En définitive, nous pouvons énoncer le théorème suivant.

**Théorème.** Soient, dans un espace euclidien à $n$ dimensions, $o_i$, $i=1, 2, \ldots, n$, $n$ multiplicités linéaires à $n-2$ dimensions, à distance finie ou non, telles que le lieu des droites s'appuyant sur elles toutes soit une multiplicité à $n-1$ dimensions $[o]$, et $AB$ un arc simple sans point sur $[o]$.

1° - si $AB$ est d'ordre borné par rapport aux faisceaux d'arêtes $o$, l'arc est rectifiable,

2° - si $AB$ est d'ordre fini par rapport aux faisceaux, l'arc est quasi-rectifiable; on peut le remplacer par un arc simple rectifiable (d'ordre borné par rapport aux faisceaux) coïncidant avec lui sauf peut-être aux points d'un ensemble ouvert, sur $AB$, dont l'image, dans une correspondance arbitrairement donnée, a une mesure aussi petite qu'on veut; de plus, entre deux points quelconques de $AB$ il y a un arc partiel rectifiable.

En prenant pour $AB$ une courbe $y=f(x)$ d'ordre fini par rapport à la direction $y=0$, on a le cas particulier suivant, qui donne une propriété intéressante de certaines fonctions continues.

Une fonction continue d'une variable qui prend chacune de ses valeurs seulement un nombre fini de fois possède une dérivée presque partout; on peut la modifier sur un ensemble de mesure aussi petite qu'on veut de manière à obtenir une fonction continue à variation bornée.

13. Il faut remarquer que la restriction relative à l'absence de points sur $[o]$ ne peut être levée. Supposons $n=2$ et soit $H$ un hyperbolôde dont les génératrices sont les courbes coordonnées $u=\alpha$ et $v=\beta$. Prenons sur $H$ un arc simple $AB$ défini par $v=f(u)$, où $f(u)$ est une fonction continue sans dérivée, cet arc ne rencontrant pas les trois génératrices $o_i$, $i=1, 2, 3$. $AB$ est d'ordre un par rapport aux faisceaux d'arêtes $o_i$. Or ici $[o_1, o_2, o_3]$ n'est autre que $H$.

14. Il est bien évident qu'on pourrait remplacer les faisceaux par des familles continues de multiplicités à $n-1$ dimensions, pas nécessairement linéaires, ni même algébriques, satisfaisant à certaines conditions.

---

**Concerning regular accessibility.**

By **Beatrice Aitchison** (Baltimore, U. S. A.).

1. This paper considers the subject of regular accessibility from a new standpoint suggested by the theorems of Moore and Menger on the arcwise connectivity of certain $G_2$ sets. A general theorem on the regular accessibility of limit points of connected, locally connected $G_2$ sets is proved. Most of the known theorems on regular accessibility are special cases or immediate consequences of this proposition, and in several cases are true in a more general form than that in which they were originally stated. Treatment of these theorems in the light of the theorem proved in this paper results in shorter and simpler proofs than have hitherto been obtained.

2. **Definitions.** The space concerned in this paper is connected, locally connected, locally compact, separable and metric. An hereditarily locally connected continuum is a locally connected continuum, every subcontinuum of which is locally connected. A set which is the product of a countable number of open sets is called a $G_2$ set. In the space considered, a $G_2$ set is equivalent to a "complete space" 1). A point $P$ is said to be accessible from a set $M$, provided that, if $X$ is any point of $M$, there exists a simple continuous arc from $X$ to $P$, contained wholly in $M$ except for $P$. A point $P$ is said to be regularly accessible 2) from a set $M$, provided that, for any given positive number $\varepsilon$ there exists a positive number $\delta$, such that any point $X$ of $M$ at a distance less than $\delta$ from $P$ may

---

1) P. Alexandroff, Comptes Rendus, 178, p. 156.
be joined to \( P \) by a simple continuous arc, of diameter less than \( \varepsilon \), and contained wholly in \( \mathcal{M} \) except for \( P \). A connected set \( \mathcal{M} \) is said to have property \( 3' \), provided that, for any positive number \( \varepsilon \), \( \mathcal{M} \) is the sum of a finite number of connected sets, each of diameter less than \( \varepsilon \).

3. Fundamental Theorems. The following theorem and lemmas will be needed in the proof of our main proposition. They will be stated without proof, except for lemma 6, as the demonstrations are either obvious or may be found in the literature.

Theorem A. If \( N \) is a connected, locally connected \( G_\delta \) set, it is arcwise connected 7).

This theorem is indispensable for our purposes, as we shall use it to prove the local arcwise connectivity of the sets to be considered.

