Finite arc-sums.

By

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The principal contribution of the present paper is a set of conditions which are necessary and sufficient in order that a continuous curve should be the sum of a finite number of arcs. In a recent paper \(^{1}\), Whyburn considered the analogous problem for a countable number of arcs. He obtained a solution by reducing the problem to the same one for the true cyclic elements of the continuous curve. A like reduction is made in this, the finite case. Conditions analogous to those of Whyburn's paper are given, wherein the word finite is used instead of countable. The conditions obtained in this manner will be shown not to be sufficient to characterize a finite arc-sum; but, due to the highly specialized nature of this type of curve, to require additional conditions to complete the sufficiency. For particular kinds of continuous curves, such as the acyclic curve and the boundary of a simply connected domain in the plane, exceedingly simple conditions are obtained.

The problem naturally arises as to the degree of complexity possible in a subset of a finite number of arcs. In theorem 6 we answer this question by showing that any closed and bounded set \(M\) in \(E^n\) is a subset of two arcs if it contains no continuum of condensation. It is obvious that this is, in a sense, an extension of the Moore-Kline theorem giving the necessary and sufficient conditions in order that a closed and bounded set \(M\) should be a subset of one arc \(^2\). Furthermore, it is an extension to any finite number of arcs, for the theorem implies that any point set which is a subset of \(n\) arcs is a subset of two arcs.

I wish to express my thanks, and to acknowledge my indebtedness to Professor R. L. Wilder for his valuable suggestions and constant encouragement.

The following lemma will be found useful.

**Lemma 1.** If no one of the closed point sets \(M_1, M_2, \ldots, M_n\), contains a continuum of condensation, then the set \(M_1 + M_2 + \cdots + M_n\) contains no continuum of condensation \(^3\).

Let \(C\) be a continuum contained in the set \(M_1 + M_n\). The set of limit points of \(M_i - C\cdot M_j\) contained in \(C\) forms a closed and totally disconnected set; likewise, the limit points of \(M_n - C\cdot M_n\) in \(C\) forms a closed and totally disconnected set. Since, as is well-known, the sum of two closed and totally disconnected sets is itself closed and totally disconnected, some point of \(C\) is not a limit point of \(M_1 + M_n - C\). The lemma follows by induction.

For the sake of completeness, we first state the following obvious consequence of Lemma 1.

**Theorem 1.** If the set \(M\) is the sum of a finite number of arcs, then \(M\) contains no continuum of condensation.

By virtue of Theorem 1 and well-known properties of continua that are not continuous curves, we have:

**Theorem 2.** If the continuum \(M\) is the sum of a finite number of arcs, then \(M\) is a perfect continuous curve \(^4\).

The properties involved in the following two theorems were proved by Urysohn \(^5\) to be properties of any continuum containing no continuum of condensation. By virtue of Theorem 1, they are properties of a finite arc-sum.

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\(^3\) A special case of this lemma was given by S. Janiszewski, *Sur les continus irreductibles entre deux points*, Journ. Ecole Polyt., (1912), pp. 79—170 Th. VII.

\(^4\) A perfect continuous curve is a continuous curve whose every subcontinuum is a continuous curve.

Theorem 3. If the continuum $M$ is the sum of a finite number of arcs, then $M$ is a regular curve.

Theorem 4. If the continuum $M$ is the sum of a finite number of arcs, then $M$ can be decomposed in a unique manner into the sum of a closed, totally disconnected set $F$ and a set, at most denumerable, of open, maximal, free arcs that are mutually exclusive. These free, open arcs contain no points of $F$, but their end points belong to $F$, and if they are infinite in number, their diameters converge to zero.

Theorem 5. For the bounded continuum $M$ in $E_n$ to be a subset of two arcs of $E_n$, it is necessary and sufficient that $M$ contain no continuum of condensation.

