Man kann also die unbestimmten Integrale nicht beschränkter Funktionen im allgemeinen nach achsenparallelen Rechtecken nicht differenzieren, oder, anders ausgedrückt: Für die Funktion

$$F(x, y) = \int_0^x \int_0^y f(\xi, \eta)\, d\xi\, d\eta$$

braucht, wenn $f(x, y)$ nicht beschränkt ist, die zweite gemischte Ableitung $f_{(x,y)}$ im Sinne des Doppellimes nicht zu existieren.

Zusatz bei der Korrektur.

Nach Vollendung der Arbeit bemerken wir, dass einige Behauptungen der Einleitung durch das neue Buch von S. Saks: Théorie de l'Intégrale, Warszawa 1933, überholt sind. Der Dichtesatz für das System der Intervalle wurde bereits von Herrn Saks bewiesen (vgl. S. 231 f.). Ferner hat Herr Saks dort auch schon bemerkt, dass die Integrale über unbeschränkte Integranden nach Intervallen nicht differenzierbar zu sein brauchen; wir verdanken einen Beweis und eine, auf dem Begriff der Baireschen Kategorien beruhende, Verschärfung dieser Behauptung einer freundlichen brieflichen Mitteilung von Herrn Saks. Schliesslich sei erwähnt, dass man nach einer Bemerkung von O. Nikodym bereits aus einem, zu anderen Zwecken konstruierten Beispiel von A. Zygmund schliessen kann, dass der Dichtesatz für das System aller Rechtecke der Ebene nicht gilt. (vgl. S. 232 des zitierten Buches).

Kopenhagen, den 30. XI. 1933.

Remark on the differentiability of the Lesbesgue indefinite integral.

By

S. Saks (Warszawa).

1. If $f(x, y)$ is a summable function of two variables then, by the well known theorem of Lebesgue, at almost every point (x, y) we have

$$\lim_{\delta(\sigma)\to 0} \frac{1}{\text{meas}\,\sigma} \iint_{\sigma} f\, dx\, dy$$

where σ denotes an arbitrary square containing the point (x, y). Under the additional assumption that $f(x, y)$ is a bounded function it has been recently proved [1]) that this theorem holds again if the square σ is replaced by an interval i. e. by a rectangle with sides parallel to the axis. However, this generalization of the Lebesgue theorem is not true for unbounded functions as it has been shown by Busemann and Feller on the basis of a general criterion concerning the differentiability of absolutely additive functions with respect to given families of open sets, and as has been also mentioned elsewhere by the author [2]). In connection with those results the following theorem might be of some interest:

A. *Let $\mathfrak{L}$ be the space of the functions $f(x, y)$ summable over the square $S=[0,1;0,1]$ (with the ordinary norm $\|f\| = \iint_S |f(x,y)|\, dx\, dy$).*

[1]) See F. Riesz, *Sur les points de densité au sens fort*, this volume, pp. 221—225; Busemann und Feller, *Zur Differentiation des Lebesguesche Integrale*, ibid., pp. 226—256; Saks, *Théorie de l'intégrale* (Monografje Matematyczne), Warszawa, 1933, p. 232.

[2]) Busemann und Feller, l. c.; Saks, l. c., p. 232.

Then, except for the functions belonging to a set of the first category of B a i r e *in* $\mathfrak{L}$, *every function* f *in* $\mathfrak{L}$ *satisfies the condition*

(1) $$\overline{\lim_{h,k\to 0}}\frac{1}{4hk}\int_{x_0-h}^{x_0+h}\int_{y_0-k}^{y_0+k} f\,dx\,dy = +\infty$$

at every point (x_0, y_0) *in* S.

2. The proof rests on the following lemma:

B. *For any positive integer* N *there exists a non-negative and measurable function* $\Phi_N(x,y)$ *which satisfies the following conditions*:

(2) $$\int\int_S \Phi_N\,dx\,dy \leqslant 2N^{-1},$$

(3) *to any point* (x_0, y_0) *in* S *there corresponds an interval* $I_0 = [x_0-h, x_0+h;\ y_0-k, y_0+k]$ *such that* $\delta(I_0)\leqslant 4\cdot N^{-1}$ *and* $\int\int_{I_0}\Phi_N\,dx\,dy \geqslant 4^{-1}N\,\mathrm{meas}\,I_0$.