Lemma 1. Any connected, locally connected, locally compact, separable, metric space \( S \) is a \( G_\delta \) set.

Lemma 2. If \( G\) and \( E \) are any two \( G_\delta \) sets in \( S \), then \( G + E \) is a \( G_\delta \) set.

Lemma 3. Every open subset of a \( G_\delta \) set is a \( G_\delta \) set.

Lemma 4. If \( G \) is a connected, locally connected \( G_\delta \) set, and \( N \) is a connected open subset of \( G \), then \( N \) is a connected, locally connected \( G_\delta \) set.

The following general proposition on regular accessibility will now be established on the basis of the above theorem and lemmas:

Theorem B. A necessary and sufficient condition that a point \( P \) be regularly accessible from a connected, locally connected \( G_\delta \) set, \( G \), is that \( G + P \) should be connected and locally connected.


\[ \text{Regular accessibility} \]

Proof. The condition is necessary, for it is obvious from the definition of regular accessibility that \( G + P \) is connected and locally connected.

The condition is also sufficient. Assume that \( G + P \) is connected and locally connected. If \( P \subseteq G \) and, given any positive number \( \varepsilon \), any connected open subset of diameter \( < \varepsilon \) of \( G \) containing \( P \) is a connected, locally connected \( G_\delta \) set (lemma 4), and hence is arcwise connected. Therefore \( P \) is regularly accessible from \( G \). If \( P \) is not contained in \( G \), it must be a boundary point of \( G \) as \( G + P \) is connected. Then \( G + P \) is a connected, locally connected \( G_\delta \) set (lemmas 2, 4). For any given positive number \( \varepsilon \) there exists a positive number \( \delta \), such that any point of \( G \) whose distance from \( P \) is less than \( \delta \) can be joined to \( P \) by a connected subset of \( G + P \) of diameter \( < \varepsilon \). Let \( N \) be the component containing \( P \) of the set of all points of \( G + P \) at a distance less than \( \varepsilon/2 \) from \( P \). Then \( N \) is a connected open subset of \( G + P \), and therefore a connected, locally connected \( G_\delta \) set (lemma 4) and is arcwise connected (Theorem A). Hence every point of \( G \) at a distance less than \( \delta \) from \( P \) can be joined to \( P \) by an arc lying wholly in \( N \) and of diameter \( < \varepsilon \). Therefore \( P \) is regularly accessible from \( G \).

Corollary. A necessary and sufficient condition that any limit point \( P \) of a connected, locally connected, locally compact, separable, metric set \( S \) be regularly accessible from \( S \) is that \( S + P \) should be locally connected 8).

Lemma 5. If a set \( R \) has property \( S \), then \( R \) is locally connected 8).

Lemma 6. If \( R \) is a connected open subset of a connected, locally connected, locally compact, separable, metric space, and has property \( S \), and if \( R \subseteq R_0 \subseteq R \), then \( R_0 \) has property \( S \).

Proof. Given any positive number \( \varepsilon \), \( R = \bigcup_{i=1}^{\infty} K_i \), where \( K_i \) is a connected subset of \( R \) of diameter \( < \varepsilon \). Every point of \( R_0 \) — \( R \) is not contained in \( R \).

8) Special cases of this theorem are Theorem 3 and its corollary in G. T. Whyburn, Concerning accessibility in the plane and regular accessibility in \( n \) dimensions, Bull. Amer. Math. Soc., vol. 34 (1928), pp. 509—510.

is a limit point of some one of these subsets. If each $K_i$ has added to its limit points in $R_0 = R$, it is still connected and of diameter $< \varepsilon$. Therefore $R_2 = R_1 \cdot R_1 \cdot K_i$ has property S.

4. Applications. Having proved these theorems, we shall now turn to some of the theorems on accessibility found in the literature. Several of these propositions originally involved plane sets alone, but are valid in $n$ dimensions. It will be noted that Theorem B of this paper holds for any number of dimensions.

Definition. A set $K$ is said to be uniformly locally connected with reference to every one of its compact subsets, provided it is true that if $M$ is any compact set whatever, and $\varepsilon$ is any positive number, then there exists a positive number $\delta$, such that every two points common to $M$ and $K$, whose distance apart is less than $\delta$, lie together in a connected subset of $K$ of diameter less than $\varepsilon$.

Theorem 1. If the domain $D$ is uniformly locally connected with reference to every one of its compact subsets, then every point of the boundary of $D$ is regularly accessible from $D$.

