \) in \( S(P, \varepsilon) \), there exists a \( \beta_1 > 0 \) and a \( \alpha \)-transformation \( T_1 \) such that \( T_1(C) \subseteq H \), and \( \gamma_1 \) links \( T_1(C) \) in \( S(P, \eta) \).

Let \( \delta_2 \) be a positive number such that \( T_1(C) \subseteq S(P, \delta_2) \). In \( D \cdot S(P, \delta_2) \) there is a \( \gamma' \) that links \( C \) in \( S(P, \delta_2) \). Then \( \gamma_1 \) and \( \gamma_2 \) are independent with respect to homologies in \( S(P, \eta_1) \) of \( C \cdot S(P, \eta) \).

For suppose we had

\[
K^{i+1} \rightarrow c_1 \gamma_1 + c_2 \gamma_2 \text{ in } S(P, \eta_1) - C \cdot S(P, \eta_1).
\]

If \( K^{i+1} \) is the maximal portion of \( K^{i+1} \) forming a chain of \( H \), then clearly \( K^{i+1} \rightarrow 0 \). Then

1) \( K^{i+1} = K^{i+1} - K^{i+1} \rightarrow c_1 \gamma_1 + c_2 \gamma_2 \)

in \( D \cdot S(P, \eta_1) \subseteq S(P, \eta_1) - T_1(C) \cdot S(P, \eta_1) \).

However, there exists a chain \( L^{i+1} \) such that

2) \( L^{i+1} \rightarrow c_3 \gamma_3 \)

in \( S(P, \delta_2) \subseteq S(P, \eta_1) - T_1(C) \cdot S(P, \eta_1) \).

From relations 1) and 2) we get

\[
K^{i+1} - L^{i+1} \rightarrow c_1 \gamma_1 \text{ in } S(P, \eta_1) - T_1(C) \cdot S(P, \eta_1),
\]

which contradicts the fact that \( \gamma_1 \) links \( T_1(C) \) in \( S(P, \eta_1) \).

Having determined, by the process just indicated, cycles \( \gamma_1, \ldots, \gamma_m \) that are linearly independent with respect to homologies in \( D \cdot S(P, \eta_2) \), we observe that since there are only a finite number of possible linear combinations of the \( \gamma \)s (mod \( m \)), there is a \( \beta_2 \) such that if \( T_2(C) \) is a \( \beta_2 \)-transformation of \( C \) into a subset of \( H \), every such linear combination links \( T_2(C) \) in \( S(P, \eta_2) \). Let \( \delta_{i+1} > \beta_2 \) be such that \( T_2(C) \subseteq S(P, \delta_{i+1}) \) and in \( D \cdot S(P, \delta_{i+1}) \) let \( \gamma_{i+1} \) be a cycle linking \( C \) in \( S(P, \varepsilon) \). Then \( \gamma_1, \ldots, \gamma_i, \gamma_{i+1} \) are linearly independent

with respect to homologies in \( S(P, \eta_i) - C \cdot S(P, \eta_1) \). For if not, we would have by reasoning as above, a chain \( K^{i+1} \) such that

3) \( K^{i+1} \rightarrow c_1 \gamma_1 + \ldots + c_i \gamma_i + c_{i+1} \gamma_{i+1} \)

in \( S(P, \eta_1) - T_2(C) \cdot S(P, \eta_1) \).

However, we have

4) \( L^{i+1} \rightarrow c_{i+1} \gamma_{i+1} \)

in \( S(P, \delta_{i+1}) \subseteq S(P, \eta_1) - T_2(C) \cdot S(P, \eta_1) \), and from relations 3) and 4) we get

\[
K^{i+1} - L^{i+1} \rightarrow c_1 \gamma_1 + \ldots + c_i \gamma_i \text{ in } S(P, \eta_1) - T_2(C) \cdot S(P, \eta_1),
\]

contradicting the fact that every linear combination of the cycles \( \gamma_1, \ldots, \gamma_i \) links \( T_2(C) \) in \( S(P, \eta_1) \).

