Finite arc-sums.

By

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The principal contribution of the present paper is a set of conditions which are necessary and sufficient in order that a continuous curve should be the sum of a finite number of arcs. In a recent paper [1]), Whyburn considered the analogous problem for a countable number of arcs. He obtained a solution by reducing the problem to the same one for the true cyclic elements of the continuous curve. A like reduction is made in this, the finite case. Conditions analogous to those of Whyburn's paper are given, wherein the word finite is used instead of countable. The conditions obtained in this manner will be shown not to be sufficient to characterize a finite arc-sum; but, due to the highly specialized nature of this type of curve, to require additional conditions to complete the sufficiency. For particular kinds of continuous curves, such as the acyclic curve and the boundary of a simply connected domain in the plane, exceedingly simple conditions are obtained.

The problem naturally arises as to the degree of complexity possible in a subset of a finite number of arcs. In theorem 6 we answer this question by showing that any closed and bounded set M in E_n is a subset of *two* arcs if it contains no continuum of condensation. It is obvious that this is, in a sense, an extension of the Moore-Kline theorem giving the necessary and sufficient conditions in order that a closed and bounded set M should be a subset of *one* arc [2]). Furthermore, it is an extension to any finite

[1]) G. T. Whyburn, *Continuous curves and arc-sums,* Fund. Math., vol 14 (1929), pp. 103–106.

[2]) R. L. Moore and J. R. Kline, *On the most general plane closed point-set through which it is possible to pass a simple continuous arc,* Annals of Math. vol. 20, pp. 218–223.

number of arcs, for the theorem implies that any point set which is a subset of n arcs is a subset of *two* arcs

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The following lemma will be found useful.

Lemma 1. *If no one of the closed point sets $M_1, M_2, \dots, M_n$, contains a continuum of condensation, then the set $M_1+M_2+\dots+M_n$ contains no continuum of condensation* [3]).

Let C be a continuum contained in the set M_1+M_2. The set of limit points of $M_1 - C \cdot M_1$ contained in C forms a closed and totally disconnected set; likewise, the limit points of $M_2 - C \cdot M_2$ in C forms a closed and totally disconnected set. Since, as is well-known, the sum of two closed and totally disconnected sets is itself closed and totally disconnected, some point of C is not a limit point of $M_1 + M_2 - C$. The lemma follows by induction.

For the sake of completeness, we first state the following obvious consequence of Lemma 1.

Theorem 1. *If the set M is the sum of a finite number of arcs, then M contains no continuum of condensation.*

By virtue of Theorem 1 and well-known properties of continua that are not continuous curves, we have:

Theorem 2. *If the continuum M is the sum of a finite number of arcs, then M is a perfect continuous curve* [4]).

The properties involved in the following two theorems were proved by Urysohn [5]) to be properties of any continuum containing no continuum of condensation. By virtue of Theorem 1, they are properties of a finite arc-sum.

[3]) A special case of this lemma was given by S. Janiszewski, *Sur les continus irréductibles entre deux points*, Journ. Ecole Polyt., (1912), pp. 79–170 Th. VII.

[4]) A *perfect continuous curve* is a continuous curve whose every subcontinuum is a continuous curve.

[5]) P. Urysohn, *Mémoire sur les multiplicités cantoriennes*, II, Verhand. Kon. Akad. v. Wet. te Amsterdam, Eerste Sectie, vol. 13 (1928), No. 4. See especially pp. 57 and 69.

Theorem 3. *If the continuum M is the sum of a finite number of arcs, then M is a regular curve.*

Theorem 4. *If the continuum M is the sum of a finite number of arcs, then M can be decomposed in a unique manner into the sum of a closed, totally disconnected set F and a set, at most denumerable, of open, maximal, free arcs that are mutually exclusive. These free, open arcs contain no points of F, but their end points belong to F, and if they are infinite in number, their diameters converge to zero.*

Theorem 5. *For the bounded continuum M in E_n to be a subset of two arcs of E_n, it is necessary and sufficient that M contain no continuum of condensation.*

The necessity of the condition follows from Theorem 1.

