En définitive, nous pouvons énoncer le théorème suivant.

Théorème. *Soient, dans un espace euclidien à n dimensions,* ω_i, $(i=1, 2, \dots, n)$, *n multiplicités linéaires à* $n-2$ *dimensions, à distance finie ou non, telles que le lieu des droites s'appuyant sur elles toutes soit une multiplicité à* $n-1$ *dimensions* $[\omega_i]$, *et* $\widehat{AB}$ *un arc simple sans point sur* $[\omega_i]$,

1º — *si* $\widehat{AB}$ *est d'ordre borné par rapport aux faisceaux d'arêtes* ω_i, *l'arc est rectifiable,*

2º — *si* $\widehat{AB}$ *est d'ordre fini par rapport aux faisceaux, l'arc est quasi-rectifiable: on peut le remplacer par un arc simple rectifiable* (d'ordre borné par rapport aux faisceaux) *coïncidant avec lui sauf peut-être aux points d'un ensemble ouvert, sur* $\widehat{AB}$, *dont l'image, dans une correspondance arbitrairement donnée, a une mesure aussi petite qu'on veut; de plus, entre deux points quelconques de* $\widehat{AB}$ *il y a un arc partiel rectifiable.*

En prenant pour $\widehat{AB}$ une courbe $y=f(x)$ d'ordre fini par rapport à la direction $y=0$, on a le cas particulier suivant, qui donne une propriété intéressante de certaines fonctions continues.

Une fonction continue d'une variable qui prend chacune de ses valeurs seulement un nombre fini de fois possède une dérivée presque partout; on peut la modifier sur un ensemble de mesure aussi petite qu'on veut de manière à obtenir une fonction continue à variation bornée.

13. Il faut remarquer que la restriction relative à l'absence de points sur $[\omega_i]$ ne peut être levée. Supposons $n=3$ et soit H un hyperboloïde dont les génératrices sont les courbes coordonnées $u=c^{te}$ et $v=c^{te}$. Prenons sur H un arc simple $\widehat{AB}$ défini par $v=f(u)$, où $f(u)$ est une fonction continue sans dérivée, cet arc ne rencontrant pas les trois génératrices ω_i, $v=v_i$, $(i=1, 2, 3)$. $\widehat{AB}$ est d'ordre *un* par rapport aux faisceaux d'arêtes ω_i. Or ici $[\omega_1, \omega_2, \omega_3]$ n'est autre que H.

14. Il est bien évident qu'on pourrait remplacer les faisceaux par des familles continues de multiplicités à $n-1$ dimensions, pas nécessairement linéaires, ni même algébriques, satisfaisant à certaines conditions.

Concerning regular accessibility.

By

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1. This paper considers the subject of regular accessibility from a new standpoint suggested by the theorems of Moore and Menger on the arcwise connectivity of certain G_δ sets. A general theorem on the regular accessibility of limit points of connected, locally connected G_δ sets is proved. Most of the known theorems on regular accessibility are special cases or immediate consequences of this proposition, and in several cases are true in a more general form than that in which they were originally stated Treatment of these theorems in the light of the theorem proved in this paper results in shorter and simpler proofs than have hitherto been obtained.

2. Definitions. The space concerned in this paper is connected, locally connected, locally compact, separable and metric. An hereditarily locally connected continuum is a locally connected continuum, every subcontinuum of which is locally connected. A set which is the product of a countable number of open sets is called a G_δ set. In the space considered, a G_δ set is equivalent to a „complete space“ [1]). A point P is said to be accessible from a set M, provided that, if X is any point of M, there exists a simple continuous arc from X to P, contained wholly in M except for P. A point P is said to be regularly accessible [2]) from a set M, provided that, for any given positive number ε there exists a positive number δ, such that any point X of M at a distance less than δ from P may

[1]) P. Alexandroff, Comptes Rendus, 178, p. 185.