The necessity of the condition follows from Theorem 1. The condition is sufficient: Let $M$ be a continuum containing no continuum of condensation. Then $M$ can be decomposed in a unique manner into the sum of a closed, totally disconnected set $F$ and a set, at most denumerable, of open, maximal, free arcs that are mutually exclusive. These free, open arcs contain no points of $F$, but their end points belong to $F$, and if they are infinite in number, their diameters converge to zero. Let $t_i$ be one of these open, free arcs; and let $P$ and $Q$ be the end points of $t_i$, in the order $P < Q$ on $t_i$. Let $P_1, P_2, P_3, \ldots$ be a sequence of points of $t_i$ having $P$ as a sequential limit point, such that on $t_i$ they are in the order $P_1 > P_2 > P_3 > \ldots > P$. Let $Q_1, Q_2, Q_3, \ldots$ be a sequence of points of $t_i$ having $Q$ as a sequential limit point, such that on $t_i$ they are in the order $Q_1 < Q_2 < Q_3 < \ldots < Q$. Let the set $K_i$ consist of the following arcs of $t_i$: $P_1 Q_1, P_2 Q_2, P_3 Q_3, \ldots$ etc. Let the set $K$ consist of the remaining arcs of $t_i$, namely: $P_1 Q_1, P_2 Q_2, P_3 Q_3, \ldots$ etc. Let the sets $K_1$ and $K_2$ be defined as follows: $K_1 = \sum K_i$, $K_2 = \sum K_{i+1}$. It is evident that every component of $K_1$ is either a point (of $F$) or an arc of some $t_i$, and no interior point of an arc component is a limit point of points of $K_2$ not on that component. By the theorem of Moore and Kline referred to in the introduction, $K_1$ is a subset of an arc.

Likewise $K_2$ is a subset of an arc. As $K_1 + K_2 = M$, it follows that $M$ is a subset of two arcs.

Theorem 5 seems hardly intuitively evident. Consider, for example, the well-known acyclic curve whose end points are identical with the Cantor ternary set on the unit interval. As the curve contains no continuum of condensation it is, by Theorem 5, a subset of the sum of two arcs.

The following corollary makes Theorem 5 seem even more surprising.

Corollary. There exist, in the plane, two arcs whose sum contains a connected subset which is not arcwise connected.

This follows from Theorem 5 in view of an example due to Whyburn of a continuum, in the plane, containing no continuum of condensation, which contains, nevertheless, a connected subset not arcwise connected.

It is possible, however, to prove a much more general theorem as follows:

Theorem 6. For the closed and bounded point set $M$ in $E_n$ to be a subset of two arcs of $E_n$, it is necessary and sufficient that $M$ contain no continuum of condensation.

The necessity again follows from Theorem 1. The condition is sufficient: Let $K$ be a set which consists of one point from each component of $M$. The set $K$ is closed, bounded and totally disconnected (for if $K$ contains a continuum $C$, then since $C \subset M$, $C \cdot K$ is at most one point and therefore $C$ would be a continuum of condensation of $K$ and of $M$). By the above mentioned theorem of Moore and Kline, $K$ is a subset of an arc $t$. By Lemma 1, $M + t$ is a continuum containing no continuum of condensation. By the preceding theorem, $M + t$, and therefore $M$, is a subset of two arcs.

The proof of the following lemma is quite obvious.

\footnote{For the generalization of this theorem to $n$ dimensions, cf. E. W. Miller, On subsets of a continuous curve which lie on an arc of the continuous curve, Amer. Jour. of Math., vol. 54 (1932), pp. 397-410, Theorem 1.}

\footnote{G. T. Whyburn, Sets of local separating points of a continuum, Bull. Amer. Math. Soc., vol. 59 (1953), pp. 97-100.}
Lemma 2. If \( P \) is a cut point of the continuous curve \( M \), and \( D \) is a component of \( M - P \) having a point in common with an arc of \( M \), then \( D \) contains an end point of the arc.

Theorem 7. If the continuous curve \( M \) is the sum of a finite number of arcs, then the nodes\(^\text{a}\) of \( M \) are finite in number.

The nodes of \( M \) fall into two classes: the end points\(^\text{a}\) of \( M \), and all true cyclic elements containing only one cut point of \( M \). It is evident from the definition that every end point of \( M \) must be an end point of at least one of the finite set of arcs making up \( M \). Thus the end points number at most \( 2n \) if \( M \) is the sum of \( n \) arcs. Let \( C \) be a true cyclic element of \( M \) containing only one point \( P \), which cuts \( M \). \( C - P \) is not a subset of one arc since \( C - P \) contains a simple closed curve. So \( C - P \) has points in common with at least two of the \( n \) arcs making up \( M \). The set \( C - P \) is a component of \( M - P \), hence, by Lemma 2, \( C - P \) contains two of the \( 2n \) end points of the \( n \) arcs whose sum is \( M \). Thus, there can exist at most \( n \) true cyclic elements of \( M \) containing but one cut point of \( M \). In fact we have shown that the number of end points plus twice the number of remaining nodes cannot be greater than \( 2n \); for, if this number is as great as \( 2n \), we shall have accounted for all \( 2n \) end points of the \( n \) arcs whose sum is \( M \).