Proof. The domain $D$ is a connected, locally connected $G_2$ set (lemma 1) and hence every connected open subset of $D$ is a connected, locally connected $G_2$ set (lemma 4). If $P$ is a point of $F(D)$, then, given any positive number $\varepsilon$, let $R$ be a compact spherical region of diameter $< \varepsilon$ with $P$ as center. As $P$ is a limit point of $D$, $D + P$ is a connected $G_2$ set. The set $R - D$ is a compact subset of $D$. As $D$ is uniformly locally connected with reference to every one of its compact subsets, for $\varepsilon' := \varepsilon/3$, there exists a positive number $\delta', \varepsilon'$, such that any two points of $R - D$ at a distance apart $< \delta'$ lie together in a connected subset of $D$ of diameter $< \varepsilon'$. Therefore every two points at a distance $< \delta/2$ from $P$ lie in a connected subset of $D$ of diameter $< \varepsilon/2$. Hence all points of $D$ at a distance $< \delta/2$ from $P$ lie in one connected subset $I$ of $D$. The set $I$ is the component of $R - D$ containing every point $X$ such that $q(XP) < \delta/2$. Then $I$ is of diameter $< \varepsilon$, and has $P$ as a limit point. Hence $I + P$ is connected. As every point of

$D + P$ at a distance $< \delta/2$ from $P$ can be joined to $P$ by a connected subset of $D + P$ of diameter $< \varepsilon$, $D + P$ is locally connected. Therefore $P$ is regularly accessible from $D$ (theorem B). But $P$ was any point of $F(D)$. Hence every point of $F(D)$ is regularly accessible from $D$.

Theorem 2. If $G$ is any $G_4$ subset of an hereditarily locally connected continuum, $M$, then the boundary points of each component $C$ of $G$ are regularly accessible from $C$.

Proof. If $P$ is a point on the boundary $B$ of a component $C$ of a $G_4$ set of $M$, then $C + P$ is a connected, locally connected $G_4$ set, as $M$ is an hereditarily locally connected continuum. Hence $P$ is regularly accessible from $C$ (theorem B). But $P$ was any point of the boundary of $C$. Hence every point of the boundary of $C$ is regularly accessible from $C$.

Corollary. If $R$ is a connected open subset of an hereditarily locally connected continuum $M$, then every point of the boundary of $R$ is regularly accessible from $R$.

Theorem 3. In order that a boundary point $P$ of a domain $D$ of a locally connected continuum $M$ be regularly accessible from $D$ it is sufficient that if $C_1$ is any spherical region with center at $P$, there should exist a concentric spherical region $C_0$ with radius smaller than that of $C_1$, such that only a finite number of the components of $D - C_0$ contain points of $C_0$.

Proof. Let $P$ be a point of $F(D)$. Then $D + P$ is a connected $G_2$ set (lemmas 1, 2), and $D$ is locally connected. By hypothesis, given any positive number $\varepsilon$, if $C_1$ is a spherical region of radius $\varepsilon/2$ with center at $P$, there exists a concentric spherical region $C_0$ with radius $\eta$ ($0 < \eta < \varepsilon/2$), such that only a finite number of the

---

11) G. T. Whyburn, Concerning the open subsets of a plane continuous curve, loc. cit., p. 653, theorem 3. As stated here, this theorem holds for $n$-space.
components of \( D \cdot C \) contain points of \( C \). As \( P \) is a limit point of \( D \), there must be at least one component of \( D \cdot C \) which has \( P \) as a limit point. Let \( Q \) be the sum of all the components of \( D \cdot C \) which have \( P \) as a limit point. Then \( Q \cup P \) is connected. There exists a spherical region \( C \) with center at \( P \) and radius \( \delta < \gamma \), such that \( C \) contains no points of \( D \cdot C \) \( \cap \) \( Q \). Then any point \( \Delta \subset D \cdot C \) can be joined to \( P \) by a connected subset, \( Q \cup P \), of \( D \) of diameter \( < \varepsilon \). Therefore \( D \cup P \) is locally connected, and \( P \) is regularly accessible from \( D \) (Theorem B).

Definitions. The point \( P \) is a regular point of a continuum \( M \), provided that, given any positive number \( \varepsilon \), there exists an open subset of \( M \), containing \( P \), of diameter \( < \varepsilon \), whose boundary is finite. A continuum \( M \) is said to be a regular curve, provided that every point of \( M \) is a regular point of \( M \). The point \( P \) is a one-dimensional point of a continuum \( M \) if, given any positive number \( \varepsilon \), there exists an open subset of \( M \), containing \( P \), of diameter less than \( \varepsilon \), whose boundary contains no nondegenerate connected sets. A continuum \( M \) is said to be a one-dimensional curve, if every point \( P \) of \( M \) is a one-dimensional point of \( M \).

