so gehört er mit jedem V gleichzeitig immer den durch Permutation aus V hervorgehenden Wahrheitsbereichen an, er enthält also mit v zugleich die Disjunktion v^* aller aus v durch Permutation hervorgehenden Sätze, und diese Disjunktion v^* ist selbst „kategorisch". Sollte es außer den so aus V hervorgehenden Wahrheitsbereichen noch weitere V_α, V_β, ... geben, denen s angehört, so entsprechen auch diesen weitere „elementar-symmetrische" oder „kategorische" Sätze v_α^*, v_β^*, v_γ^*, ... und s ist dann und nur dann wahr, wenn einer dieser Sätze wahr ist, d. h. s ist „äquivalent" der Disjunktion aller dieser kategorischen Sätze, w. z. b. w.

Die vorstehenden Ausführungen bilden erst den *Anfang* einer noch nicht abgeschlossenen Untersuchung, welche die Begründung einer „infinitistischen" echt mathematischen Syllogistik und Beweistheorie zum Ziele hat. Einer ehrenvollen Einladung der Redaktion folgend, habe ich hier meine vorläufigen Ergebnisse für diesen Festband zusammengestellt in der Hoffnung, in einer weiteren Mitteilung die erforderlichen Ergänzungen nachholen zu können.

The mathematical structure of Lewis's theory of strict implication.

By

Edward V. Huntington (Cambridge, U. S. A.).

Introduction.

The development of any abstract mathematical theory may be described, in the last analysis, as a process of writing down, one after another, a series of expressions — this process involving the motor activity of some human agent. The human agent, in actually working out the development of the theory, constantly passes judgment upon a variety of expressions, deciding (on the basis of previously agreed upon rules of procedure) which expressions are to be written down as „accepted" expressions in the theory, and which are to be rejected.

The term „expression" is here used in a general sense, to denote any sequence of a finite number of marks on paper, to be read, say, from left to right. The marks which occur in existing abstract mathematical theories may be roughly described as of two kinds: (1) letters, each of which may stand for an „element" or „class of elements" within the system; and (2) signs, each of which may stand for an „operation" or „relation" among the elements. In any particular mathematical theory the marks which are to appear in the theory are (or should be) listed in advance, to indicate the „universe of discourse" within which the theory is to be developed.

In order to start such a theory going at all, one must have at least one expression which is agreed upon as an „accepted" expression. In any particular theory, the expressions which are accepted as the starting point of that theory are (or should be) listed in advance and may be called a „set of formal postulates" for the system in question.

In order to develop the theory beyond this initial step, one must have at least one „rule of procedure" which authorizes one to proceed from expressions already „accepted" to new expressions. In any particular theory, the rules of procedure which are agreed upon as the rules to be followed in that theory should also be listed in advance as an integral part of the theory.

The purpose of the present note is to indicate how C. I. Lewis's theory of „strict implication", originally presented as a concrete theory of logical propositions, would look if expressed in the form of an abstract mathematical theory in the sense described above. It will be found that a comparison between Lewis's theory of „strict" implication and the earlier Whitehead-Russell theory of „material" implication is greatly facilitated by expressing both theories in the abstract mathematical form.

Note. The theory of strict implication was first proposed in Lewis's „*Survey of Symbolic Logic*" in 1918, and was developed in revised form in Lewis and Langford's „*Symbolic Logic*" (The Century Co., New York) in 1932. The explicit introduction of the „subclass T" was suggested by E. V. Huntington in 1933 (Trans. Amer. Math. Soc., vol. 35, pp. 274—304, 557, 971; see also Bull. Amer. Math. Soc., vol. 40, pp. 127—143, 1934). The concept of „effective implication" introduced in the present paper on the basis of „effective equality", without reference to „negation" or „truth values", is possibly new. In the preparation of the present paper, I am indebted not only to Professor Lewis but also to Dr. W. V. Quine and Dr. S. MacLane for helpful suggestions.

Expressions in Lewis's System.

The expressions which occur in Lewis's theory are constructed chiefly out of five kinds of marks, which may be described as follows.

1 a) a, b, c etc., denote unspecified objects, which may be spoken of as „elements" belonging to some unspecified class K.

1 b) $a = b$ denotes an unspecified relation between the elements a and b, which may be spoken of as the relation of „effective equality" with respect to some unspecified property.

2) $a \times b$, or simply ab (read: a times b) denotes an object determined by the elements a and b in accordance with some unspecified binary rule of operation.