[2]) G. T. Whyburn, *Concerning the open subsets of a plane continuous curve*, Proc. Nat. Acad. Sc., vol. 13 (1927), p. 650

be joined to P by a simple continuous arc, of diameter less than ε, and contained wholly in M except for P. A connected set M is said to have property S [3]), provided that, for any positive number ε, M is the sum of a finite number of connected sets, each of diameter less than ε.

3. Fundamental Theorems. The following theorem and lemmas will be needed in the proof of our main proposition. They will be stated without proof, except for lemma 6, as the demonstrations are either obvious or may be found in the literature.

Theorem A. If N is a connected, locally connected G_δ set, it is arcwise connected [4]).

This theorem is indispensable for our purposes, as we shall use it to prove the local arcwise connectivity of the sets to be considered.

Lemma 1. Any connected, locally connected, locally compact, separable, metric space S is a G_δ set.

Lemma 2. If G and E are any two G_δ sets in S, then $G+E$ is a G_δ set.

Lemma 3. Every open subset of a G_δ set is a G_δ set.

Lemma 4. If G is a connected, locally connected G_δ set, and N is a connected open subset of G, then N is a connected, locally connected G_δ set.

The following general proposition on regular accessibility will now be established on the basis of the above theorem and lemmas:

Theorem B. A necessary and sufficient condition that a point P be regularly accessible from a connected, locally connected G_δ set, G, is that $G+P$ should be connected and locally connected.

[3]) W. Sierpiński, *Sur une condition pour qu'un continu soit une courbe jordanienne.* Fund. Math., vol. 1 (1920), p. 44, and also R. L. Moore, *Concerning connectedness im kleinem and a related property*, Fund. Math., vol. 3 (1922), p. 232.

[4]) R. L. Moore, Bull. Amer. Math. Soc., vol. 33 (1927), p. 141 (abstract). See also K. Menger, *Über die Dimension von Punktmengen III, Zur Begründung einer axiomatischer Theorie der Dimension*, Monatshefte für Math. und Phys., vol. 36 (1929), p. 210, and C. Kuratowski, *Sur les espaces complets*, Fund. Math., vol. 15 (1930), pp. 306—309.

Proof. The condition is necessary, for it is obvious from the definition of regular accessibility that $G+P$ is connected and locally connected.

The condition is also sufficient. Assume that $G+P$ is connected and locally connected. If $P \subset G$, and, given any positive number ε, any connected open subset of diameter $<\varepsilon$ of G containing P is a connected, locally connected G_δ set (lemma 4), and hence is arcwise connected. Therefore P is regularly accessible from G. If P is not contained in G, it must be a boundary point of G as $G+P$ is connected. Then $G+P$ is a connected, locally connected G_δ set (lemmas 2, 4). For any given positive number ε there exists a positive number δ, such that any point of G whose distance from P is less than δ can be joined to P by a connected subset of $G+P$ of diameter $<\varepsilon$. Let N be the component containing P of the set of all points of $G+P$ at a distance less than $\varepsilon/2$ from P. Then N is a connected open subset of $G+P$, and is therefore a connected, locally connected G_δ set (lemma 4) and is arcwise connected (theorem A). Hence every point of G at a distance less than δ from P can be joined to P by an arc lying wholly in N and of diameter $<\varepsilon$. Therefore P is regularly accessible from G.

Corollary. A necessary and sufficient condition that any limit point P of a connected, locally connected, locally compact, separable, metric set S be regularly accessible from S is that $S+P$ should be locally connected [5]).

Lemma 5. If a set R has property S, then R is locally connected [6]).

Lemma 6. If R is a connected open subset of a connected, locally connected, locally compact, separable, metric space, and has property S, and if $R \subset R_0 \subset \overline{R}$, then R_0 has property S.

Proof. Given any positive number ε, $R = \sum_{i=1}^{n} K_i$ where K_i is a connected subset of R of diameter $<\varepsilon$. Every point of $R_0 - R$

[5]) Special cases of this theorem are Theorem 3 and its corollary in G. T. Whyburn, *Concerning accessibility in the plane and regular accessibility in n dimensions.* Bull. Amer. Math. Soc., vol. 34 (1928), pp. 509—510.