The condition is sufficient: Let M be a continuum containing no continuum of condensation. Then M can be decomposed in a unique manner into the sum of a closed, totally disconnected set F and a set, at most denumerable, of open, maximal, free arcs that are mutually exclusive. These free, open arcs contain no points of F, but their end points belong to F, and if they are infinite in number, their diameters converge to zero [6]). Let t_i be one of these open, free arcs; and let P and Q be the end points of t_i, in the order $P<Q$ on t_i. Let $P_1, P_2, P_3, \dots$ be a sequence of points of t_i having P as a sequential limit point, such that on t_i they are in the order $P_1>P_2>P_3>\dots>P$. Let $Q_1, Q_2, Q_3, \dots$ be a sequence of points of t_i having Q as a sequential limit point, such that on t_i they are in the order $P_1<Q_1<Q_2<Q_3<\dots<Q$. Let the set K_{1i} consist of the following arcs of t_i: P_1Q_1, P_2P_3, Q_2Q_3, P_4P_5, $Q_4Q_5, \dots$ etc. Let the set K_{2i} consist of the remaining arcs of t_i, namely: P_1P_2, Q_1Q_2, P_3P_4, Q_3Q_4, $P_5P_6, \dots$ etc. Let the sets K_1 and K_2 be defined as follows: $K_1=\overline{\sum_i K_{1i}}$, $K_2=\overline{\sum_i K_{2i}}$. It is evident that every component of K_1 is either a point (of F) or an arc (of some t_i), and no interior point of an arc component is a limit point of points of K_1 not on that component. By the theorem [6]) of Moore and Kline referred to in the introduction, K_1 is a subset of an arc.

[6]) For the generalization of this theorem to n dimensions, cf. E. W. Miller, *On subsets of a continuous curve which lie on an arc of the continuous curve*, Amer. Jour. of Math., vol. 54 (1932), pp. 397—416, Theorem I.

Likewise K_2 is a subset of an arc. As $K_1+K_2=M$, it follows that M is a subset of two arcs.

Theorem 5 seems hardly intuitively evident. Consider, for example, the well-known acyclic curve whose end points are identically the Cantor terniary set on the unit interval. As the curve contains no continuum of condensation it is, by Theorem 5, a subset of the sum of two arcs.

The following corollary makes Theorem 5 seem even more surprising.

Corollary. *There exist, in the plane, two arcs whose sum contains a connected subset which is not arcwise connected.*

This follows from Theorem 5 in view of an example [7]) due to Whyburn of a continuum, in the plane, containing no continuum of condensation, which contains, nevertheless, a connected subset not arcwise connected.

It is possible, however, to prove a much more general theorem as follows:

Theorem 6. *For the closed and bounded point set M in E_n to be a subset of two arcs of E_n, it is necessary and sufficient that M contain no continuum of condensation.*

The necessity again follows from Theorem 1.

The condition is sufficient: Let K be a set which consists of one point from each component of M. The set $\overline{K}$ is closed, bounded and totally disconnected (for if $\overline{K}$ contains a continuum C, then since $C\subset M$, $C\cdot K$ is at most one point and therefore C would be a continuum of condensation of $\overline{K}$ and of M). By the above mentioned theorem of Moore and Kline, $\overline{K}$ is a subset of an arc t. By Lemma 1, $M+t$ is a continuum containing no continuum of condensation. By the preceding theorem, $M+t$, and therefore M, is a subset of two arcs.

The proof of the following lemma is quite obvious.

[7]) G. T. Whyburn, *Sets of local separating points of a continuum*, Bull. Amer. Math. Soc., vol. 39 (1933), pp. 97—100.

Lemma 2. If P is a cut point of the continuous curve M, and D is a component of $M - P$ having a point in common with an arc t of M, then D contains an end point of the arc t.

Theorem 7. If the continuous curve M is the sum of a finite number of arcs, then the nodes[8]*) of M are finite in number.*

The nodes of M fall into two classes: the end points[9]) of M, and all true cyclic elements containing only one cut point of M. It is evident from the definition that every end point of M must be an end point of at least one of the finite set of arcs making up M. Thus the end points number at most $2n$ if M is the sum of n arcs. Let C be a true cyclic element of M containing only one point P which cuts M. $C-P$ is not a subset of one arc since $C-P=C$ contains a simple closed curve. So $C-P$ has points in common with at least two of the n arcs making up M. The set $C-P$ is a component of $M-P$, hence, by Lemma 2, $C-P$ contains two of the $2n$ end points of the n arcs whose sum is M. Thus, there can exist at most n true cyclic elements of M containing but one cut point of M. In fact we have shown that the number of end points plus twice the number of remaining nodes cannot be greater than $2n$; for, if this number is as great as $2n$, we shall have accounted for all $2n$ end points of the n arcs whose sum is M.

