Concerning a problem of K. Borsuk.

By


The following problem was recently proposed by M. K. Borsuk: Is every (compact) subcontinuum C of \( E_n \), which cuts \( E_n \) and for which there exists for every \( \varepsilon > 0 \) an \( \varepsilon \)-transformation \(^3\) of \( C \) into a set \( C' \) such that \( C \cdot C' = 0 \), an \((n-1)\)-dimensional manifold?

Although, as pointed out below, the answer to this question is negative, even (if \( n > 2 \)) for the case where \( C \) is a Jordan continuum \(^4\) (unless restrictions be put upon the Brouwer numbers \( P^r(C) \) where \( r > 0 \)), we are able to obtain several positive results through either of two devices, viz., by further restrictions on the nature of \( C \), or by modification of the type of transformation.

\(^{1}\) Fund. Math., 29 (1938), p. 285, Problem 54. Since this paper was presented for publication, M. Borsuk has communicated to me that it was his intention to formulate this problem only for "lokal zusammenziehbar" continua, the words "lokal zusammenziehbar" having been omitted through an oversight (for definition see Fund. Math., vol. 19, p. 236). Thus, M. Borsuk's problem should really be stated as "let every lokal zusammenziehbar" Teilkontinuum \( C \)... eine \((n-1)\)-dimensionale Mannigfaltigkeit?" However, as M. Borsuk points out in his communication, since all Betti numbers of a finite dimensional, lokal zusammenziehbar compact space are finite (for proof see M. Borsuk's note "Zur kombinatorischen Eigenschaften der Retrakte", Fund. Math., vol. 51), my Theorem 3 below furnishes an affirmative answer to M. Borsuk's problem (amended as just stated) for the case \( n = 3 \).

\(^{2}\) A domain \( D \) is uniformly locally connected if for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that if \( P \) and \( Q \) are points of \( D \) whose distance apart is \( < \delta \), then \( P \) and \( Q \) may be joined by an arc of \( D \) of diameter \( < \varepsilon \).

\(^{3}\) The precise definition is given below.

\(^{4}\) That is, a continuous mapping \( f \) of \( C \) such that if \( P \subset C \), then \( \varepsilon(P,f(P)) < \varepsilon \).

\(^{5}\) By Jordan continuum (= continuous curve, = Peano continuum) we mean a locally connected, compact continuum.

It follows easily from the conditions stated in the problem that \( C \) is a common boundary of two and only two domains of \( E_n \), and if \( C \) is a Jordan continuum then these domains are uniformly locally connected \(^6\). Thus, when \( C \) is a Jordan continuum and \( n = 2 \), \( C \) is a simple closed curve; and if \( n = 3 \) and \( p^1(C) \) is finite, \( C \) is a closed 2-dimensional manifold.

We also consider the problem: In \( E_n \), let \( C \) be a compact continuum which cuts \( E_n \) and which may be deformed continuously without meeting itself \(^8\); is \( C \) an \((n-1)\)-manifold? We show that \( C \) is a Jordan continuum whose complement is two uniformly locally connected domains having \( C \) as a common boundary, and for \( n = 2 \) \( 3 \) is a closed \((n-1)\)-manifold. In every case, whether \( C \) cuts \( E_n \) or not, its points are regularly accessible \(^10\) from its complement.

Theorem 1. In \( E_n \), let \( C \) be a compact continuum such that for every \( \varepsilon > 0 \) there exists an \( \varepsilon \)-transformation of \( C \) into a set \( C' \) such that \( C \cdot C' = 0 \). Then the complement of \( C \) is either one domain whose boundary is \( C \), or two domains of which \( C \) is the common boundary.