[6]) R. L. Moore, *Concerning connectedness im kleinem and a related property*, loc. cit., pp. 233—234, theorem 2.

is a limit point of some one of these subsets. If each K_i has added to it its limit points in $R_0 - R$, it is still connected and of diameter $< \varepsilon$. Therefore $R_0 = R_0 \cdot \sum_{i=1}^{n} \overline{K}_i$ has property S.

4. Applications. Having proved these theorems, we shall now turn to some of the theorems on accessibility found in the literature. Several of these propositions originally involved plane sets alone, but are valid in n dimensions. It will be noted that Theorem B of this paper holds for any number of dimensions.

Definition. A set K is said to be uniformly locally connected with reference to every one of its compact subsets, provided it is true that if M is any compact set whatever, and ε is any positive number, then there exists a positive number δ, such that every two points common to M and K, whose distance apart is less than δ, lie together in a connected subset of K of diameter less than ε.

Theorem 1. If the domain D is uniformly locally connected with reference to every one of its compact subsets, then every point of the boundary of D is regularly accessible from D [7]).

Proof. The domain D is a connected, locally connected G_δ set (lemma 1) and hence every connected open subset of D is a connected, locally connected G_δ set (lemma 4). If P is a point of $F(D)$, then, given any positive number ε, let R be a compact spherical region of diameter $\leqslant \varepsilon$ with P as center. As P is a limit point of D, $D+P$ is a connected G_δ set. The set $R \cdot D$ is a compact subset of D. As D is uniformly locally connected with reference to every one of its compact subsets, for $\varepsilon' = \varepsilon/3$, there exists a positive number $\delta_{\varepsilon'}$, such that any two points of $R \cdot D$ at a distance apart $< \delta$ lie together in a connected subset of D of diameter $< \varepsilon'$. Therefore every two points at a distance $< \delta/2$ from P lie in a connected subset of D of diameter $< \varepsilon/3$. Hence all points of D at a distance $< \delta/2$ from P lie in one connected subset I of D. The set I is the component of $R \cdot D$ containing every point X such that $\varrho(XP) < \delta/2$. Then I is of diameter $< \varepsilon$, and has P as a limit point. Hence $I+P$ is connected. As every point of $D+P$ at a distance $< \delta/2$ from P can be joined to P by a connected subset of $D+P$ of diameter $< \varepsilon$, $D+P$ is locally connected. Therefore P is regularly accessible from D (theorem B). But P was any point of $F(D)$. Hence every point of $F(D)$ is regularly accessible from D.

Theorem 2. If G is any G_δ subset of an hereditarily locally connected continuum, M, then the boundary points of each component C of G are regularly accessible from C [8]).

Proof. If P is a point on the boundary B of a component C of a G_δ set of M, then $C+P$ is a connected, locally connected G_δ set, as M is an hereditarily locally connected continuum [9]). Hence P is regularly accessible from C (theorem B). But P was any point of the boundary of C. Hence every point of the boundary of C is regularly accessible from C.

Corollary. If R is a connected open subset of an hereditarily locally connected continuum M, then every point of the boundary of R is regularly accessible from R [10]).

Theorem 3. In order that a boundary point P of a domain D of a locally connected continuum M be regularly accessible from D it is sufficient that if C_1 is any spherical region with center at P, there should exist a concentric spherical region C_2 with radius smaller than that of C_1, such that only a finite number of the components of $D \cdot C_1$ contain points of C_2 [11]).

Proof. Let P be a point of $F(D)$. Then $D+P$ is a connected G_δ set (lemmas 1, 2), and D is locally connected. By hypothesis, given any positive number ε, if C_1 is a spherical region of radius $\varepsilon/2$ with center at P, there exists a concentric spherical region C_2 with radius η $(0 < \eta < \varepsilon/2)$, such that only a finite number of the

[7]) G. T. Whyburn, *Concerning continua in the plane*, Trans. Amer. Math. Soc., vol. 29 (1927), p. 375. As stated here, this theorem holds for n-space, and also gives regular accessibility.

