Concerning Fréchet limit-spaces.

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I. This note attempts to enumerate all possible Fréchet limit-spaces [1]) which possess one or more of the following five properties:

A) The closure of every set is closed [2]).

B) Every non-denumerable set contains an element of condensation [3]).

C) Every well-ordered series of decreasing closed sets is denumerable.

D) Every well-ordered series of increasing closed sets is denumerable.

E) Every set Q contains a denumerable set D such that Q is contained in the closure of D.

Although the task is facilitated by means of a simple lemma (§ IV), the enumeration remains incomplete.

[1]) That is to say, a space wherein, for every sequence of elements selected from the space it is known whether or not it has a limit, this limit being a unique element of the space. These spaces will be referred to hereafter as L-spaces, and sequences which have a limit will be called „convergent".

[2]) A set is closed if it contains all of its limiting elements, that is, the limits of all convergent sequences of elements taken from the set. The set of all limiting elements of a set Q is called the derived set of Q, and denoted by Q'. The closure of the set Q is the set $Q + Q'$, and is denoted by $\overline{Q}$. If the space has the property that any subsequence of a convergent sequence is also convergent, then we can use, instead of this property, the property: A') The derived set of any set is closed.

[3]) The element b is an element of condensation of the set Q if b is a limiting element of $Q - D$, where D is any denumerable subset of Q.

II. W. Sierpiński [4]) demonstrated that, for any L-space, the following implications hold:

a) B implies C;
b) E implies D;
c) A and C together imply B;
d) A and D together imply E.

Hence it follows that, of the thirty-two ($= 2^5$) conceivable L-spaces described in § I, the following nineteen are impossible [5]):

(2), (5), (12), (13), (14), (15), (24), (25), (35), (124), (125), (134), (135), (235), (245), (1234), (1235), (1245), (1345).

III. There remain to be considered the following thirteen spaces:

(0), (1), (3), (4), (23), (34), (45), (123), (145), (234), (345), (2345), (12345).

Examples of some of these spaces are readily obtainable. Thus:

1) The Euclidian plane, interpreted in an obvious manner as an L-space, is clearly of type (12345).

2) Consider an L-space whose elements consist of the elements of the double sequence $\{b_{ij}\}$, the simple sequence $\{b_i\}$, and the element b, all these elements being distinct. For a fixed i, we define:

$$\lim_{j\to\infty} b_{ij} = b_i,$$

and

$$\lim_{i\to\infty} b_i = b;$$

no other sequences are „convergent". This space is denumerable, and so certainly possesses the properties B, C, D, E. However, it does not possess property A; thus, the closure of the set $Q = [b_{ij}]$ is the set $\overline{Q} = [b_{ij}] + [b_i]$, which is not closed, since b does not belong to $\overline{Q}$, although it is a limiting element of $\overline{Q}$. This L-space is thus of type (2345).

[4]) *Sur l'équivalence de trois propriétés des ensembles abstraits*, Fund. Math. II, pp. 179—188.

[5]) Hereafter the symbol (134), for example, denotes an L-space possessing the first, third, and fourth of the given five properties, and no others; similarly for the other symbols, except that (0) denotes an L-space which lacks all these properties.

3) W. Sierpiński[6]) constructed examples of L-spaces which are of type (123) and (145), respectively.

4) C. Kuratowski[7]) constructed a space of type (34).

IV. We consider now the following situation. Let L_1 and L_2 be two given L-spaces with distinct elements. In terms of these we define an L-space L_3 as follows:

a) The elements of L_3 consist of those and only those elements which are in L_1 or in L_2:

b) The „convergent" sequences of L_3 consist of those and only those sequences which converge in L_1 or in L_2, the limit of a „convergent" sequence in L_3 being identical with the limit of the same sequence in L_1 or L_2.

If we call the space L_3 the „sum-space" of the spaces L_1 and L_2, then we can state the following lemma:

Lemma. *If L_3 is the sum-space of L_1 and L_2, then a necessary and sufficient condition for any one of the properties A, B, C, D and E to hold in L_3 is that it holds in L_1 and in L_2.*

The proof of this lemma is immediate.

V. The above lemma, and the spaces already constructed in § III, enable us to construct all but two of the remaining L-spaces.

1) $(123)+(145)=(1)$ [8]).
2) $(123)+(34)=(3)$.
3) $(145)+(34)=(4)$.
4) $(2345)+(123)=(23)$.
5) $(2345)+(145)=(45)$.
6) $(23)+(45)=(0)$.

VI. To complete the enumeration, we have to answer the following questions:

Is it true that any Fréchet L-space possessing properties B and D also possesses property E?

[6]) In the article cited in footnote [4]).

[7]) *Une remarque sur les classes (L) de M. Fréchet*, Fund. Math. III, pp. 41—43.

[8]) This means: The sum-space of two L-spaces of type (123) and (145), respectively, is of type (1). The remaining „sums" in this paragraph are similarly defined.

Is it true that any Fréchet L-space possessing properties C and E also possesses property D?

The author is unable to answer these questions. It is clear that if there is an L-space of type (234), then the first question is to be answered in the negative; similarly, the existence of (345) would show that the second question is likewise to be answered negatively. Conversely, if the theorems suggested by these questions are true, then L-spaces of type (234) or (345) cannot exist.