Theorem 8. If the continuous curve \( M \) is the sum of \( n \) arcs, then every true cyclic element of \( M \) is the sum of \( n \) arcs; and, for all but a finite number of true cyclic elements, these arcs begin and end at cut points of \( M \).

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In the theory of cyclic elements, as developed by Whyburn, a true cyclic element has the property that, for every pair of its points, \( x \) and \( y \), it contains every arc of the continuous curve from \( x \) to \( y \). Let \( C \) be a true cyclic element of \( M \), and \( t_1, t_2, ..., t_n \) be the arcs of which \( M \) is the sum. Let \( x_i \) and \( y_i \) be the first and last points respectively of \( C \) on the arc \( t_i \). Since \( C \) contains every arc of \( M \) joining \( x_i \) and \( y_i \), it must contain the sub-arc of \( t_i \) joining \( x_i \) and \( y_i \). It is easy to see that \( x_i \) is a cut point of \( M \) if it is not an end point of \( t_i \); similarly \( y_i \) is a cut point of \( M \) if it is not an end point of \( t_i \). Since there exist only \( 2n \) end points of the arcs \( t_i \), for all but \( 2n \) of the true cyclic elements, the end points of these sub-arcs are cut points of \( M \). If an arc \( t_i \) has but one point in common with \( C \), join this point to some other point of \( C \) by an arc; in case \( C \) is not a node, let this other point be a cut point of \( M \). If an arc \( t_i \) has no points in common with \( C \), corresponding to \( t_i \) select some arc of \( C \); in case \( C \) is not a node, select this arc so that its end points are cut points of \( M \). Evidently \( C \) is the sum of the \( n \) arcs constructed in this manner. Since the nodes are finite in number, by Theorem 7, it is evident that, for all but a finite number of true cyclic elements, the end points of these arcs are cut points of \( M \).

By the order of a cyclic element we shall mean the number of complementary components of the cyclic element. Thus, end points and end elements are of order one.

Theorem 9. If the continuous curve \( M \) is the sum of \( n \) arcs, then the order of any cyclic element of \( M \) is at most \( 2n \).

Let \( C \) be a cyclic element of \( M \), and \( D \) a component of \( M - C \). \( D \) is a complementary component of a cut point of \( M \), namely, its boundary in \( C \). By Lemma 2, it follows that \( D \) contains at least one end point of the \( 2n \) end points of the \( n \) arcs whose sum makes up \( M \). Thus, there can exist at most \( 2n \) components of \( M - C \).

Theorem 10. If the continuous curve \( M \) is the sum of \( n \) arcs, and \( C \) is a true cyclic element of \( M \) of order \( 2n \), then \( C \) is the only true cyclic element of \( M \).

Since there are $2^n$ components of $M - C$, each component contains one and only one end point of the $2^n$ end points of the $n$ arcs whose sum is $M$. By Lemma 2, it follows that each component of $M - C$ can contain points in common with but one of the $n$ arcs, and is, therefore, a subset of that arc. Thus, every component of $M - C$ is a semi-open arc, and $C$ is the only true cyclic element of $M$.

**Theorem 11.** If the acyclic continuous curve $M$ is the sum of $n$ arcs, then the branch points of $M$ number at most $2n - 2$.

Let $t_i$ be one of the $n$ arcs; set $t_i = M_i$. Since $M$ is connected, $t_i$ has points in common with at least one of the remaining $n - 1$ arcs. Let such an arc be $t_j$; set $t_i + t_j = M_i$. In general, since $M$ is connected, $M_i$ must contain points in common with at least one of the remaining $n - k$ arcs. Let such an arc be $t_{i+k}$; set $t_i + t_j + \ldots + t_{i+k} = M_{k+1}$. Evidently each of the sets $M_i, M_{i+1}, \ldots, M_k = M$ is an acyclic curve, since each is a subcontinuum of an acyclic curve; and each is obtained from its predecessor by the addition of an arc. The theorem is true for the acyclic curve $M_i$. Let us assume the truth of the theorem for the acyclic curve $M_k$; we shall show that the theorem is true for the acyclic curve $M_{k+1}$. If $M_{k+1}$ has two points in common with $t_{i+k}$, every point of $t_{i+k}$ between these two points must also be point of $M_k$, otherwise $M_{k+1}$ would contain a simple closed curve. Therefore the only branch points which can be created by the addition of $t_{i+k}$ to $M_k$ are the first and last points of $M_k$ on the arc $t_{i+k}$. So $M_{k+1}$ contains, at most, two branch points more than $M_k$.