Proof. Let \( D \) be a domain complementary to \( C \). Denoting the boundary of \( D \) by \( B \), suppose \( P \) is a point of \( C \) not in \( B \), and let \( Q \) be an arbitrary fixed point of \( D \). Then \( B \) separates \( P \) and \( Q \) in \( E_n \). Let \( f(C) \) be an \( \varepsilon \)-transformation of \( C \) such that \( C \cdot C' = 0 \), and with \( \varepsilon \) small enough that \( f(P) \) is not in \( D \) and \( B \) does not meet \( P \cdot Q \) during the rectilinear deformation \(^7\) of \( B \) into \( f(B) \). By a theorem of Pontrjagin \(^9\), \( f(B) \) separates \( P \) and \( Q \) in \( E_n \). Now

\(^{6}\) A domain \( D \) is uniformly locally connected if for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that if \( P \) and \( Q \) are points of \( D \) whose distance apart is \( < \delta \), then \( P \) and \( Q \) may be joined by an arc of \( D \) of diameter \( < \varepsilon \).

\(^{7}\) That is, a deformation \( F(P,\delta) \), \( 0 \leq \delta \leq 1 \) then \( P \) and \( Q \) may be joined by an arc which lies wholly in \( S(P,\varepsilon) \) and meets \( C \) only in \( P \).

\(^{8}\) That is, a deformation \( F(P,\delta) \), \( 0 \leq \delta \leq 1 \), such that if \( P \subset B \) the point \( P \) moves along the straight line joining \( P \) and \( f(P) \) in such a way that \( F(P,\delta) \) is in the segment \( f(P,\delta) \) of \( f(P) \).

\(^{9}\) L. Pontrjagin. "{Z}"um Alexandrovschen Dualit"atssatz. G~tt. Nach., Math.-Phys. Kl., 1917, pp. 316—322, Sitz. IV. I have made considerable use, both in the present paper and in others, of this fundamental and important theorem of Pontrjagin, so that I feel I should point out what seems to me a defect in its proof and a method for rectifying it. I refer to the last sentence of Pontrjagin's proof, wherein it is asserted that from \( \eta \cdot 2^{n-3} \cdot k^{n+1} - 0 \) it follows that
$C'$ is a continuum, and consequently lies either wholly in $D$ or wholly in $E - D$. The former case has been ruled out, since $f(P)$ is not in $D$. In the latter case, $D + C$ is a continuum joining $P$ and $Q$ and not meeting $f(B)$, so that this case is impossible. Consequently $B = C$.

There are at most two domains complementary to $C$. For suppose there are three domains $D_i (i = 1, 2, 3)$; let $P_i$ denote a point of $D_i$. From the three points $P_i$ we form three sets of pairs of points, $(P_i, P_j)$. Let $f$ denote a transformation of $C$ into a set $C'$ such that $C' \cap C = \emptyset$ and such that $C'$ separates each of the pairs $(P_i, P_j)$ (See preceding paragraph) As above, $C'$ must lie wholly in one domain complementary to $C$, say in $D_1$. Then $D_2 + D_3 = C$ is a continuum that does not meet $C'$ and yet contains both $P_2$ and $P_3$, thus violating the fact that $C'$ separates $P_2$ and $P_3$. Consequently $C$ has at most two complementary domains. This completes the proof.

Before considering the case where $C$ is a Jordan continuum, we note the reason for stipulating this condition. In $E_a$, let $C$ consist of the following set of points: 1) All points on the curve $y = \sin 1/x$ where $0 < x \leq 1/\pi$, 2) All points $(0, y)$ for $-1 \leq y \leq +1$, and 3) An arc which joins the points $(0, 1/(1/\pi))$ (without otherwise containing any points 1 - 2) and lies, except for these points, wholly in the fourth quadrant of the plane. For any $\varepsilon > 0$, there exists, in either of the domains complementary to $C'$, $\varepsilon - 1$, $C \not\subseteq \emptyset$. This overlooks the fact that $\varepsilon - 1$ and $C$ may be mutually exclusive and yet after deformation of $\varepsilon - 1$, $K'', \emptyset$, as may be shown by simple examples, I should like to suggest substituting the following argument for the last paragraph of Pontrjagin's proof:

Suppose $\Gamma \not\subseteq 0$ in $E - \emptyset$; there exists, then, a complex $K + 1 \subseteq E - \emptyset$ which is bounded by $\Gamma$. Let $F$ be a polyhedral neighborhood of $E$ excluding $\Gamma$ and so small that during the deformation $\Delta$, $F$ does not meet $\Gamma$; also such that for every point $x \subseteq \Delta (P)$, the relation $e(x, \phi) < e$ holds, where $e(K'' + 1, \emptyset) = e$. Clearly $\Delta (P'' + 1, \emptyset) \subseteq F'$. The cycle $\Gamma'$ links $P''$, since it links $F'$. Therefore by Satz II there exists a cycle $\varepsilon - 1 \subseteq P''$ which is linked with $\Gamma'$. The deformation $\Delta$ carries $\varepsilon - 1$ into a (possibly singular) cycle $\varepsilon - 1$ which is linked with $\Gamma' = \Delta (\Gamma')$. Consequently $\varepsilon - 1$, $K'' + 1 = 0$. However, we have

$$\varepsilon - 1 \subseteq \Delta (P'') \subseteq E - K'' + 1,$$

and thus the supposition that $\Gamma \not\subseteq 0$ in $E - \emptyset$ leads to a contradiction.

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A homeomorph $C' = f(C)$ which constitutes an $\varepsilon$-transformation of $C$ into $C'$.

An analogous example may be given in $E_{a_1}$, using a surface which has one of its cross-sections similar to the example in the preceding paragraph, and in which the analogue of the continuum of condensation, $M$, described in 2) above is such that for every $\varepsilon > 0$ there is a rectangular parallelopiped with two of its dimensions $< \varepsilon$ enclosing $M$.

It is to be noted from the above examples that 1) to require that $C'$ be a homeomorph of $C$ is not in itself a requirement strong enough to make $C$ an $(n - 1)$-manifold, and 2) in each case $C$ is not a Jordan continuum.

Theorem 2. Under the same hypothesis as in Theorem 1, with the additional condition that $C$ be a Jordan continuum which cuts $E_a$, the domains complementary to $C$ are uniformly locally connected.

For the proof of Theorem 2 we require the following lemma:

Lemma. Let $C$ be any closed point set in $E_{a_1}$, a point of $C$, $e$ an arbitrary positive number, and $\Gamma'$ a cycle that links $C$ in $S(A, e)$.

Then for any positive number $e < e$ such that $\Gamma' \subseteq S(A, e)$, there exists a positive number $\eta$ such that if $f(C)$ is any $\eta$-transformation of $C$, then $\Gamma'$ links $f(C)$ in $S(A, e)$.

Proof of Lemma. Let $\eta$ be such that $0 < \eta < \frac{1}{4}(e - e')$, as well as such that a rectilinear deformation or $C$ into $f(C)$ does not meet $\Gamma'$.

Let $\Delta$ denote a rectilinear deformation such as that just mentioned. Let $F = F'(A, e)$. Using a method of proof employed by Pontrjagin, the deformation $\Delta$ may be extended to an $\eta$-deformation $\Delta'$ of $F + C$ which agrees on $C$ with $\Delta$ and carries no point of $F$ into $S(A, e')$. By the deformation theorem of Pontrjagin the set $F' + C'$ (where $F'$ and $C'$ denote the sets into which $\Delta$ carries $F$ and $C$ respectively) is linked by $\Gamma'$. But then $\Gamma'$ links $C'$ in $S(A, e')$; since if it bounded a complex in $S(A, e')$ that did not meet $C'$, such a complex would not meet $F' + C'$

Proof of Theorem 2. Denoting the domains complementary to $C$ by $D_1$ and $D_2$, suppose that $D_1$ is not uniformly locally connected. Then there exist an $\varepsilon > 0$ and a point $A$ of $C$ such that

$^7$ That is, $\Gamma'$ does not bound a complex in $S(A, e)$ that does not meet $C$.