Indeed, divide the square S into N^2 equal and non-overlapping squares $S_1, S_2, \ldots, S_{N^2}$. By virtue of an elementary B o h r's lemma [1]) each square S_i may be considered as the sum of a finite number of intervals

$$I_{i,1}^{(1)},\ I_{i,2}^{(1)},\ldots,\ I_{i,r}^{(1)};\quad I_{i,1}^{(2)},\ I_{i,2}^{(2)},\ldots,\ I_{i,r}^{(2)};\ldots;\quad I_{i,1}^{(q)},\ldots,\ I_{i,r}^{(q)};\quad J_{i,1},\ldots,\ J_{i,p}$$

such that

(4) $$\mathrm{meas}\,I_{i,1}^{(j)} = \ldots = \mathrm{meas}\,I_{i,r}^{(j)} = N^{-2}\,\mathrm{meas}\,V_i^{(j)},$$

where $V_i^{(j)} = \sum_{n=1}^{r} I_{i,n}^{(j)}$, $j = 1, 2, \ldots, q$ *and* $i = 1, 2, \ldots, N^2$;

(5) $$\sum_{i,n}\mathrm{meas}\,J_{i,n} \leqslant N^{-2};$$

(6) $$\mathrm{meas}\,E_i^{(j)} > 0 \quad \textit{where} \quad E_i^{(j)} = \prod_{n=1}^{r} I_{i,n}^{(j)};$$

(7) *the sets* $V_i^{(j)}$ *and* $J_{i,n}$ *are non-overlapping.*

[1]) See C a r a t h é o d o r y, *Vorlesungen über reelle Funktionen*, 2. Aufl., Leipzig, 1927, p. 689—691.

Let us put now

$$\Phi_N(x,y) = \frac{1}{N}\frac{\mathrm{meas}\,V_i^{(j)}}{\mathrm{meas}\,E_i^{(j)}} \text{ for } (x,y)\in E_i^{(j)},\ j=1,2,\ldots,q \text{ and } i=1,2,\ldots,N^2,$$

$\Phi_N(x,y) = N$ for $(x,y)\in J_{i,n}$, $n=1,2,\ldots,p$ and $i=1,2,\ldots,N^2$, and $\Phi_N(x,y)=0$ elsewhere in S. Then in view of (7) and (5)

$$\int\int_S \Phi_N\,dx\,dy = N^{-1}\sum_{i,j}\mathrm{meas}\,V_i^{(j)} + N\sum_{i,n}\mathrm{meas}\,J_{i,n} \leqslant 2N^{-1}.$$

Let (x_0, y_0) be a point in S belonging to an interval $I_{i_0,l_0}^{(j_0)}$; it follows from (4) that

$$\int\int_{I_{i_0,l_0}^{(j_0)}} \Phi_N\,dx\,dy = N^{-1}\,\mathrm{meas}\,V_{i_0}^{(j_0)} \geqslant N\,\mathrm{meas}\,I_{i_0,l_0}^{(j_0)}.$$

Denote by $I_0 = [x_0-h, x_0+h;\ y_0-k, y_0+k]$ the smallest interval containing $I_{i_0,l_0}^{(j_0)}$, with center (x_0, y_0). Since the function $\Phi_N(x,y)$ is non-negative and the diameter of $I_{i_0,l_0}^{(j_0)}\subset S_{i_0}$ less than $2N^{-1}$ we see at once that the interval I_0 satisfies the condition (3). The same obviously holds if (x_0, y_0) belongs to an interval J_{i_0,l_0} and if I_0 denotes the smallest interval of center (x_0, y_0), containing J_{i_0,l_0}.

We are now prepared to prove Theorem A. Let $\mathfrak{A}_n$ be the set of functions f in the space $\mathfrak{L}$ with the following property:

($\mathcal{P}$) *there exists a point* (x_0, y_0) *in* S *such that for any interval* I *of center* (x_0, y_0) *and diameter* $\leqslant n^{-1}$ *we have*

$$\left|\int\int_I f\,dx\,dy\right| \leqslant n\cdot\mathrm{meas}\,I.$$

$\mathfrak{A}_n$ are closed sets. We shall prove that they are non dense. Indeed, let $f\in\mathfrak{A}_n$ and let, for an arbitrarily given $\varepsilon>0$, $f_0(x,y)$ be a bounded measurable function such that