The analogy existing between acyclic curves and continuous curves when considered in terms of their cyclic elements has been stressed by several writers, particularly by Whyburn. It is of interest to observe that the analogy persists in a very detailed fashion when we compare acyclic curves which are the sum of a finite number of arcs with general continuous curves which are the sum of a finite number of arcs. Since the node of the general continuous curve is the analogue of the end point of the acyclic curve, the theorem that the end points of an acyclic curve are finite in number, if the curve is the sum of a finite number of arcs, suggests the analogous theorem that the nodes of a continuous curve are finite in number under the same hypothesis. This we proved in Theorem 7.

Let us define a branch element as a cyclic element belonging to no other cyclic element such that the order of the given cyclic element is greater than two. Evidently a branch element is the analogue of the branch point of the acyclic curve. Analogous to Theorem 11, we would have the following theorem: If the continuous curve $M$ is the sum of $n$ arcs, then the branch elements of $M$ number at most $2n - 2$. In Theorem 12 we shall not prove quite so much, but the theorem will be found more useful in the proof of Theorem 13. But, first, we will need the following lemma.

**Lemma 3.** If $P$ is a cut point of the continuous curve $M$, and $D$ is a component of $M - P$, then $D + P$ contains at least one node of $M$.

If $D + P$ contains no cut point of itself, then $D + P$ is itself a node of $M$. Suppose, then, that $D + P$ contains a cut point of itself. Then $D + P$ is a continuum which is the sum of a collection of cyclic elements of $M$, and has more than one cyclic element of itself. By a theorem \(^\text{11)}\) due to Whyburn, $D + P$ contains at least two nodes of itself. Since $P$ is a non-cut point of $D + P$, not more than one of these nodes can contain $P$; the other node is evidently a node of $M$.

**Theorem 12.** If the nodes of the continuous curve $M$ are finite in number, then the branch elements of $M$ are finite in number.

Under the hypothesis of the theorem, the order of a cyclic element of $M$ is always finite. For suppose $C$ is a cyclic element of $M$ such that $M - C$ has infinitely many components, $D_1, D_2, \ldots$, where the boundary of $D_i$ is a single point $P_i$ of $C$; by Lemma 3, $D_i + P_i$ contains a node of $M$ for every $i$. But this means that $M$ contains infinitely many nodes. Hence, the order of a cyclic element of $M$ is always finite.

Suppose, contrary to the conclusion of the theorem, that $M$ contains infinitely many branch elements: $C_1, C_2, C_3, \ldots$. Let $C_n = C_1$. Since $C_n$ has only a finite number of complementary components,

it can have points in common with only a finite number of the branch elements of the above sequence; for, suppose two branch elements $C_j$ and $C_k$ of the same complementary component of $C_n$ contain the boundary point $P$ of the component in common with $C_n$; but this is impossible since $P$ must cut $M$ between $C_j$ and $C_k$. Consequently, infinitely many branch elements of $M$ lie wholly in some component of $M - C_n$; denote such a component by $X_1$. Let $C_n$ be the branch element of the above sequence of lowest subscript lying in $X_1$. By reasoning similar to the above, some component of $X_4 - C_n$ must contain infinitely many branch elements of the above sequence; denote this component by $X_4$. Let $C_n$ be the branch element of the above sequence of lowest subscript lying in $X_4$. In general, let $C_n$ be the branch element of the above sequence of lowest subscript lying in $X_{k-1}$. Some component of $X_{k-1} - C_n$ must contain infinitely many branch elements of the above sequence; denote this component by $X_k$. In this manner we define an infinite subsequence of branch elements: $C_n, C_m, C_m, \ldots$, and an infinite set of connected, open sets: $X_1, X_2, \ldots$, such that:

$$X_k = \bigcap_{k=1}^{\infty} C_n$$ and $X_1 \supset X_{k+1}$. Since $X_{k+1}$ is open in $X_k$ and has only one boundary point in $X_k$, it follows that $X_k - X_{k+1}$ is a connected set. Also $X_k - X_1$ is connected. Since $M - X_1 = (M - X_1) + (X_2 - X_1) + (X_3 - X_2) + \ldots + (X_k - X_{k-1})$, it follows that $M - X_1$ has at most $k$ components.