[8]) R. L. Wilder, *Concerning perfect continuous curves*, Proc. Nat. Acad. Sc., vol. 16 (1930), p. 236, theorem 2.

[9]) See R. L. Wilder, loc. cit., p. 235.

[10]) G. T. Whyburn, *Concerning the open subsets of a plane continuous curve*, loc. cit., p. 653, theorem 3. As stated here, this theorem holds for n-space.

[11]) W. L. Ayres, *Concerning continuous curves in metric space*, Amer. Jour. Math., vol. 51 (1929), p. 578. This theorem as here stated gives regular accessibility.

components of $D \cdot C_1$ contain points of C_2. As P is a limit point of D, there must be at least one component of $D \cdot C_1$ which has P as a limit point. Let Q be the sum of all the components of $D \cdot C_1$ which have P as a limit point. Then $Q + P$ is connected. There exists a spherical region C_3 with center at P and radius $\delta \leqq \eta$, such that C_3 contains no points of $D \cdot C_1 - Q$. Then any point $X \subset D \cdot C_3$ can be joined to P by a connected subset, $Q + P$, of $D + P$ of diameter $< \varepsilon$. Therefore $D + P$ is locally connected, and P is regularly accessible from D (theorem B).

Definitions. The point P is a regular point of a continuum M, provided that, given any positive number ε, there exists an open subset of M, containing P, of diameter $< \varepsilon$, whose boundary is finite. A continuum M is said to be a regular curve, provided that every point of M is a regular point of M. The point P is a one-dimensional point of a continuum M if, given any positive number ε, there exists an open subset of M, containing P, of diameter less than ε, whose boundary contains no nondegenerate connected sets. A continuum M is said to be a one-dimensional curve, if every point P of M is a one-dimensional point of M.

The following theorems 4—6 are true in Euclidean space.

Theorem 4. If P is a regular point of a continuum M, then P is regularly accessible from every complementary domain of M to whose boundary it belongs [12]).

Proof. Let P be a regular point of a continuum M, which is contained in $F(R)$, where R is a complementary domain of M. Then R is a connected, locally connected G_δ set (lemmas 1, 4). Hence $R + P$ is a connected G_δ set. Given any positive number ε, let D be an $\varepsilon/3$ neighborhood of P, whose boundary contains only a finite number of points of M. Let Q be the sum of all those components of $\overline{D} \cdot R$ which have P as a limit point. Then $Q + P$ is connected. Either P is not a limit point of $\overline{D} \cdot R - Q$ or there exists

[12]) G. T. Whyburn, *Concerning Menger regular curves*, Fund. Math., vol. 12 (1928), pp. 275, theorem 4. A special case of this theorem is theorem 27 of H. M. Gehman, *Concerning endpoints of continuous curves and other continua*, Trans. Amer. Math. Soc., vol. 30 (1928), p. 83. In this case the regular point is an end point.

a sequence $\{A_i\}$ of distinct components of $\overline{D} \cdot R$, not contained in Q, the superior limit of which contains P.

If P is not a limit point of $\overline{D} \cdot R - Q$, there exists a positive number δ such that the δ neighborhood of P contains no points of $\overline{D} \cdot R - Q$. Then every point X of R at a distance $< \delta$ from P can be joined to P by a connected subset, $Q + P$, of $R + P$, of diameter $< \varepsilon$. Hence $R + P$ is locally connected, and P is regularly accessible from R (theorem B).

Suppose, however, that there does exist an infinite collection $\{A_i\}$ of distinct components of $\overline{D} \cdot R$, not contained in Q, the superior limit of which contains P. If the containing space is a plane, D may be chosen so that $F[D]$ is a simple closed curve, containing only a finite number of points of M. As R is connected, each A_i has a point a_i contained in $F(D)$ which must belong to R. Some two of these points a_i and a_j must lie between two points of the finite set of points of M on $F(D)$. Then $A_i + a_i a_j + A_j$, where $a_i a_j$ is an arc on $F(D)$ containing no points of M, is a connected set of diameter $< \varepsilon$. Therefore the two components A_i and A_j are not distinct, and the assumption that this case is possible leads to a contradiction.