$^{14}$ Loc. cit., p. 281 (proof of Hiltens).
in every \( S(A, \delta) \), for \( 0 < \delta < \epsilon \), there exist two points of \( D_1 \) that cannot be joined by a subarc of \( D_1 \cdot S(A, \epsilon) \).

Since \( C \) is a Jordan continuum there is a \( \delta > 0 \) such that \( C \cdot S(A, \delta) \subseteq M \), where \( M \) is the component of \( C \cdot S(A, \epsilon) \) determined by \( A \). Let \( P \) and \( Q \) denote points of \( D_1 \cdot S(A, \delta/2) \) that cannot be joined by a subarc of \( D_1 \cdot S(A, \epsilon) \). Also, let \( R \) be a point of \( D_1 \cdot S(A, \delta/2) \).

Obviously \( C \) separates each of the point pairs \( (P, Q) \), \( (P, R) \), \( (Q, R) \) in \( S(A, \epsilon) \). Let \( H \) be a subarc of \( M \) that contains all points of \( M \cdot S(A, \delta) \). Also, let \( \epsilon' \) be a number such that \( \epsilon > \epsilon' > 0 \) and \( S(A, \epsilon') \) contains \( H \). By the above Lemma there exists \( \eta > 0 \) such that any \( \eta \)-transformation of \( C \) separates each of the above point-pairs in \( S(A, \epsilon') \). Let \( f \) denote such an \( \eta \)-transformation with the added stipulations that a) \( C \cdot f(C) = 0 \) (permissible by hypothesis), b) \( \eta < \delta/2 \), and c) \( f(H) \subseteq S(A, \epsilon') \). We note that

\[
f(C) \cdot S(A, \delta/2) \subseteq f(H) \subseteq S(A, \epsilon').
\]

There are two cases to be considered. 1) Suppose \( f(C) = C' \subset D_1 \). On the straight line interval \( PQ \) let \( a_1 \) be the first point of \( M \) in the order from \( P \) to \( Q \), and \( a_5 \) the first point of \( M \) in the reverse order. Obviously \( a_1 \) and \( a_5 \) are points of \( H_1 \), and consequently \( P_a + H + Q = Q_a \) is a subarc of \( S(A, \epsilon') \) joining \( P \) and \( Q \) and containing no point of \( C' \). But this contradicts the fact that \( C' \) separates \( P \) and \( Q \) in \( S(A, \epsilon') \). 2) Suppose \( C \subset D_1 \). Let \( b \) be the first point of \( M \) on the straight line interval \( PR \) in the order from \( R \) to \( P \). On the respective straight line intervals \( PR \) and \( QR \), let \( a_1 \) and \( a_5 \) be the first points of \( M \) in the order named. Then on \( P_a - a_1 \) there exists a point \( c_1 \) of \( f(H) \), else \( P_a + H + R \) is a subarc of \( S(A, \epsilon') \) joining \( P \) and \( R \) and not meeting \( C' \). Similarly, on \( Q_a - a_5 \) there is a point \( c_5 \) of \( f(H) \). But then \( P_{c_1} + f(H) + Q_{c_5} \) is a subarc of \( S(A, \epsilon') \) that contains no point of \( C' \), contradicting the fact that \( C' \) separates \( P \) and \( Q \) in \( S(A, \epsilon') \).

In either case, then, we have a contradiction, and consequently \( D_1 \) is uniformly locally connected. Similarly \( D_3 \) is uniformly locally connected.