Let us now proceed to define an infinite set of mutually exclusive domains of $M$ such that the boundary of each domain is a single point. Since $C_n$ is a branch element it has at least three complementary components; denote by $D_1$ and $D_2$ the two components not containing the connected set $X_1$. Proceed to $C_1$; denote by $D_1$ that component of $M - C_1$ having no points in common with either of the connected sets $M - X_1$ or $X_1$. If, after considering each of the branch elements $C_n, C_m, \ldots, C_k$ in order, we have obtained $k$ domains $D_1, D_2, \ldots, D_k$ of our sequence, and one of the components of $M - C_{n+k}$ contains no points of either of the sets $M - X_1$ or $X_{n+k}$, then $D_{n+k}$ shall be that component. If the set $M - X_1$ is connected, it follows, since $X_{n+k}$ is connected and $M - C_{n+k}$ has at least three components, that such a component must exist.

We have to consider the case where $M - X_1$ is not connected, and every component of $M - C_{n+k}$ has points in common with either $M - X_1$ or $X_{n+k}$. Suppose $M - X_1$ has $k$ components ($1 \leq k \leq 1$). The components of $M - C_{n+k}$ not containing $X_{n+k}$ are joined by the connected set $X_1 - X_{n+k}$ in $M - X_{n+k}$. Each of these components contains at least one component of $M - X_1$. Therefore, at least two components of $M - X_1$ are joined by the connected set $X_1 - X_{n+k}$ in $M - X_{n+k}$. Since $M - X_{n+k} = (M - X_1) + (X_1 - X_{n+k})$, it follows that $M - X_{n+k}$ has at least one less component than $M - X_1$ (i.e. at most $k - 1$ components). Again, if no component of $M - C_{n+k}$ contains no point of either of the sets $M - X_{n+k}$ or $X_{n+k}$, then the components of $M - X_{n+k}$ number at most $k - 2$. Proceeding in this fashion we shall either find a complementary component of some branch element which, by definition, will be $D_{n+k}$, or we shall reach the branch element $C_{n+k}$ which will have a complementary component not containing points of either of the sets $X_{n+k}$ or $M - X_{n+k} - 1$; for $X_{n+k} - X_{n+k}$ will be connected, since it will have at most $k - (k - 1) = 1$ components. This component of $M - C_{n+k}$ is, by definition, $D_{n+k}$.

Thus, under the supposition that there exists an infinite set of branch elements of $M$, we have proven the existence of an infinite set of mutually exclusive domains each having but one point for a boundary. By Lemma 3, each of these domains plus its boundary point contains a node of $M$. But this implies the existence of an infinite set of distinct nodes of $M$, contrary to the hypothesis of the theorem. The contradiction proves the theorem.

**Corollary.** If the continuous curve $M$ is the sum of a finite number of arcs, then the branch elements of $M$ are finite in number.

This follows from Theorem 12 in view of Theorem 7.

**Theorem 13.** In order that the continuous curve $M$ should be the sum of a finite number of arcs, it is necessary and sufficient that (1) the end points of $M$ should be finite in number, (2) each true cyclic element should be the sum of a finite number of arcs, in every case numbering not more than a fixed integer $n$, and for all but a finite number of the true cyclic elements these arcs should begin and end at cut points of $M$.

In Theorems 7 and 8 we have shown the conditions to be necessary.
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The conditions are sufficient: According to the cyclic element theory \(1^4\), a continuous curve is the sum of its end points, its cut points, and its true cyclic elements. We shall show that each of these sets in turn is a subset of a finite number of arcs lying in \(M\).

The end points of \(M\) are a subset of a finite number of arcs of \(M\). It is sufficient to choose some end point of \(M\) and join it by arcs of \(M\) to each of the other end points.