We thus show the existence of an infinite set of such cycles \( \gamma \) every linear combination of which links \( C \) is \( S(P, \eta) \). As the \( \gamma \)s are thereby linearly independent with respect to homologies in \( E_n - [C \cdot S(P, \eta) + F(P, \eta)] \), they are uniquely linked with a set of cycles \( a_k^{n-i+1} \) of \( C \cdot S(P, \eta) + F(P, \eta) \).

As a Victoria cycle, let \( a_k^{n-i+1} = (a_k, b_1, a_1, \ldots, b_{m-1}, a_m, \ldots) \), where \( a_k \) is an \( e_n \)-cycle, \( \lim e_n = 0 \). Let \( h_{k,n} \) denote the portion of \( a_k \)

whose abstract \( (n - i - 1) \)-cells lie wholly in \( S(P, \theta) \). Then \( K_{k,n} \), where \( h_{k,n} \rightarrow K_{k,n} \), may, without loss of generality, be assumed to lie on \( C \cdot F(P, \theta) \), and since the sequence \( K_{k,1}, K_{k,2}, \ldots, K_{k,n} \) contains a convergent subsequence forming a Victoria cycle \( I^{n-i-2} \) of \( C \cdot F(P, \theta) \), we can assume for the sake of brevity, that \( I^{n-i-2} \) is the former sequence itself.

Since there are infinitely many cycles \( I^{n-i-2} \), there exists, by virtue of the above Lemma, a relation

\[
B^{n-i-1} \rightarrow c_1 I_{2}^{n-i-1} + \ldots + c_{m} I_{m}^{n-i-1} \text{ in } C \cdot [S(P, \theta) - S(P, \delta_i)],
\]

where the elements of the chain \( B^{n-i-1} \) are \( h_1, b_1, \ldots, b_m, \ldots \) and \( b_m \rightarrow c_k K_{k,m} + \ldots + c_{m} K_{m,m} \). We assume that from the sequence of cycles \( c_k h_{k,n} + \ldots + c_{m} h_{m,n} - b_m \), there has been chosen a convergent

11) If \( a_k^{n-i+1} \) lay wholly in \( S(P, \theta) \), it would bound on \( C \cdot S(P, \eta) \).

14) Actually, we should choose the convergent subsequence \( I^{n-i} = (K_{k,1}, K_{k,2}, \ldots, K_{k,n}) \), and henceforth replace \( e_n^{n-i} \) by the homologous cycle formed by the sequence \( (a_k, b_1, a_1, \ldots, b_{m-1}, a_m, \ldots) \). And when other convergent sequences are selected hereafter, it will usually mean that the cycles just treated will have to be replaced again by subsequences to harmonize with the new cycles.
an \( \eta \)-transformation of \( C \), then any i-cycle of \( C' \) of diameter \( < \delta \)
bound a chain of \( C' \) of diameter \( < \varepsilon \) (by elementary continuity
considerations). Now let \( P \) be a point of \( C \) and \( \gamma' \) a cycle of \( H \cdot S(P, \delta) \).
Without loss of generality we may assume that \( |\gamma'| \) is a connected point
set, and it is easily shown \(^1\) that there exists a positive number \( \theta \)
such that a \( \theta \)-transformation of \( C \) into a subset of \( H \) yields a set
which separates \( |\gamma'| \) from \( C \); let \( C' \) denote the result of such
a transformation. By methods similar to those used in the proof
of Theorem 1, we may show the existence of chains \( E_i^{m}, \overline{E}_i^{m} \) (with
however, the roles of \( C \) and \( C' \) reversed) such that \( E_i^{m} + \overline{E}_i^{m} \to \gamma_i \)
in \( S(P, \varepsilon) - C \cdot S(P, \varepsilon) \), for \( m \) great enough. Accordingly, \( H \) is
uniformly locally i-connected as stated above, and with the assumption
that \( p_i^{m}(C) \) is finite the theorem follows \(^1\).