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As $T_{n-2}$ lies wholly in $S(A, \eta)$, there is a complex $K_{n-1}^{1}$ such that

$$K_{n-2}^{1} \rightarrow T_{n-2}^{1} \quad [S(A; \eta); E_{n} - K].$$

As $T_{n-2}$ links $C$, it follows from the Alexander Addition Theorem (19) that the cycle $K_{n-1}^{1} + K_{n-1}^{1}$ links $K - C - K$. Thus, there exist points $P$ and $Q$ of $K - C - K$ that are separated in $E_{n}$ by $K - C - K$. However, consider the set $F = K \cdot (K_{n-1}^{1} + K_{n-1}^{1})$. Clearly $F \subseteq L$. Now $K - C - K \subseteq K - L$, and therefore $P$ and $Q$ are points of $K - L$. But then $K - L$ is a connected set containing $P$ and $Q$ and not meeting $K_{n-1}^{1} + K_{n-1}^{1}$. Thus we conclude that $T_{n-2}$ must link $K$.

It follows (20), then, that $T_{n-2}$ links a simple closed curve $J$ of $K$. Furthermore, since $D_{n}$ is uniformly locally connected, there exists in $D_{n} \cdot S(A, \eta)$ an irreducible cycle $\gamma$ which is linked with $T_{n-2}$ (11). That $\gamma$ links $C$ follows immediately, since if it bounded in $E_{n} - C$ it would bound in $D_{n}$ and hence in $E_{n} - T_{n-2}$.

Definition. Let $C$ denote a subset of any topological space $S$. We say that there exists a deformation of $C$ during the course of which $C$ does not meet itself, provided there exist a continuous function $f(P, t), P \subseteq C, 0 \leq t \leq 1$, such that $f(P, t) \subseteq S, f(P, 0) = P \subseteq C$, and such that $C \cdot f(P, t) = 0$ if $t > 0$.

For the sake of brevity, we shall call a deformation of $C$ during the course of which $C$ does not meet itself, a $\Delta$-deformation (22). In case $C$ is a set for which there exists a $\Delta$-deformation, we shall call $C$-deformable.


20) By Theorem 3 of my paper On the linking of Jordan continua in $E_{n}$ by $(n-2)$-cycles, to appear soon in Annals of Math.

21) The cycle $\gamma$ is obtained by approximation to $J$, using the uniform local connectedness of $D_{n}$. See my paper referred to in 19. Also, see in regard to the mutual linking of $T_{n-2}$ and $\gamma L$. Pontrjagin, loc. cit., Sats III.

22) Instead of requiring, in the definition of $\Delta$-deformation, that $C \cdot f(P, t) = 0$ if $t > 0$, we might stipulate instead that for given $P, f(P, t) \subseteq C$ implies that $f(P, t) = P$ for $0 \leq t \leq t_{1}$, so that it is not necessary that all points of $C$ leave their initial position simultaneously. However, his existence of such a de-

formation implies the existence of a $\Delta$-deformation $\Phi(C, t)$. For given $P, t_{1}$ be the greatest value of $t$ such that $f(P, t) = P$. Then, let $\Phi(P, t') = f(P, t' + t_{1}(1 - t_{1}))$ for $0 \leq t' \leq 1$. Then $\Phi(C, t')$ is a $\Delta$-deformation of $C$.
Theorem 5. In $E_n$, let $C$ be a compact continuum which cuts $E_n$ and is $\Delta$-deformable in $E_n$. Then $C$ is a Jordan continuum whose complement is the sum of two and only two uniformly locally connected domains of which $C$ is the common boundary.

Proof. That $E_n - C$ is the sum of just two domains $D_1$ and $D_2$ of which $C$ is the common boundary follows from Theorem 1. We shall show that these domains are uniformly locally connected, from which the conclusion that $C$ is a Jordan continuum follows.

By hypothesis $C$ is deformable into a set $C'$ by means of a $\Delta$-deformation, which we denote briefly by $\Delta$, where $C'$, being a connected point set, must lie in $D_1$, say. We note then that $f(P, 0)$, where $f$ is the function defining $\Delta$, lies in $D_1$ for all $t$ such that $0 < t$.