The cut points of \(M\) are a subset of a finite number of arcs of \(M\). Let \(P_1, P_2, \ldots, P_j\) be the end points of \(M\). Since a node has at most one cut point, it is evident, if condition 2 is to be satisfied, that the nodes of \(M\) must be finite in number. Let the nodes of \(M\) that are true cyclic elements be \(N_1, N_2, \ldots, N_k\). Let \(P_{j+1}\) be a non-cut point of \(M\) in \(N_1\); let \(P_{j+2}\) be a non-cut point of \(M\) in \(N_2\); etc. The case where \(M\) contains no cut point is trivial. Assuming, then, that \(M\) contains a cut point, we have from Lemma 5 that \(M\) contains at least two nodes. Consequently, the set \(P_1, P_2, \ldots, P_{j+2}\) contains at least two points. Let \(P\) be some point of this set. Join \(P\) by a finite number of arcs of \(M\) to each of the other points of this set. Let \(Q\) be any cut point of \(M\). Then \(M - Q = D_1 + D_2 + D_3 + \ldots\) where the \(D_i\)'s are components and \(D_1\) contains \(P\). By Lemma 3, \(D_1 + Q\) contains at least one node of \(M\); hence, \(D_1\) contains at least one point other than \(P\) of the set \(P_1, P_2, \ldots, P_{j+2}\). The arc of the above set joining \(P\) to this point must necessarily contain \(Q\). Thus, the cut points of \(M\) are a subset of a finite number of arcs of \(M\).

The true cyclic elements of \(M\) are a subset of a finite number of arcs of \(M\). If the true cyclic elements are finite in number the proposition is immediate. Suppose \(M\) contains an infinity of true cyclic elements. As pointed out above, the nodes of \(M\) are finite in number. By Theorem 12, we have immediately that the branch elements of \(M\) are finite in number. If \(C\) is a true cyclic element such that the arcs of which \(C\) is the sum do not all begin and end at cut points of \(M\), then condition 2 states that the class of all true cyclic elements such as \(C\) forms a finite set. These finite cases may obviously be disregarded, since they are the sum of a finite number of arcs of \(M\). There only remains for consideration an infinity of true cyclic elements \(C_1, C_2, \ldots\) of \(M\) such that, for every \(i\), \(C_i\) has exactly two complementary components, and contains exactly two points \(P\) and \(Q\) which cut \(M\), and \(C_i\) is the sum of \(n\) or less arcs all of which join \(P\) to \(Q\). Let \(t_1, t_2, \ldots, t_j\) be the finite set of arcs of \(M\) of which the cut points are a subset. Associate with \(t_i\) every true cyclic element of the above sequence for which \(t_i\) contains \(P_j\) or \(Q_j\) or both. Of those that remain, associate with \(t_i\) those for which \(t_i\) contains \(P_j\) and \(Q_j\) or both; etc. Thus, every true cyclic element of the above sequence is associated with one and only one of the arcs \(t_i\). If \(t_i\) has only a finite number of true cyclic elements associated with it, they are all, obviously, the sum of a finite number of arcs of \(M\). Suppose \(t_i\) has infinitely many true cyclic elements associated with it. Now the end points of \(t_i\) are either end points of \(M\) or non-cut points of certain nodes of \(M\). So, if \(t_i\) contains a non-cut point, of some true cyclic element \(C_j\), \(t_i\) will have to contain both \(P_j\) and \(Q_j\). Furthermore, in only a finite number of cases can \(t_i\) contain, for some \(C_j\), \(P_j\) or \(Q_j\) and not the other. For, suppose \(t_i\) contains infinitely many of the points \(P_j\) and not one of the corresponding points \(Q_j\). For every \(j\), let \(D_j\) be the component of \(M - P_j\) containing \(Q_j - P_j\). Let us show the domains \(D_j\) to be mutually exclusive. Since \(t_i\) does not contain \(Q_j\) and contains a point \(P_j\) exterior to that component of \(M - C_j\) having \(Q_j\) for a boundary, it follows that \(t_i\) has no point in common with that component. Now \(D_j\) is the sum of \(C_j - P_j\) and the component of \(M - C_j\) having \(Q_j\) as its boundary. Consequently, since \(t_i(\overline{C_j - P_j}) = 0\), we have \(t_i \cdot D_j = 0\). Suppose two domains \(D_a\) and \(D_b\) have points in common. Since \(t_i(\overline{D_a + D_b}) = 0\), and since \(t_i \cdot \overline{P_a + P_b}\), it follows that neither \(D_a\) nor \(D_b\) contains the boundary point of the other. But this implies both \(D_a \supset D_b\) and \(D_b \supset D_a\) which implies the identity of the true cyclic elements \(C_a\) and \(C_b\). Thus, the domains \(D_j\) are mutually exclusive. By Lemma 3, \(D_j + P_j\) contains a node of \(M\); this would mean that \(M\) contains infinitely many nodes, which is impossible. Thus, in only a finite number of cases does \(t_i\) contain \(P_j\) or \(Q_j\) and fail to contain the other. Neglecting those true cyclic elements associated with \(t_i\) for which this occurs, we have left a countable collection of true cyclic elements \(C_1, C_2, \ldots\) associated with \(t_i\) such that \(t_i\) contains, for every \(j\), both \(P_j\) and \(Q_j\). Assign an order to the \(n\) or less arcs of which \(C_j\) is the sum. Evidently \(t_i\) has the arc \(P_jQ_j\) in common with \(C_j\). For every \(j\), replace the arc \(P_jQ_j\) of \(t_i\) by the arc of \(C_j\) numbered 1. Let \(t_i\) be the set constructed in this manner. Since the replacement arcs are non-overlapping, and since their diameters converge to zero, it fol-
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the independence of the condition. It is to be observed that the set \( S \) is not the sum of any finite number of simple closed curves. This fact suggests the following series of theorems. We proceed to establish that the property of a cyclically connected continuous curve of being the sum of a finite number of arcs from \( P \) to \( Q \), for every pair of distinct points \( P \) and \( Q \), and the property of being the sum of a finite number of simple closed curves are equivalent.