In Euclidean n-space ($n > 2$) by a theorem of Urysohn [13]), D may be chosen so that $F(D) - F(D) \cdot M$ is a connected open subset of $F(D)$ containing no points of M. As every A_i must have at least one point in $F(D) - F(D) \cdot M$, $A_i + A_j + F(D) - F(D) \cdot M$ is a connected set of diameter $< \varepsilon$. Therefore the two components A_i and A_j are not distinct. Hence the assumption that such a collection of components exists leads to a contradiction.

Theorem 5. If P is any one-dimensional point of a continuum M in E_n ($n > 2$), then P is regularly accessible from every complementary domain of M in E_n to whose boundary it belongs [14]).

Proof. If we bear in mind that the theorem of Urysohn referred to in theorem 4 holds if $F(D) \cdot M$ is a null-dimensional set in n-space ($n > 2$), the proof of this theorem is obtained in the same manner as that of theorem 4.

[13]) P. Urysohn, *Mémoire sur les multiplicités cantoriennes*, Fund. Math., vol. 7 (1925), p. 65.

[14]) G. T. Whyburn, loc. cit., p. 276, theorem 5.

Theorem 6. In a space of n dimensions ($n > 2$), all points of a one-dimensional continuum M are regularly accessible [15].

Proof. By a theorem of Urysohn [16], it is seen that there is just one complementary domain of M. Hence every point of M is regularly accessible from $S - M$ (theorem 5).

Theorem 7. In order that every point of the boundary B of a connected open subset R of a locally connected continuum M should be regularly accessible from R, it is sufficient that R should have property S [17].

Proof. The set R is a connected, locally connected G_δ set (lemma 1). If $P \subset B$, then $R + P$ is a connected G_δ set (lemma 2), has property S (lemma 6), and is therefore locally connected (lemma 5). Hence P is regularly accessible from R (theorem B). But P was any point of B. Hence every point of B is regularly accessible from R.

5. In consequence of a theorem of R. L. Moore [18], that if a simply connected, bounded plane domain R has a locally connected continuum as its boundary, R has property S, and of theorem B, we obtain at once the following theorem concerning accessibility in the plane:

Theorem 8. If P is a point of a locally connected continuum M which is the boundary of a plane region R, P is regularly accessible from R [19].

[15]) C. Zarankiewicz, *Sur les points de division dans les ensembles connexes*, Fund. Math. vol. 9 (1927), p. 166. This theorem was originally stated for ordinary accessibility.

[16]) P. Urysohn, loc. cit., p. 94. The common boundary of any two domains in n dimensions ($n > 2$) is at least two dimensional.

[17]) G. T. Whyburn, *Concerning the open subsets of a plane continuous curve*, loc. cit., pp. 650—651, theorem 1. As stated here this theorem holds in n-space.

[18]) R. L. Moore, *Concerning connectedness im kleinem, etc.*, loc. cit., pp. 236—237, theorem 4.

[19]) A. Schoenflies, *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten*, Zweiter Teil, Leipzig, 1908, p. 237. A special case when M is a simple closed curve is considered by C. Jordan (*Cours d'analyse*, 2 Ed., vol. 1, p. 92) and by R. L. Moore (*On the foundations of plane analysis situs*, Trans. Amer. Math. Soc., vol. 17, (1916), pp. 153—156).

Proof. The set R has property S. Hence $R+P$ has property S (lemma 6) and is a connected, locally connected G_δ set (lemmas 1, 2, 4). Therefore P is regularly accessible from R (theorem B).

Theorem 8 is also an immediate consequence of theorem 4 and the theorem of G. T. Whyburn [20]) which states that if the locally connected continuum M is the boundary of a plane region R, then M is a regular curve. For then under the conditions of theorem 8, every point of M is a regular point, and therefore, by theorem 4, is regularly accessible from R.

[20]) See *Concerning Menger regular curves*, loc. cit., p. 267.

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