3) a' (read: a prime) denotes an object determined by the element a in accordance with some unspecified unary rule of operation.

4) a in T denotes an unspecified property of the element a, which may be spoken of as membership in some unspecified subclass T.

5) $a \prec b$ (read: a hook b) denotes an object determined by the elements a and b in accordance with some unspecified binary rule of operation.

Parentheses are also used, with obvious meaning; and abbreviations are introduced „by definition" in the usual manner.

The notation $(K, \times, ', T, \prec)$ serves to symbolize the „universe of discourse" with which the theory is concerned.

Informal postulates, or rules of procedure, in Lewis's system.

The following rules of procedure in Lewis's system are of the nature of instructions addressed to a human agent as to how he is to proceed in developing the theory, and are therefore expressed primarily in ordinary language. For brevity, each rule is re-stated in a semi-symbolic form; but the arrow „$\rightarrow$" used in these re-statements is to be understood as a merely verbal abbreviation, and is not to be classed among the formal „marks" employed in the symbolic expressions of the system.

Thus, the notation $X \rightarrow Y$ is to be read: „If we find X established, we are thereupon authorized to write down Y"; or simply „X leads to Y". Obviously, whenever $X \rightarrow Y$, and $Y \rightarrow Z$ then $X \rightarrow Z$. (At any particular point in the development, the question whether X really does lead to Y is to be determined, of course, solely on the basis of the postulates of the system).

The rules of procedure which appear, explicitly or implicitly, in Lewis's theory may be classified as follows.

Rules of class-closure.

A. If a is an element of K, then we are to regard a' also as an element of K: Briefly:

$$(a \text{ in } K) \rightarrow (a' \text{ in } K).$$

B. If a and b are elements of K, then we are to regard $a \times b$ also as an element of K: Briefly:

$$[(a \text{ in } K) \text{ and } (b \text{ in } K)] \rightarrow (ab \text{ in } K).$$

C. If a and b are elements of K, then we are to regard $a \prec b$ also as an element of K: Briefly:

$$[(a \text{ in } K) \text{ and } (b \text{ in } K)] \to (a \prec b \text{ in } K).$$

D. If a is an element of T, then we are to regard a as also an element of K: Briefly:

$$(a \text{ in } T) \to (a \text{ in } K).$$

Rule of adjunction.

E. Whenever, for any particular elements a and b, we find in one place that „a in T" has been established as an accepted expression and in another place that „b in T" has been established as an accepted expression, we may thereupon write down the expression „ab in T" as an accepted expression. Briefly:

$$[(a \text{ in } T) \text{ and } (b \text{ in } T)] \to (ab \text{ in } T).$$

Rule of inference.

F. Whenever, for any particular elements a and b, we find in one place that „a in T" has been established as an accepted expression and in another place that „$a \prec b$ in T" has been established as an accepted expression, we may thereupon write down the expression „b in T" as an accepted expression. Briefly:

$$[(a \text{ in } T) \text{ and } (a \prec b \text{ in } T)] \to (b \text{ in } T).$$

Rule of equivalence.

G. Whenever, for any particular elements a and b, we find in one place that „$a \prec b$ in T" has been established as an accepted expression and in another place that „$b \prec a$ in T" has been established as an accepted expression, we are thereupon authorized to write down the expression $a = b$ as an accepted expression. Briefly:

$$[(a \prec b \text{ in } T) \text{ and } (b \prec a \text{ in } T)] \to (a = b).$$

Rule of replacement of equals.

H. Whenever, for any particular elements a and b, we find that „$a = b$" has been established as an accepted expression, we are thereupon authorized to replace a by b (or b by a) at any point in any expression of the system. Briefly:

$$(a = b) \to (\text{„}a \text{ and } b \text{ are interchangeable"}).$$

And conversely: („a and b interchangeable") $\to (a = b)$.

Rule of existence.

J. There exists at least one pair of elements in K such that the expression „$[(a \prec b)' (a \prec b')']$ in T" is an accepted expression. Briefly:

$$(\exists a, b \text{ in } K): \ [(a \prec b)' (a \prec b')'] \text{ in } T.$$

Formal postulates in Lewis's system.

In the following formal postulates, it is understood that each postulate is an „accepted expression" in the theory. It is also understood that each postulate remains an accepted expression no matter what element is put for a, no matter what element is put for b, etc. In other words, in these expressions, the letters a, b, c, etc. are variables, each of which may stand for any element of K.