**Theorem 14.** If the cyclically connected continuous curve \( M \) is the sum of a finite number of simple closed curves, then, if \( R \) and \( Q \) are any pair of distinct points of \( M \), \( M \) is the sum of a finite number of arcs whose end points are \( P \) and \( Q \).

Let \( J \) be a simple closed curve of \( M \). Select the points \( P \) and \( Q \). We shall show that \( J \) is a subset of two arcs of \( M \) whose end points are \( P \) and \( Q \). If \( P \) and \( Q \) are points of \( J \) it is obvious that this is true. Suppose \( Q \) is on \( J \) and \( P \) is not. Let \( P' \) be a point of \( J \) distinct from \( Q \). According to a theorem of Ayres \( 19 \), \( M \) contains an arc \( PP'Q \). Let \( P'' \) be the first point of this arc on \( J \). Evidently \( P'' \neq Q \). Then the subarc \( PP'' \) (of \( PP'Q \)) together with the two arcs of \( J \) from \( P'' \) to \( Q \) evidently form two arcs of \( M \) from \( P \) to \( Q \) whose sum contains \( J \).

Suppose, then, that neither \( P \) nor \( Q \) is on \( J \). Let \( X \) be a point of \( J \). As above, \( M \) contains an arc \( PXQ \). Let \( P' \) and \( Q' \) be the first and last points respectively of \( J \) on the arc \( PXQ \). If \( P' \neq Q' \), then the subarc \( PP' \) (of \( PXQ \)) plus the subarc \( Q'Q \) (of \( PXQ \)) plus the two arcs of \( J \) from \( P \) to \( Q \) evidently form two arcs of \( M \) from \( P \) to \( Q \) whose sum contains \( J \). Consider the case \( P' = Q' = X \). Let \( X' \) be a point of \( J \) distinct from \( X \). As above, \( M \) contains an arc \( PX'Q \). If, as we may assume, \( X' \) is the only point \( J \) has in common with \( PX'Q \), let \( Y \) be the last point of the subarc \( PX' \) (of \( PX'Q \)) on the arc \( PXQ \). Suppose \( Y \) is on the open interval \( XQ \) of \( PXQ \); then the subarc \( YQ \) (of \( PXQ \)) plus the subarc \( YX' \) (of \( PX'Q \)) forms an arc \( QX' \) having no point in common with the arc \( PX \) (of \( PXQ \)). Suppose, next, that \( Y \) is on the interval \( PX \) of \( PXQ \); then the subarc \( PY \) (of \( PXQ \)) plus the subarc \( YX' \)

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(of \(P X' Q\) forms an arc \(P X\) having no point in common with the arc \(X Q\) of \(P X Q\)). In any case \(M\) contains two mutually exclusive arcs from \(P\) to \(J\) and from \(Q\) to \(J\). These two arcs plus the two arcs of \(J\) from \(X\) to \(X'\) evidently form two arcs from \(P\) to \(Q\) whose sum contains \(J\). Thus, \(J\) is a subset of the sum of two arcs from \(P\) to \(Q\) for every pair of distinct points \(P\) and \(Q\). And, if \(M\) is the sum of \(n\) simple closed curves, \(M\) is the sum of \(2n\) arcs whose end points are \(P\) and \(Q\).