Remarks. On comparing Theorems 2 and 3 we note that in an
even-dimensional \( E_n \), Theorem 2, except for its restriction on the
type of chains, is strongest. In Theorem 2 this case requires the local i-connectedness of \( C \) only for \( 0 \leq i \leq (n-2)/2 \),
whereas Theorem 3 requires the finiteness of \( p_i^{m}(C) \) as well as the
stronger type of transformation. However, aside from its freedom
from restriction on the type of chains employed, Theorem 3 yields
additional information in the case of an odd-dimensional \( E_n \), since
it assumes the local i-connectedness of \( C \) only for \( 0 \leq i \leq (n-3)/2 \),
whereas Theorem 2 requires local i-connectedness of \( C \) for \( i = (n-1)/2 \)
as well.

We call attention also to the fact that in the case of \( E_n \),
Theorem 1 is a stronger theorem than Theorem 2, being in this
case the same as Theorem 3 of C. P. B.

In connection with the distinction between "free" and "free in
the strong sense", it would be interesting to know under what
conditions the first implies the second. A related question is the
following: If \( C \) is a compact, closed subset of \( E_n \), and \( C' = f(C) \)
the result of a continuous transformation \( f \) under what conditions
\( f \) can be \( \varepsilon \)-extended, for arbitrary \( \varepsilon > 0 \), into a homeomorphism
of \( C \) ? For instance, when will there exist a \( \delta > 0 \) such that if \( f \)
is a \( \delta \)-transformation of \( C \), then \( f \) can be \( \varepsilon \)-extended, for arbitrary
\( \varepsilon > 0 \), into a homeomorphism of \( C \) ?

\(^{1}\) By definition, the general element of the cycle in the left-hand member
of this relation is \( e_i, x_{n,m} + \ldots + e_i x_{n,m} \).
As we have already noted in Theorem 1 of C. P. B., a compact continuum which cuts $E_n$ is the common boundary of two domains $D$ and $H$. If for every $\varepsilon > 0$ there exist $\varepsilon$-transformations of $C$ into subsets of both $D$ and $H$ that do not meet $C$, we shall say, for brevity, that $C$ is two-way free. Concerning such continua we may state:

**Theorem 4a.** In $E_n$, where $n$ is odd and $> 1$, let $C$ be a compact continuum which cuts $E_n$, is two-way free, and is locally $i$-connected for $0 \leq i \leq (n - 3)/2$ ($= f$). Then if $p^{+i}(C)$ is finite, $C$ is a g. c. $(n - 1)$-m.

**Theorem 4b.** In $E_n$, where $n$ is even and $> 2$, let $C$ be a compact continuum which cuts $E_n$, is two-way free, and is locally $i$-connected for $0 \leq i \leq (n - 2)/2$. Then $C$ is a g. c. $(n - 1)$-m.

Theorems 4a and 4b are proved by proceeding, as in the proof of Theorem 1, to obtain local connectedness properties of both the complementary domains, and by applying the results stated in the reference given in footnote 11).

In C. B. P. it was shown 11) that for $n = 2, 3$, a compact continuum which cuts $E_n$ and is continuously deformable without meeting itself is a closed $(n - 1)$-manifold. This result is contained in the following theorem:

**Theorem 5.** In $E_n$, a compact continuum which cuts $E_n$ and is continuously deformable without meeting itself is a g. c. $(n - 1)$-m.

**Proof.** Denoting the continuum by $C$, and the complementary domains of which it is the common boundary (Theorem 1, C. P. B.) by $D$ and $H$, we may suppose the deformation of $C$ to take place in $H$. If $D$ is not uniformly locally $i$-connected for such an $i$ that $0 \leq i \leq n - 2$, there exist an $\varepsilon > 0$, a point $A$ of $C$, and a sequence of $i$-cycles $\gamma'_i$ of $D$ whose diameters converge to zero and have the point $A$ as topological limit, and each of which links $C$ in $S(A, \varepsilon)$. By practically the same argument as given for the case $i = 0$ in the fourth and fifth paragraphs of the proof of Theorem 5 in C. P. B., we may show a contradiction. The theorem then follows from Principal Theorem C of G. C. M.

11) P. 166, Corollary, and Theorem 7.