Suppose $D_1$ is not uniformly locally connected. Then there exist an $\epsilon > 0$, a point $A$ of $C$, and a sequence of pairs of points $(P_i, Q_i)$, $i = 1, 2, 3, \ldots$, belonging to $D_1 \cdot S(A, \epsilon)$, such that $\lim_{i \to \infty} P_i = A$, $\lim_{i \to \infty} Q_i = A$, and each pair of points $(P_i, Q_i)$ is separated in $E_n$ by $\Delta$.

It is well-known \(^{19}\) that $\Delta$ may be extended into a deformation $\Delta'$ of the set $H = C + F(A, \epsilon)$. Although $\Delta'$ is not a $\Delta$-deformation of $H$, it agrees, on $C$, with $\Delta$. There exists a value of $t = t_1 > 0$ such that during that part of the deformation $\Delta'$ which takes place over the interval $0 \leq t \leq t_1$, $F(A, \epsilon)$ does not meet $A$, and, indeed, does not enter a certain neighborhood $S(A, \eta)$. Let us denote the part of the deformation $\Delta'$ just referred to by $\Delta''$. Then $\Delta''$ carries $C$ into a set $C''$ of $D_2$, and $H$ into a set $H''$.

Let $\delta$ be a positive number such that $\delta < \eta$ and there are no points of $C''$ in $S(A, \delta)$. Consider a fixed pair of points $(P_1, Q_1)$ in $S(A, \delta)$. As stated above, $H$ separates $(P_1, Q_1)$ in $E_n$; but $H''$ has no points in $S(A, \delta)$, so that $H''$ does not separate $(P_1, Q_1)$. Then by the theorem of Pontrjagin referred to above, this is the case, $H$ must meet either $P_1$ or $Q_1$ during the deformation $\Delta''$. However, $C$ deforms entirely in $D_2$ (whereas $P_1$ and $Q_1$ lie in $D_1$).


and $F(A, \epsilon)$ does not enter $S(A, \delta)$ at all. Consequently the supposition that $D_1$ is not uniformly locally connected leads to a contradiction.

Let us now consider $D_2$. Let $A$ denote any point of $C$. We shall extend $\Delta$ into a deformation $\Delta'$ as defined above, and we now select $t = t_1 > 0$ so that during the portion $\Delta''$ of $\Delta'$ over the interval $0 \leq t \leq t_1$, $C$ deforms into a set $C''$, $A$ goes into a point $A''$ in $S(A, \epsilon)$ in such a way that its path of deformation, $A\Delta''A''$, does not leave $S(A, \epsilon)$, and $F(A, \epsilon)$ does not meet $A$. As the path $A\Delta''A''$ is connected, it lies, except for $A$, wholly in one component, $D_2$ of $D_2 \cdot S(A, \epsilon)$. Also, there exists an $\eta > 0$ such that 1) $F(A, \epsilon)$ does not enter $S(A, \eta)$ during the deformation $\Delta''$ and 2) every point in $C \cdot S(A, \eta)$ is deformed into a point in $D$ without leaving $S(A, \epsilon)$.

Since no point of $C$ meets $C$ again during the deformation $\Delta''$ every point of $C \cdot S(A, \eta)$ must lie on the boundary of $D$. This clearly implies that either a) $D$ is the only component of $D_2 \cdot S(A, \epsilon)$ that has points in $S(A, \eta)$, or b) if another such component exists, such points of its boundary as lie in $S(A, \eta)$ are also on the boundary of $D$. If case a) holds, $D_2$ is clearly uniformly locally connected.