Theorem 8. If the continuous curve M is the sum of n arcs, then every true cyclic element of M is the sum of n arcs; and, for all but a finite number of true cyclic elements, these arcs begin and end at cut points of M.

[8]) A *node* of a continuous curve is a cyclic element of the continuous curve which neither is itself a cut point nor contains more than one cut point of the continuous curve. Cf. G. T. Whyburn, *Concerning the structure of a continuous curve*, Amer. Jour. of Math., vol. 50 (1928), p. 178.

[9]) An *end point*, of a continuous curve M is a point of M contained in no arc of M as an interior point. The definition given here is the one most convenient for the proof. It has been shown by W. L. Ayres that an end point in this sense is equivalent to an end point in the ordinary sense. Cf. W. L. Ayres, *Concerning continuous curves in metric space*, Amer. Jour. of Math. vol. 51 (1929), pp. 577—594, Theorem 5.

In the theory of cyclic elements, as developed by Whyburn[10]), a true cyclic element has the property that, for every pair of its points, x and y, it contains every arc of the continuous curve from x to y. Let C be a true cyclic element of M, and $t_1, t_2, \dots, t_n$ be the arcs of which M is the sum. Let x_i and y_i be the first and last points respectively of C on the arc t_i. Since C contains every arc of M joining x_i and y_i, it must contain the sub-arc of t_i joining x_i and y_i. It is easy to see that x_i is a cut point of M if it is not an end point of t_i; similarly y_i is a cut point of M if it is not an end point of t_i. Since there exist only $2n$ end points of the arcs t_i, for all but $2n$ of the true cyclic elements, the end points of these sub-arcs are cut points of M. If an arc t_j has but one point in common with C, join this point to some other point of C by an arc; in case C is not a node, let this other point be a cut point of M. If an arc t_k has no points in common with C, corresponding to t_k select some arc of C; in case C is not a node, select this arc so that its end points are cut points of M. Evidently C is the sum of the n arcs constructed in this manner. Since the nodes are finite in number, by Theorem 7, it is evident that, for all but a finite number of true cyclic elements, the end points of these arcs are cut points of M.

By the *order* of a cyclic element we shall mean the number of complementary components of the cyclic element Thus, end points and end elements are of order one.

Theorem 9. If the continuous curve M is the sum of n arcs, then the order of any cyclic element of M is at most 2n.

Let C be a cyclic element of M, and D a component of $M-C$. D is a complementary component of a cut point of M, namely, its boundary in C. By Lemma 2, it follows that D contains at least one end point of the $2n$ end points of the n arcs whose sum makes up M. Thus, there can exist at most $2n$ components of $M-C$.

Theorem 10. If the continuous curve M is the sum of n arcs, and C is a true cyclic element of M of order 2n, then C is the only true cyclic element of M.

[10]) For an exposition of this theory cf. C. Kuratowski and G T. Whyburn, *Sur les éléments cycliques et leurs applications*, Fund. Math., vol. 16 (1930), pp. 305—331.

Since there are $2n$ components of $M-C$, each component contains one and only one end point of the $2n$ end points of the n arcs whose sum is M. By Lemma 2, it follows that each component of $M-C$ can contain points in common with but one of the n arcs, and is, therefore, a subset of that arc. Thus, every component of $M-C$ is a semi-open arc, and C is the only true cyclic element of M.

Theorem 11. *If the acyclic continuous curve M is the sum of n arcs, then the branch points of M number at most $2n-2$.*

Let t_1 be one of the n arcs; set $t_1 = M_1$. Since M is connected, t_1 has points in common with at least one of the remaining $n-1$ arcs. Let such an arc be t_2; set $t_1 + t_2 = M_2$. In general, since M is connected, M_k must contain points in common with at least one of the remaining $n-k$ arcs. Let such an arc be t_{k+1}; set $t_1 + t_2 + \ldots + t_{k+1} = M_{k+1}$. Evidently each of the sets $M_1, M_2, \ldots, M_n = M$ is an acyclic curve, since each is a subcontinuum of an acyclic curve; and each is obtained from its predecessor by the addition of an arc. The theorem is true for the acyclic curve M_1. Let us assume the truth of the theorem for the acyclic curve M_k; we shall show that the theorem is true for the acyclic curve M_{k+1}. If M_k has two points in common with t_{k+1}, every point of t_{k+1} between these two points must also be point of M_k, otherwise M_{k+1} would contain a simple closed curve. Therefore the only branch points which can be created by the addition of t_{k+1} to M_k are the first and last points of M_k on the arc t_{k+1}. So M_{k+1} contains, at most, two branch points more than M_k.