[The numbering of the postulates has been arranged to correspond with Lewis's list B1—B9 on page 493 of „*Symbolic Logic*". But Postulate 5 has recently been shown to be redundant (J. C. C. McKinsey, Bull. Amer. Math. Soc., vol. 40, p. 425—427, 1934), and Postulate 9 appears to belong among the informal rules rather than among the formal postulates. Also, Postulate 8 may be replaced by the simpler form 8a: $[(a \prec a') \prec (a \prec b)]$ in T]

1. $(ab \prec ba)$ in T.
2. $(ab \prec a)$ in T.
3. $(a \prec aa)$ in T.
4. $[(ab)c \prec a(bc)]$ in T.
5. $[a \prec (a')']$ in T.
6. $[(a \prec b)(b \prec c) \prec (a \prec c)]$ in T.
7. $[a(a \prec b) \prec b]$ in T.
8. $[(a \prec b')' \prec (a \prec a')']$ in T.
9. Same as the „*rule of existence*" above.

10a. $\{(a \prec b) \prec [(ab') \prec (ab')']\}$ in T.

10b. $\{[(ab') \prec (ab')'] \prec (a \prec b)\}$ in T.

On the basis of these postulates, and introducing the mark a^* through the following abbreviative definition:

11. Def. a^* [read: a star] $= (a \prec a')$,

we can easily establish the following theorem:

12. $(a \prec b) = (ab')^*$.

If now we compare the abstract mathematical system defined by the foregoing rules and postulates with Lewis's system of strict implication as set forth in Lewis and Langford's „*Symbolic Logic*“ (1932), we find that the element „a star“ corresponds exactly to Lewis's „curl diamond a“; the expression „$a \prec b$ in T“ corresponds to the assertion „a strictly implies b“; the formulas 11 and 12 show how each of the operators $\prec$ and $*$ may be defined in terms of the other; and the postulates 1—9 (omitting the symbol „in T“) correspond exactly to Lewis's postulates B1—B9. Hence the structural identity of the abstract system here presented and Lewis's system of strict implication is seen to be established.

In view of 11 and 12, as we have already noted, the system may be expressed either in terms of the base $(K, \times, ', T, \prec)$, with $*$ treated as a derived concept, or in terms of the base $(K, \times, ', T, *)$, with $\prec$ treated as a derived concept. The latter plan is, in effect, the one presented by Lewis in „*Symbolic Logic*“; but in view of the importance of the symbol $\prec$ in the usual interpretation of the system, the plan adopted above would appear to be the more direct.

Further definitions and theorems.

Just as the initial postulates of the theory are of two kinds: (1) the *informal postulates* or rules of procedure; and (2) the *formal postulates* or symbolic expressions of the form „$(\ldots)$ in T“; so the further development of the theory leads to theorems of two kinds: (1) *informal theorems*, which are to be added to our list of rules of procedure; and (2) *formal theorems*, which are to be added to our list of accepted symbolic expressions.

For brevity and convenience, the theory makes use also of the following „abbreviative definitions“, which define the marks $a+b$, $a \supset b$, $a = b$, and $a \equiv b$ as elements of the class K.

13. Def. $(a+b) = (a' b')'$. [Read: a plus b].
14. Def. $(a \supset b) = (a b')'$. [Read: a horseshoe b).
15. Def. $(a \equiv b) = [(a \supset b)\ (b \supset a)]$. [Read: a triple b].
16. Def. $(a \equiv b) = [(a \prec b)\ (b \prec a)]$. [Read: a quad b].

(The symbol „$\equiv$“ is here used in place of one of the symbols „$=$“ which appear ambiguously in Lewis's definition 11·03).

The following theorems (not explicitly mentioned in „*Symbolic Logic*“) may be noted.

17. $a a' = b b'$.
18. $(a a' \text{ in } T) \rightarrow (b \text{ in } T)$, no matter what element is put for b.
19. $(a \equiv b \text{ in } T) \rightarrow (a = b)$.

Further, the following fundamental theorem has recently been established by the present writer (Bull. Amer. Math. Soc., vol. 40, p. 729—735, 1934):

20. $(a \prec b) = (a \equiv ab)$; that is, $(a \prec b) = [(a \prec ab)\ (ab \prec a)]$.

From this it is easy to deduce the following rules of procedure:

21. $(a \prec b \text{ in } T) \rightarrow (a = ab)$.
22. $(a = ab) \rightarrow (a \prec b \text{ in } T)$.