Consider case b). There exists $\eta' > \eta$ such that no point of $C$ exterior to $S(A, \eta)$ enters $S(A, \eta')$ during the deformation $\Delta''$ and $H' \cdot S(A, \eta') = 0$. Suppose $P'$ another component of $D_2 \cdot S(P, \epsilon)$ having points in $S(A, \eta')$. Let $P$ and $P'$ denote points of $D_1$ and $D_2$, respectively, in $S(A, \eta')$. Since, after the deformation $\Delta''$, the set $H'$ does not separate $P$ and $P'$, and $F(A, \epsilon)$ does not enter $S(A, \eta')$, $C$ must meet either $P$ or $P'$ during the deformation. Suppose $x$ a point of $C$ that meets $P'$ during the deformation (we recall that no point of $C$ can meet $P$). Then $x$ lies in $S(A, \eta)$, and we have its path of deformation $xP'x''$ passing through $P'$, with $x''$ in $D$, yet not leaving $S(A, \epsilon)$ nor meeting $C$. This is impossible, and consequently there are no components of $D_2 \cdot S(P, \epsilon)$ with points in $S(A, \eta')$ other than $D$, and accordingly, since $A$ is any point of $C$, $D_2$ is uniformly locally connected.

Corollary. In the plane, let $C$ be a compact continuum which cuts $E_n$ and is $\Delta$-deformable. Then $C$ is a simple closed curve.

As another result of Theorem 5, it is easily shown by a stan-
standard procedure that every point of $C$ is regularly accessible from its complement. Is the same true in case $C$ does not cut $E_n$? By Theorem 1, every point of $C$ is a boundary point of the one domain constituting the complement of $C$ in this case. By following through the proof for the uniform local connectedness of $D_0$ in Theorem 5 above, we see that in this case $(E_n - C) \cdot S(A, \eta)$ has at most two components with points in $S(A, \eta)$, and from this that $C$ is regularly accessible. Thus we have the theorem

**Theorem 6.** In $E_n$, let $C$ be a $\Delta$-deformable continuum. Then every point of $C$ is regularly accessible from its complement.

For the special case of $n = 3$ we now prove the following theorem:

**Theorem 7.** In $E_3$ let $C$ be a compact and $\Delta$-deformable continuum which cuts $E_3$. Then $C$ is a closed 2-dimensional manifold.

Proof. Since, by Theorem 5, $C$ is the common boundary of two uniformly locally connected domains $D_1$ and $D_2$, it is necessary only to prove that for some $\epsilon > 0$ there is no 1-cycle of diameter $< \epsilon$ that links $C$. Suppose this not to be the case. Since, by Theorem 1, $E_3 - C$ consists of just the two domains $D_1, D_2$, the hypothesis of Theorem 4 is satisfied for at least one of these domains, say $D_1$, and we accordingly obtain the point $A$ and the properties stated in the conclusion of Theorem 4.

By the hypothesis there exists a $\Delta$-deformation $f(C, t), 0 \leq t \leq 1$, and we denote $f(C, 1)$ by $C'$. The set $C'$ lies wholly in one of the domains complementary to $C$, say in $D_1$. Because of the continuity of $f$, we note that all of the sets $f(C, t)$ lie in $D_1$ in this case.

By Theorem 4, there exists in $D_1 \cdot S(A, \eta)$, where $\eta$ is such that $C' \cdot S(A, \eta) = 0$, a 1-cycle $\gamma^1$ which links $C$. Obviously, however, $\gamma^1$ does not link $C'$ since it bounds in $S(A, \eta)$. But by the deformation theorem of Pontrjagin, $\gamma^1$ must link $C'$, since during the deformation $C$ does not meet $\gamma^1$.

If $C'$ lies in $D_1$, a contradiction is established in the same manner. We must conclude, then, that there exists an $\epsilon > 0$ such that no 1-cycle of $E_n - C$ links $C$, and accordingly $C$ is a closed two-dimensional manifold.

**Problem:** In $E_n, n > 3$, is a compact and $\Delta$-deformable continuum an $(n - 1)$-dimensional closed manifold?

$^{13)}$ By Theorem 21 of my paper on the properties ..., referred to in footnote 13).