The analogy existing between acyclic curves and continuous curves when considered in terms of their cyclic elements has been stressed by several writers, particularly by Whyburn. It is of interest to observe that the analogy persists in a very detailed fashion when we compare acyclic curves which are the sum of a finite number of arcs with general continuous curves which are the sum of a finite number of arcs. Since the node of the general continuous curve is the analogue of the end point of the acyclic curve, the theorem that the end points of an acyclic curve are finite in number, if the curve is the sum of a finite number of arcs, suggests the analogous theorem that the nodes of a continuous curve are finite in number under the same hypothesis. This we proved in Theorem 7.

Let us define a *branch element* as a cyclic element belonging to no other cyclic element such that the order of the given cyclic element is greater than two. Evidently a branch element is the analogue of the branch point of the acyclic curve. Analogous to Theorem 11, we would have the following theorem: *If the continuous curve M is the sum of n arcs, then the branch elements of M number at most $2n-2$.* In Theorem 12 we shall not prove quite so much, but the theorem will be found more useful in the proof of Theorem 13. But, first, we will need the following lemma.

Lemma 3. *If P is a cut point of the continuous curve M, and D is a component of $M-P$, then $D+P$ contains at least one node of M.*

If $D+P$ contains no cut point of itself, then $D+P$ is itself a node of M. Suppose, then, that $D+P$ contains a cut point of itself. Then $D+P$ is a continuum which is the sum of a collection of cyclic elements of M, and has more than one cyclic element of itself. By a theorem [11]) due to Whyburn, $D+P$ contains at least two nodes of itself. Since P is a non-cut point of $D+P$, not more than one of these nodes can contain P; the other node is evidently a node of M.

Theorem 12. *If the nodes of the continuous curve M are finite in number, then the branch elements of M are finite in number.*

Under the hypothesis of the theorem, the order of a cyclic element of M is always finite. For suppose C is a cyclic element of M such that $M-C$ has infinitely many components, $D_1, D_2, \ldots$, where the boundary of D_i is a single point P_i of C; by Lemma 3, $D_i + P_i$ contains a node of M for every i. But this means that M contains infinitely many nodes. Hence, the order of a cyclic element of M is always finite.

Suppose, contrary to the conclusion of the theorem, that M contains infinitely many branch elements: $C_1, C_2, C_3, \ldots$ Let $C_{n_1} = C_1$. Since C_{n_1} has only a finite number of complementary components,

[11]) G. T. Whyburn, *Concerning the structure of a continuous curve*, loc. cit., Theorem 14.

it can have points in common with only a finite number of the branch elements of the above sequence; for, suppose two branch elements C_j and C_k of the same complementary component of C_{n_1} contain the boundary point P of the component in common with C_{n_1}; but this is impossible since P must cut M between C_j and C_k. Consequently, infinitely many branch elements of M lie wholly in some component of $M-C_{n_1}$; denote such a component by X_1. Let C_{n_2} be the branch element of the above sequence of lowest subscript lying in X_1. By reasoning similar to the above, some component of $X_1-C_{n_2}$ must contain infinitely many branch elements of the above sequence; denote this component by X_2. Let C_{n_3} be the branch element of the above sequence of lowest subscript lying in X_2. In general, let C_{n_i} be the branch element of the above sequence of lowest subscript lying in X_{i-1}. Some component of $X_{i-1}-C_{n_i}$ must contain infinitely many branch elements of the above sequence; denote this component by X_i. In this manner we define an infinite subsequence of branch elements: $C_{n_1}, C_{n_2}, \dots$, and an infinite set of connected, open sets: $X_1, X_2, \dots$, such that: $X_i \cdot \sum_{k=1}^{i} C_{n_k} = 0$, $X_i \supset \sum_{k=i+1}^{\infty} C_{n_k}$, and $X_i \supset \sum_{k=i}^{\infty} X_k$. Since X_{j+1} is open in X_j and has only one boundary point in X_j, it follows that $X_j - X_{j+1}$ is a connected set. Also $M-X_1$ is connected. Since

$$M-X_i=(M-X_1)+(X_1-X_2)+(X_2-X_3)+\dots+(X_{i-1}-X_i),$$

it follows that $M-X_i$ hat at most i components.