Before proceeding to the converse, we must, first, prove a lemma.

**Lemma 4.** If \(X\) is an interior point of an arc \(t\) of the cyclically connected continuous curve \(M\), then \(X\) is contained in an open interval \(I_0\) of \(t\) such that there exists a simple closed curve \(J_0\) of \(M\) containing \(I_0\).

Let \(A\) and \(B\) be the end points of \(t\). Since \(M - X\) is connected, \(M - X\) contains an arc \(t'\) from \(A\) to \(B\). The two arcs \(t\) and \(t'\) evidently contain a simple closed curve \(J_0\) containing \(X\); and obviously \(J_0\) contains an open interval \(I_0\) of \(t\) containing \(X\).

**Theorem 15.** If \(t\) is an arc of the cyclically connected continuous curve \(M\), then \(t\) is a subset of a countable number of simple closed curves of \(M\).

Let \(A\) and \(B\) be the end points of \(t\). The open interval \(t - A - B\) can be covered by a countable number of the intervals \(I_0\) of Lemma 4; since \(A\) and \(B\) lie together on a simple closed curve of \(M\), we have immediately that \(t\) is a subset of a countable number of simple closed curves of \(M\).

**Corollary.** If the cyclically connected continuous curve \(M\) is the sum of a finite or countable number of arcs, then \(M\) is the sum of a countable number of simple closed curves.

**Theorem 16.** If the cyclically connected continuous curve \(M\) has the propery that, for every pair of distinct points \(P\) and \(Q\) of \(M\), \(M\) is the sum of a finite number of arcs whose end points are \(P\) and \(Q\), then \(M\) is the sum of a finite number of simple closed curves.

Let \(X\) be a point of \(M\). Choose \(P\) and \(Q\) to be any pair of distinct points of \(M\) other than \(X\). Then \(M\) is the sum of \(n\) arcs \(t_1, t_2, \ldots, t_n\) whose end points are \(P\) and \(Q\). Let \(t_{m_1}, t_{m_2}, \ldots, t_{m_k}\) be the arcs of this set containing \(X\). By Lemma 4, for every arc \(t_{m_i}\) there exists an open interval \(I_{m_i}\) of \(t_{m_i}\) containing \(X\), and a simple closed curve \(J_{m_i}\) of \(M\) containing \(I_{m_i}\). We have then: \(\bigcup_{i=1}^{k} J_{m_i} \supseteq \bigcup_{i=1}^{k} I_{m_i}\). Let \(S_x\) be a neighborhood of \(X\) (relative to \(M\)) containing no point of the closed set \(M - \bigcup_{i=1}^{k} J_{m_i}\). We must then have: \(S_x \subseteq \bigcup_{i=1}^{k} J_{m_i}\). Cover every point \(X\) of \(M\) by the corresponding region \(S_x\) of \(M\). Applying the Borel theorem, we find that \(M\) is covered by a finite number of the regions \(\Sigma S_x\). And since each region \(S_x\) is covered by a finite number of simple closed curves of \(M\), we have that \(M\) is the sum of a finite number of simple closed curves.

**Theorem 17.** In order that the continuous curve \(M\) should be the sum of a finite number of arcs, it is sufficient that (1) the nodes of \(M\) be finite in number, (2) each true cyclic element be the sum of a finite number of simple closed curves, numbering in every case not more than a fixed integer \(n\).

Condition 1 of Theorem 13 is satisfied. Let \(C\) be a true cyclic element of \(M\). By Theorem 14, \(C\) is the sum of \(2n\) arcs whose end points are \(P\) and \(Q\) where \(P\) and \(Q\) are any pair of distinct points of \(M\). Whenever possible select \(P\) and \(Q\) so that they may be cut points of \(M\); this may be done in every case where \(C\) is not a node. Evidently condition 2 of Theorem 13 is satisfied.

**Theorem 18.** For the boundary \(M\) of a bounded, simply connected domain in the plane to be the sum of a finite number of arcs, it is necessary and sufficient that the nodes of \(M\) be finite in number.

The necessity follows from Theorem 7, the sufficiency from Theorem 17 and the fact that the true cyclic elements of such a boundary are themselves simple closed curves.\(^{15}\)