(8) $$\|f-f_0\| = \int\int_S |f-f_0|\,dx\,dy \leqslant \varepsilon.$$

Then, preserving the notations adopted in Lemma B and putting $\varphi_N = f_0 + \Phi_N$ we have, by (8) and (2),

(9) $$\|\varphi_N - f\| \leqslant \varepsilon + 2N^{-1} \leqslant 2\varepsilon \quad \textit{for} \quad N \geqslant 2\varepsilon^{-1}.$$

Let M be the upper bound of $|f_0(x,y)|$. By virtue of (3), if $N > 4(M+n)$, then to any point (x_0, y_0) there corresponds an interval I_0 with center (x_0, y_0) such that

$$\delta(I_0) \leqslant 4 \cdot N^{-1} < n^{-1} \quad \text{and} \quad \int\int_{I_0} |\varphi_N|\, dx\, dy \geqslant (4^{-1}N - M) \operatorname{meas} I_0 > n \cdot \operatorname{meas} I_0.$$

Hence, for the values of N sufficiently large, φ_N does not belong to $\mathfrak{A}_n$ and by (9) f is the limit point of a sequence of elements not belonging to $\mathcal{C}_n$. Thus $\mathcal{C}_n$ are non dense sets, $\sum_n \mathcal{C}_n$ is a set of the first category; and, as the complementary set $\mathfrak{L} - \sum_n \mathcal{C}_n$ coincides with that of functions satisfying the condition (1), Theorem A is established.

Theorem A suggests the problem as to the existence of a measurable summable function $f(x,y)$ such that

$$+\infty > \overline{\lim_{h,k\to 0}} \frac{1}{4hk} \int_{x_0-h}^{x_0+h}\int_{y_0-k}^{y_0+k} f(x,y)\, dx\, dy > \underline{\lim_{h,k\to 0}} \frac{1}{4hk} \int_{x_0-h}^{x_0+h}\int_{y_0-k}^{y_0+k} f(x,y)\, dx\, dy > -\infty$$

everywhere on a set of positive measure of points (x_0, y_0). This problem seems to be open.

3. The following interesting remark which brings to light a further difference between the Fourier series of one variable and those of several ones is due to Zygmund[1]). Let, for any summable function $f(x,y)$, of period 2π with respect to each variable, $\sigma_{m,n}(f; x_0, y_0)$ denote the mn-th Fejér sum of $f(x,y)$ at (x_0, y_0); viz.

$$\sigma_{m,n}(f; x_0, y_0) = \\ = \frac{1}{4mn\pi^2} \int_{x_0-\pi}^{x_0+\pi}\int_{y_0-\pi}^{y_0+\pi} f(x,y) \left(\frac{\sin \frac{m(x-x_0)}{2}}{\sin \frac{x-x_0}{2}} \right)^2 \left(\frac{\sin \frac{n(y-y_0)}{2}}{\sin \frac{y-y_0}{2}} \right)^2 dx\, dy.$$

Then *there exists a summable periodic function* $f(x,y)$ *such that at every point* (x_0, y_0)

(10) $$\overline{\lim_{m,n}}\, \sigma_{m,n}(f; x_0, y_0) = +\infty.$$

[1]) This remark answers a question set by L. Tonelli (*Serie trigonometriche*, B logna (1928), footnote of p. 495).

Indeed, let f be a summable periodic function satisfying (1) everywhere, and let (x_0, y_0) be an abitrary point. We can obviously suppose $f(x,y)$ to be non negative. Then it easily follows from (1) that there exist two sequences of positive integers $\{m_k\}$ and $\{n_k\}$ such that $m_k \to \infty$, $n_k \to \infty$ and

(11) $$m_k n_k \cdot \int_{x_0-1/m_k}^{x_0+1/m_k}\int_{y_0-1/m_k}^{y_0+1/m_k} f(x,y)\, dx\, dy \to +\infty.$$

As $4\pi^2 \sigma_{m_k,n_k}(f; x_0, y_0)$ is greater than, or equal to, the left-hand expression in (11) the formula (10) follows at once[1]).

[1]) It remains however unknown if there exists a summable function $f(x,y)$ such that $\overline{\lim_m}\, \sigma_{m,m}(f; x, y) = +\infty$ at all points of a set of positive measure.