Let us now proceed to define an infinite set of mutually exclusive domains of M, such that the boundary of each domain is a single point. Since C_{n_1} is a branch element it has at least three complementary components; denote by D_1 and D_2 the two components not containing the connected set X_1. Proceed to C_{n_2}; denote by D_3 that component of $M-C_{n_2}$ having no points in common with either of the connected sets $M-X_1$ or X_2. If, after considering each of the branch elements $C_{n_1}, C_{n_2}, \dots, C_{n_i}$ in order, we have obtained k domains $D_1, D_2, \dots, D_k$ of our sequence, and one of the components of $M-C_{n_{i+1}}$ contains no points of either of the sets $M-X_i$ or X_{i+1}, then D_{k+1} shall be that component. If the set $M-X_i$ is connected, it follows, since X_{i+1} is connected and $M-C_{n_{i+1}}$ has at least three components, that such a component must exist. We have to consider the case where $M-X_i$ is not connected, and every component of $M-C_{n_{i+1}}$ has points in common with either $M-X_i$ or X_{i+1}. Suppose $M-X_i$ has k components $(1<k\leq i)$. The components of $M-C_{n_{i+1}}$ not containing X_{i+1} are joined by the connected set X_i-X_{i+1} in $M-X_{i+1}$. Each of these components contains at least one component of $M-X_i$. Therefore, at least two components of $M-X_i$ are joined by the connected set X_i-X_{i+1} in $M-X_{i+1}$. Since $M-X_{i+1}=(M-X_i)+(X_i-X_{i+1})$, it follows that $M-X_{i+1}$ has at least one less component than $M-X_i$ (i. e. at most $k-1$ components). Again, if no component of $M-C_{n_{i+2}}$ contains no point of either of the sets $M-X_{i+1}$ or X_{i+2}, then the components of $M-X_{i+2}$ number at most $k-2$. Proceeding in this fashion we shall either find a complementary component of some branch element which, by definition, will be D_{k+1}, or we shall reach the branch element $C_{n_{i+k}}$ which will have a complementary component not containing points of either of the sets X_{i+k} or $M-X_{i+(k-1)}$; for $M-X_{i+(k-1)}$ will be connected, since it will have at most $k-(k-1)=1$ components. This component of $M-C_{n_{i+k}}$ is, by definition, D_{k+1}.

Thus, under the supposition that there exists an infinite set of branch elements of M, we have proven the existence of an infinite set of mutually exclusive domains each having but one point for a boundary. By Lemma 3, each of these domains plus its boundary point contains a node of M. But this implies the existence of an infinite set of distinct nodes of M, contrary to the hypothesis of the theorem. The contradiction proves the theorem.

Corollary. *If the continuous curve M is the sum of a finite number of arcs, then the branch elements of M are finite in number.*

This follows from Theorem 12 in view of Theorem 7.

Theorem 13. *In order that the continuous curve M should be the sum of a finite number of arcs, it is necessary and sufficient that* (1) *the end points of M should be finite in number,* (2) *each true cyclic element should be the sum of a finite number of arcs, in every case numbering not more than a fixed integer n, and for all but a finite number of the true cyclic elements these arcs should begin and end at cut points of M*

In Theorems 7 and 8 we have shown the conditions to be necessary.

The conditions are sufficient: According to the cyclic element theory [10]), a continuous curve is the sum of its end points, its cut points, and its true cyclic elements. We shall show that each of these sets in turn is a subset of a finite number of arcs lying in M.

The end points of M are a subset of a finite number of arcs of M. It is sufficient to choose some end point of M and join it by arcs of M to each of the other end points.