Effective implication vs. strict implication.

In order to make clear the significance of this last result, it will be convenient to introduce a new term, „a effectively implies b“, symbolized by „$a < b$“, to indicate that the element a and the element ab are „effectively equal“ in the sense explained above. Thus:

23. Def. $(a < b) \rightleftarrows (a = ab)$.

Here the notation $a < b$ may be read: a effectively implies b; or simply: a within b; the „mutual arrow“ notation $X \rightleftarrows Y$ means that whenever we find X established, we may thereupon write down Y, and whenever we find Y established, we may thereupon write down X; and the „effective equality“ notation $x = y$ means that the elements x and y may be interchanged at any point in any expression in the system.

On the basis of this definition of „effective implication“, we may establish the following theorem in Lewis's system:

24. Whenever in the course of the development of the theory we find, for any given elements a and b, that „$a \prec b$ in T“ has been established as an accepted expression, we may thereupon write down the memorandum $a < b$, meaning thereby that $a = ab$; and conversely, if we find that $a < b$ has been established, we may thereupon write down „$a \prec b$ in T“ as an accepted expression. Briefly:

$$(a \prec b \text{ in } T) \rightleftarrows (a < b).$$

Concrete interpretations of Lewis's abstract system.

Any system $(K, \times, ', T, \prec)$ which satisfies the set of postulates above (including the rules of procedure A—J as well as the formal postulates 1—10) may be called a Lewis system. In other words, the given set of postulates provides a means of classifying any system $(K, \times, ', T, \prec)$ which may be presented, and deciding whether it shall or shall not be called a Lewis system — this purely classifying function being, in fact, the characteristic function performed by any mathematical set of postulates.

Example 1.

The example of a Lewis system $(K, \times, ', T, \prec)$ in which Lewis himself was primarily interested is the following.

$K =$ the class of propositions, $p, q, r, \ldots$ (the relation $p = q$ meaning that p and q are equal in the „effects" which they produce).
$pq =$ the proposition „p and q".
$p' =$ the proposition „not p"
$T =$ the class of propositions which are „asserted".
$p \prec q =$ the proposition „p strictly implying q" (in the intuitive sense envisaged by Lewis).

In this case the derived concepts are to be interpreted, according to their definitions, as follows:

$p^* =$ the proposition „p impossible".
$p + q =$ the proposition „p or q".
$p \supset q =$ the proposition „p materially implying q".
$p \equiv q =$ the proposition „p materially equivalent to q".
$p \overline{\equiv} q =$ the proposition „p strictly equivalent to q".

Example 2.

A second, loss familiar example of a Lewis system $(K, \times, ', T, \prec)$ is the following (so-called „dual") interpretation.

$K =$ the class of propositions, $p, q, r, \ldots$, as above.
$pq =$ the proposition „p or q".
$p' =$ the proposition „not p".
$T =$ the class of propositions which are „denied".
$p \prec q =$ the proposition „p not being strictly implied by q".

In this case the derived concepts are to be interpreted as follows:

$p^* =$ the proposition „p not necessarily true".
$p + q =$ the proposition „p and q".
$p \supset q =$ the proposition „p not being materially implied by q".
$p \equiv q =$ the proposition „p not being materially equivalent to q".
$p \overline{\equiv} q =$ the proposition „p not being strictly equivalent to q".

A finite example of a Lewis system.

An example of a Lewis system, which brings out explicitly the distinctions between the various operations considered in the theory, is the following finite system $(K, \times, ', T, \prec)$, containing only four elements. (This example is an elaboration of the first example mentioned on page 493 of „*Symbolic Logic*").

$K =$ a class of four numbers, 1, 2, 3, 4, — „effective equality" between two elements in this case being simply equality in numerical value.
$T =$ the subclass comprising the two elements 1 and 2.
pq, p', and $p \prec q =$ the elements indicated in the following tables. (In the double-entry tables, p is on the left, and q along the top).