The following theorems 4–6 are true in Euclidean space.

Theorem 4. If \( P \) is a regular point of a continuum \( M \), then \( P \) is regularly accessible from every complementary domain of \( M \) to whose boundary it belongs.\(^{13}\).

Proof. Let \( P \) be a regular point of a continuum \( M \), which is contained in \( F(D) \), where \( D \) is a complementary domain of \( M \). Then \( R \) is a connected, locally connected \( G_2 \) set (lemmas 1, 4). Hence \( R \cup P \) is a connected \( G_2 \) set. Given any positive number \( \varepsilon \), let \( B \) be an \( \varepsilon/3 \) neighborhood of \( P \), whose boundary contains only a finite number of points of \( M \). Let \( Q \) be the sum of all those components of \( D \cdot R \) which have \( P \) as a limit point. Then \( Q \cup P \) is connected. Either \( P \) is not a limit point of \( D \cdot R \) \( \cap \) \( Q \) or there exists a sequence \( \{A_i\} \) of distinct components of \( D \cdot R \), not contained in \( Q \), the superior limit of which contains \( P \).

If \( P \) is not a limit point of \( D \cdot R \) \( \cap \) \( Q \), there exists a positive number \( \delta \) such that the \( \delta \) neighborhood of \( P \) contains no points of \( D \cdot R \) \( \cap \) \( Q \). Then every point \( X \) of \( R \) at a distance \( < \delta \) from \( P \) can be joined to \( P \) by a connected subset, \( Q \cup P \), of \( R \) \( \cup \) \( P \), of diameter \( < \varepsilon \). Hence \( R \cup P \) is locally connected, and \( P \) is regularly accessible from \( R \) (Theorem B).

Suppose, however, that there does exist an infinite collection \( \{A_i\} \) of distinct components of \( D \cdot R \), not contained in \( Q \), the superior limit of which contains \( P \). If the containing space is a plane, \( D \) may be chosen so that \( F(D) \) is a simple closed curve, containing only a finite number of points of \( M \). As \( R \) is connected, each \( A_i \) has a point \( a_i \) contained in \( F(D) \) which must belong to \( R \). Some two of these points \( a_i \) and \( a_j \) must lie between two points of the finite set of points of \( M \) on \( F(D) \). Then \( A_i \cup a_i \cup a_j \cup A_j \), where \( a_i \cup a_j \) is an arc on \( F(D) \) containing no points of \( M \), is a connected set of diameter \( < \varepsilon \). Therefore the two components \( A_i \) and \( A_j \) are not distinct, and the assumption that this case is possible leads to a contradiction.

In Euclidean \( n \)-space \( (n > 2) \) by a theorem of Urysohn\(^{14}\), \( D \) may be chosen so that \( F(D) \) \( \cap \) \( F(D) \cdot M \) is a connected open subset of \( F(D) \) containing no points of \( M \). As every \( A_i \) must have at least one point in \( F(D) \) \( \cap \) \( F(D) \cdot M \), \( A_i \cup A_j \cup F(D) \) \( \cap \) \( F(D) \cdot M \) is a connected set of diameter \( < \varepsilon \). Therefore the two components \( A_i \) and \( A_j \) are not distinct. Hence the assumption that such a collection of components exists leads to a contradiction.

Theorem 5. If \( P \) is any one-dimensional point of a continuum \( M \) in \( E_n \) \( (n > 2) \), then \( P \) is regularly accessible from every complementary domain of \( M \) in \( E_n \) to whose boundary it belongs.\(^{15}\).

Proof. If we bear in mind that the theorem of Urysohn referred to in theorem 4 holds if \( F(D) \cdot M \) is a null-dimensional set in \( n \)-space \( (n > 2) \), the proof of this theorem is obtained in the same manner as that of theorem 4.


\(^{15}\) G. T. Whyburn, loc. cit., p. 276, theorem 5.
Regular accessibility.

Proof. The set \( R \) has property \( S \). Hence \( R \cup P \) has property \( S \) (lemma 6) and is a connected, locally connected \( G_2 \) set (lemmas 1, 2, 4). Therefore \( P \) is regularly accessible from \( R \) (theorem B).

Theorem 8 is also an immediate consequence of theorem 4 and the theorem of G. T. Whyburn \(^{29} \) which states that if the locally connected continuum \( M \) is the boundary of a plane region \( R \), then \( M \) is a regular curve. For then under the conditions of theorem 8, every point of \( M \) is a regular point, and therefore, by theorem 4, is regularly accessible from \( R \).

\(^{29} \) See Concerning Menger regular curves, loc. cit., p. 267.

The Johns Hopkins University.