The cut points of M are a subset of a finite number of arcs of M. Let $P_1, P_2, \ldots, P_j$ be the end points of M. Since a node has at most one cut point, it is evident, if condition 2 is to be satisfied, that the nodes of M must be finite in number. Let the nodes of M that are true cyclic elements be $N_1, N_2, \ldots, N_k$. Let P_{j+1} be a non-cut point of M in N_1; let P_{j+2} be a non-cut point of M in N_2; etc. The case where M contains no cut point is trivial. Assuming, then, that M contains a cut point, we have from Lemma 3 that M contains at least two nodes. Consequently, the set $P_1, P_2, \ldots, P_{j+k}$ contains at least two points. Let P be some point of this set. Join P by a finite number of arcs of M to each of the other points of this set. Let Q be any cut point of M. Then $M-Q = D_1 + D_2 + D_3 + \ldots$ where the D's are components and D_1 contains P. By Lemma 3, $D_2 + Q$ contains at least one node of M; hence, D_2 contains at least one point other than P of the set $P_1, P_2, \ldots, P_{j+k}$. The arc of the above set joining P to this point must necessarily contain Q. Thus, the cut points of M are a subset of a finite number of arcs of M.

The true cyclic elements of M are a subset of a finite number of arcs of M. If the true cyclic elements are finite in number the proposition is immediate. Suppose M contains an infinity of true cyclic elements. As pointed out above, the nodes of M are finite in number. By Theorem 12, we have immediately that the branch elements of M are finite in number. If C is a true cyclic element such that the arcs of which C is the sum do not all begin and end at cut points of M, then condition 2 states that the class of all true cyclic elements such as C forms a finite set. These finite cases may obviously be disregarded, since they are the sum of a finite number of arcs of M. There only remains for consideration an infinity of true cyclic elements $C_1, C_2, \ldots$ of M such that, for every i, C_i has exactly two complementary components, and contains exactly two points P_i and Q_i which cut M, and C_i is the sum of n or less arcs all of which join P_i to Q_i. Let $t_1, t_2, \ldots, t_k$ be the finite set of arcs of M of which the cut points are a subset. Associate with t_1 every true cyclic element of the above sequence for which t_1 contains P_i or Q_i or both. Of those that remain, associate with t_2 those for which t_2 contains P_i and Q_i or both; etc. Thus, every true cyclic element of the above sequence is associated with one and only one of the arcs t_i. If t_i has only a finite number of true cyclic elements associated with it, they are all, obviously, the sum of a finite number of arcs of M. Suppose t_i has infinitely many true cyclic elements associated with it. Now the end points of t_i are either end points of M or non-cut points of certain nodes of M. So, if t_i contains a non-cut point, of some true cyclic element C_j, t_i will have to contain both P_j and Q_j. Furthermore, in only a finite number of cases can t_i contain, for some C_j, P_j or Q_j and not the other. For, suppose t_i contains infinitely many of the points P_j and not one of the corresponding points Q_j. For every j, let D_j be the component of $M-P_j$ containing $C_j - P_j$. Let us show the domains ΣD_j to be mutually exclusive. Since t_i does not contain Q_j and contains a point P_j exterior to that component of $M - C_j$ having Q_j for a boundary, it follows that t_i has no point in common with that component. Now D_j is the sum of $C_j - P_j$ and the component of $M - C_j$ having Q_j as its boundary. Consequently, since $t_i \cdot (C_j - P_j) = 0$, we have $t_i \cdot D_j = 0$. Suppose two domains D_h and D_k have points in common. Since $t_i \cdot (D_h + D_k) = 0$, and since $t_i \supset P_h + P_k$, it follows that neither D_h nor D_k contains the boundary point of the other. But this implies both $D_h \supset D_k$ and $D_k \supset D_h$ which implies the identity of the true cyclic elements C_h and C_k. Thus, the domains ΣD_j are mutually exclusive. By Lemma 3, $D_j + P_j$ contains a node of M; this would mean that M contains infinitely many nodes, which is impossible. Thus, in only a finite number of cases does t_i contain P_j or Q_j and fail to contain the other. Neglecting those true cyclic elements associated with t_i for which this occurs, we have left a countable collection of true cyclic elements $C_1^i, C_2^i, \ldots$ associated with t_i such that t_i contains, for every j, both P_j^i and Q_j^i. Assign an order to the n or less arcs of which C_j^i is the sum. Evidently t_i has the arc $P_j^i Q_j^i$ in common with C_j^i. For every j, replace the arc $P_j^i Q_j^i$ of t_i by the arc of C_j^i numbered 1. Let t_i^1 be the set constructed in this manner. Since the replacement arcs are non-overlapping, and since their diameters converge to zero, it follows that t_i^1 is an arc. Construct the arc t_i^2 in a like manner by replacing, for every j, the arc $P_j^i Q_j^i$ of t_i by the arc of C_j^i numbered 2. In general, construct the arc t_i^k by replacing, for every j, the arc $P_j^i Q_j^i$ of t_i by the arc of C_j^i numbered k. In case no such arc of C_j^i exists the replacement is not made. At most n distinct arcs $t_i^1, t_i^2, \ldots, t_i^n$ can be constructed in this manner. It is obvious that the true cyclic elements associated with t_i are, except for a finite number of them, a subset of this set of n arcs. Thus, we have shown that all but a finite number of the true cyclic elements of M are a subset of the finite set of arcs $\sum_{i=l}^{h}\sum_{k=1}^{n} t_i^k$. This completes the proof of the theorem.