$K = 1, 2, 3, 4.$
$T = 1, 2.$

pq	1	2	3	4
1	1	2	3	4
2	2	2	4	4
3	3	4	3	4
4	4	4	4	4

p	p'
1	4
2	3
3	2
4	1

$p \prec q$	1	2	3	4
1	2	4	4	4
2	2	2	4	4
3	2	4	2	4
4	2	2	2	2

Recalling the definitions of the elements p^*, $p + q$, $p \supset q$, $p \equiv q$, and $p \overline{\equiv} q$, namely:

$$p^* = (p \prec p'); \quad (p + q) = (p' q')'; \quad (p \supset q) = (p q')';$$

$(p \equiv q) = [(p \supset q)(q \supset p)]$; and $(p \overline{\equiv} q) = [(p \prec q)(q \prec p)]$; we have

p	p^*
1	4
2	4
3	4
4	2

$p+q$	1	2	3	4
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
4	1	2	3	4

$p \supset q$	1	2	3	4
1	1	2	3	4
2	1	1	3	3
3	1	2	1	2
4	1	1	1	1

$p \equiv q$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

$p \overline{\equiv} q$	1	2	3	4
1	2	4	4	4
2	4	2	4	4
3	4	4	2	4
4	4	4	4	2

In the following „relation tables“, a dot in row p and column q indicates that p stands to q in the relation described in the heading of the table.

Assertion of „Strict implication“ with respect to T ($p \prec q$ in T)

	1	2	3	4
1	.			
2	.	.		
3	.		.	
4	.	.	.	.

Assertion of „Strict equivalence“ with respect to T. ($p \equiv q$ in T)

	1	2	3	4
1	.			
2		.		
3			.	
4				.

Assertion of „Strict impossibility“ with respect to T. (p^* in T)

1	
2	
3	
4	.

„Effective“ implication. ($p = pq$)

	1	2	3	4
1	.			
2	.	.		
3	.		.	
4	.	.	.	.

„Effective“ equality. ($p = q$)

	1	2	3	4
1	.			
2		.		
3			.	
4				.

Assertion of „Material implication“ with respect to T. ($p \supset q$ in T)

	1	2	3	4
1	.	.		
2	.	.		
3	.	.	.	.
4	.	.	.	.

Assertion of „Material equivalence“ with respect to T. ($p \equiv q$ in T)

	1	2	3	4
1	.	.		
2	.	.		
3			.	.
4			.	.

Assertion of „Material negation“ with respect to T. (p' in T)

1	
2	
3	.
4	.

While the *element* $p \prec q$ cannot be defined in terms of the „Boolean“ symbols K, $\times$, $'$, $=$, these tables show that the *relation* „$p \prec q$ in T“ is identical with the Boolean relation „$p = pq$“ (at least in this example); and Theorem 24 shows that this will hold true in every example of a Lewis system.

Similarly, the relation „$p \equiv q$ in T“ is seen to be identical with the relation „$p = q$“; and Theorem 19 shows that this will be true not only in this example but in every example of a Lewis system.

February, 1935, Harvard University.

Sur l'existence des plans tangents aux surfaces applicables sur le plan.

Par

Henri Lebesgue (Paris).

Considérons une surface rectifiable $x(u,v)$, $y(u,v)$, $z(u,v)$; c'est-à-dire un système de trois fonctions continues telles que, lorsque le point de coordonnées rectangulaires u, v décrit une courbe de longueur finie ou infinie l, le point dont les coordonnées rectangulaires spatiales sont les valeurs des fonctions x, y, z décrive une courbe de longueur au plus égale à Kl, K étant un nombre fixe, indépendant de la courbe considérée. Si l'on passe des variables u et v aux variables $u_1 = Ku$ et $v_1 = Kv$, la courbe décrite par le point x, y, z sera de longueur au plus égale à celle décrite par le point u_1, v_1. Je suppose ce changement effectué, c'est-à-dire K ramené à l'unité.

Alors on a, pour tout système d'accroissements δu et δv,

$$\frac{[x(u+\delta u, v+\delta v)-x(u,v)]^2}{\delta u^2+\delta v^2}+\frac{[y(u+\delta u, v+\delta v)-y(u,v)]^2}{\delta u^2+\delta v^2}+$$
$$+\frac{[z(u+\delta u, v+\delta v)-z(u,v)]^2}{\delta u^2+\delta v^2}\leqslant 1.$$

Posons

$$\xi(u_0, v_0, \varphi, \varrho)=\frac{x(u_0+\varrho\cos\varphi,\ v_0+\varrho\sin\varphi)-x(u_0,v_0)}{\varrho},$$

et définissons de façon analogue les fonctions $\eta(u_0, v_0, \varphi, \varrho)$, $\zeta(u_0, v_0, \varphi, \varrho)$. Il résulte de l'inégalité précédente que ces trois fonctions sont, en valeur absolue, au plus égales à 1.