Since an acyclic curve contains no true cyclic elements, we have immediately:

Corollary. *For the acyclic continuous curve M to be the sum of a finite number of arcs, it is necessary and sufficient that the end points of M be finite in number.*

It may very well occur that the true cyclic element C of the continuous curve M, while being the sum of a finite number of arcs, is not the sum of any finite number of arcs whose end points are cut points of M. The following example illustrates the point. Let the set Q consist of the following arcs in the cartesian plane: the arc made up of the point $(0,0)$ and all points of the curve $y = x \sin 1/x$ $(0 < x \leqq 1/\pi)$; the arc $y = x$ $(0 \leqq x \leqq 1/\pi)$; the arc $y = -x$ $(0 \leqq x \leqq 1/\pi)$; and the arc $x = 1/\pi$ $(-1/\pi \leqq y \leqq 1/\pi)$. Let P, A and B be the points $(0,0)$, $(1/\pi, 1/\pi)$ and $(1/\pi, -1/\pi)$, respectively. Now Q is the sum of three arcs from P to A, or from P to B; but it is not the sum of any finite number of arcs from A to B. For, any arc of Q with an end point at A which contains a complete oscillation of the $x \sin 1/x$ curve finds its other end point separated by itself from B. It is a simple matter to construct a continuous curve M containing an infinity of true cyclic elements, all similar to the set Q, but whose diameters converge to zero, and such that the only cut points of M contained in each true cyclic element are the points corresponding to the points A and B of Q. Such a continuous curve would satisfy all but the latter part of condition 2 of Theorem 13; this example establishes

the independence of the condition. It is to be observed that the set Q is not the sum of any finite number of simple closed curves. This fact suggests the following series of theorems. We proceed to establish that the property of a cyclicly connected continuous curve of being the sum of a finite number of arcs from P to Q, for *every* pair of distinct points P and Q, and the property of being the sum of a finite number of simple closed curves are equivalent.

Theorem 14. *If the cyclicly connected continuous curve M is the sum of a finite number of simple closed curves, then, if R and Q are any pair of distinct points of M, M is the sum of a finite number of arcs whose end points are P and Q.*

Let J be a simple closed curve of M. Select the points P and Q. We shall show that J is a subset of two arcs of M whose end points are P and Q. If P and Q are points of J it is obvious that this is true. Suppose Q is on J and P is not. Let P' be a point of J distinct from Q. According to a theorem of Ayres[12]), M contains an arc $PP'Q$. Let P'' be the first point of this arc on J. Evidently $P'' \neq Q$. Then the subarc PP'' (of $PP'Q$) together with the two arcs of J from P'' to Q evidently form two arcs of M from P to Q whose sum contains J.

Suppose, then, that neither P nor Q is on J. Let X be a point of J. As above, M contains an arc PXQ. Let P' and Q' be the first and last points respectively of J on the arc PXQ. If $P' \neq Q'$, then the subarc PP' (of PXQ) plus the subarc $Q'Q$ (of PXQ) plus the two arcs of J from P' to Q' evidently form two arcs of M from P to Q whose sum contains J. Consider the case $P' = Q' = X$. Let X' be a point of J distinct from X. As above, M contains an arc $PX'Q$. If, as we may assume, X' is the only point J has in common with $PX'Q$, let Y be the last point of the subarc PX' (of $PX'Q$) on the arc PXQ. Suppose Y is on the open interval XQ of PXQ; then the subarc YQ (of PXQ) plus the subarc YX' (of $PX'Q$) forms an arc QX' having no point in common with the arc PX (of PXQ). Suppose, next, that Y is on the interval PX of PXQ; then the subarc PY (of PXQ) plus the subarc YX'

[12]) W. L. Ayres, *Continuous curves which are cyclicly connected*, Bulletin de l'Académie Polonaise des Sciences et des Lettres, (1928), pp. 127—142.

(of $PX'Q$) forms an arc PX' having no point in common with the arc XQ (of PXQ). In any case M contains two mutually exclusive arcs from P to J and from Q to J. These two arcs plus the two arcs of J from X to X' evidently form two arcs from P to Q whose sum contains J. Thus, J is a subset of the sum of two arcs from P to Q for every pair of distinct points P and Q. And, if M is the sum of n simple closed curves, M is the sum of $2n$ arcs whose end points are P and Q.

Before proceeding to the converse, we must, first, prove a lemma.

Lemma 4. *If X is an interior point of an arc t of the cyclicly connected continuous curve M, then X is contained in an open interval I_x of t such that there exists a simple closed curve J_x of M containing I_x.*

Let A and B be the end points of t. Since $M-X$ is connected, $M-X$ contains an arc t' from A to B. The two arcs t and t' evidently contain a simple closed curve J_x containing X; and obviously J_x contains an open interval I_x of t containing X.

Theorem 15. *If t is an arc of the cyclicly connected continuous curve M, then t is a subset of a countable number of simple closed curves of M.*

Let A and B be the end points of t. The open interval $t-A-B$ can be covered by a countable number of the intervals I_x of Lemma 4; since A and B lie together on a simple closed curve of M, we have immediately that t is a subset of a countable number of simple closed curves of M.

Corollary. *If the cyclicly connected continuous curve M is the sum of a finite or countable number of arcs, then M is the sum of a countable number of simple closed curves.*

Theorem 16. *If the cyclicly connected continuous curve M has the property that, for every pair of distinct points P and Q of M. M is the sum of a finite number of arcs whose end points are P and Q, then M is the sum of a finite number of simple closed curves.*

Let X be a point of M. Choose P and Q to be any pair of distinct points of M other than X. Then M is the sum of n arcs $t_1, t_2, \ldots, t_n$ whose end points are P and Q. Let $t_{x_1}, t_{x_2}, \ldots, t_{x_k}$ be the arcs of this set containing X. By Lemma 4, for every arc t_{x_i}, there exists an open interval I_{x_i} of t_{x_i} containing X, and a simple closed curve J_{x_i} of M containing I_{x_i}. We have then: $\sum_{i=1}^{k} J_{x_i} \supset \sum_{i=1}^{k} I_{x_i}$. Let S_x be a neighborhood of X (relative to M) containing no point of the closed set $M-\sum_{i=1}^{k} I_{x_i}$. We must then have: $S_x \subset \sum_{i=1}^{k} J_{x_i}$. Cover every point X of M by the corresponding region S_x of M. Applying the Borel theorem, we find that M is covered by a finite number of the regions ΣS_x. And since each region S_x is covered by a finite number of simple closed curves of M, we have that M is the sum of a finite number of simple closed curves.

Theorem 17. *In order that the continuous curve M should be the sum of a finite number of arcs, it is sufficient that* (1) *the nodes of M be finite in number,* (2) *each true cyclic element be the sum of a finite number of simple closed curves, numbering in every case not more than a fixed integer n.*

Condition 1 of Theorem 13 is satisfied. Let C be a true cyclic element of M. By Theorem 14, C is the sum of $2n$ arcs whose end points are P and Q where P and Q are any pair of distinct points of M. Whenever possible select P and Q so that they may be cut points of M; this may be done in every case where C is not a node. Evidently condition 2 of Theorem 13 is satisfied.

Theorem 18. *For the boundary M of a bounded, simply connected domain in the plane to be the sum of a finite number of arcs, it is necessary and sufficient that the nodes of M be finite in number.*

The necessity follows from Theorem 7, the sufficiency from Theorem 17 and the fact that the true cyclic elements of such a boundary are themselves simple closed curves [13]).

[13]) See R. L. Wilder, *Concerning continuous curves*, Fund. Math., vol. 7 (1925), pp. 340—377, Theorem 4.