Memoir on the Analytical Operations and Projective Sets (II).

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CHAPTER II.

Generalised Souslin's Operations.

§ 1. Definition and Immediate Consequences.

29. We have studied in the preceding chapter the properties of the δs-operations of Mr. Hausdorff, which are, as we have seen, the most general positive analytical operations effected upon a countable infinity of sets. There are however certain particular classes of these operations which present a special interest. Such are, for instance, the generalised Souslin's operations which are defined as follows:

Definition 18 [1]). Let

$$\{E_{n_1,n_2,\dots,n_k}\} \quad (k=1, 2,\dots, \ n_i=1, 2,\dots)$$

be a system of sets depending of corteges $(n_1, n_2,\dots, n_k)$, N a set of irrational numbers, $\mathfrak{N}\sim N$ the corresponding set of sequences of positive integers. *Then*

$$(29,1) \qquad \Omega_N(\{E_{n_1,n_2,\dots,n_k}\}) = \sum_{(n_1,n_2,\dots)\in\mathfrak{N}} \prod_k E_{n_1,n_2,\dots,n_k}.$$

The corresponding operation upon classes (see def. 4) we shall denote S_N, i. e.

[1]) Cp. Sierpiński VI. (List of Literature, Fund. Math. **18**, p. 218).

$S_N(\mathcal{K})$ is the class of all the sets of the form

$$\Omega_N(\{E_{n_1,n_2,\dots,n_k}\})$$

where all the

$$E_{n_1,n_2,\dots,n_k}\in\mathcal{K}.$$

Remark. In case $N=J$ (i. e. if $\mathfrak{N}\sim N$ is the set of *all* the sequences of natural numbers) the operation Ω_N becomes the well-known operation (A) of Mr. Souslin

$$\sum_{(n_1,n_2,\dots)} \prod_k E_{n_1,n_2,\dots,n_k}.$$

By Theorem I Cor. (Art. 6) the operation Ω_N is equivalent with a δs-operation $\Phi_{N'}$. We shall prove now that we can suppose N' homeomorphic to N.

Theorem XIX. *Let all the corteges $(n_1, n_2,\dots, n_k)$ be enumerated and let $\nu(n_1, n_2,\dots, n_k)$ be the natural number corresponding to $(n_1, n_2,\dots, n_k)$. Then there exists such N' homeomorphic to N that*

$$(29,2) \qquad \Omega_N(\{E_{n_1,n_2,\dots,n_k}\}) = \underset{\nu(n_1,n_2,\dots,n_k)}{\Phi_{N'}}(\{E_{n_1,n_2,\dots,n_k}\}).$$

Proof. Let

$$\xi=\frac{1|}{|n_1}+\frac{1|}{|n_2}+\dots\in J.$$

Denote

$$\varphi\,\xi=\frac{1\quad|}{|\nu(n_1)}+\frac{1\qquad|}{|\nu(n_1, n_2)}+\dots$$

Then $N'=\varphi N$ is evidently homeomorphic to N and satisfies (29, 2).

Corollary. *For any $\mathcal{K}$ we have*

$$S_N(\mathcal{K})=\mathcal{H}_{N'}(\mathcal{K}).$$

30. Here are some simple properties of Ω_N-functions.

1. For any N and $\mathcal{K}$ we have

$$(30,1) \qquad \mathcal{H}_N(\mathcal{K})\subset S_N(\mathcal{K}).$$

Let, in fact, $H\in\mathcal{H}_N(\mathcal{K})$. Then $H=\Phi_N(\{E_n\})$; $E_n\in\mathcal{K}$.

Denoting $E^*_{n_1,n_2,\dots,n_k} = E_{n_k}$ we have

$$H = \Omega_N(\{E^*_{n_1,n_2,\dots,n_k}\}) \in S_N(\mathcal{K}) \quad \text{q. e. d.}$$

$$(30,2) \quad 2. \quad \Omega_{\Sigma N_\xi}(\{E_{n_1,n_2,\dots,n_k}\}) = \sum_\xi \Omega_{N_\xi}(\{E_{n_1,n_2,\dots,n_k}\})$$

$$(30,3) \quad \Omega_N(\{E_{n_1,n_2,\dots,n_k} \cdot P\}) = \Omega_N(\{E_{n_1,n_2,\dots,n_k}\}) \cdot P$$

$$(30,4) \quad \Omega_N(\{E_{n_1,n_2,\dots,n_k} + P\}) = \Omega_N(\{E_{n_1,n_2,\dots,n_k}\}) + P$$

$$(30,5) \quad \Omega_N(\{E_{n_1,n_2,\dots,n_k} \times P\}) = \Omega_N(\{E_{n_1,n_2,\dots,n_k}\}) \times P$$

$$(30,6) \quad 3. \quad \sum_i \Omega_{N_i}(\{E^i_{n_1,n_2,\dots,n_k}\}) = \Omega_N(\{H_{n_1,n_2,\dots,n_k}\})$$

where

$$H_{2^{i-1}(2n_1-1),n_2,\dots,n_k} = E^i_{n_1,n_2,\dots,n_k}$$

and N is the set of all the numbers

$$\frac{1|}{|2^{i-1}(2n_1-1)} + \frac{1|}{|n_2} + \frac{1|}{|n_3} + \dots$$

such that

$$\frac{1|}{|n_1} + \frac{1|}{|n_2} + \dots \in N_i.$$

The proof is immediate.

On the contrary in general there is no theorem analogical to Th. VI (see Art. 34 below).

4. **Definition 10 bis.** The scheme for $\{E_{n_1,n_2,\dots,n_k}\}$ is defined by the formula

$$(30,7) \quad S(\{E_{n_1,n_2,\dots,n_k}\}) = \prod_k \sum_{n_1,n_2,\dots,n_k} E_{n_1,n_2,\dots,n_k} \times \delta_{n_1,n_2,\dots,n_k}.$$

Theorem V bis.

$$(30,8) \quad \Omega_N(\{E_{n_1,n_2,\dots,n_k}\}) = Pr_R(S(\{E_{n_1,n_2,\dots,n_k}\}) \cdot (R \times N)).$$

This theorem is analogical to Theorem V (Art. 9) and the proof is the same.

31. We shall give here some definitions that will be needed later on.

Definition 19. A class $\mathcal{K}$ of sets is a *d-class* if it possesses the following two properties:

1) $\sum_{E \in \mathcal{K}} E \in \mathcal{K}$. It is evidently the greatest set in $\mathcal{K}$.

2) If $A \in \mathcal{K}$ and $B \in \mathcal{K}$ then $AB \in \mathcal{K}$.

The family of all the d-classes we shall denote $[d]$.

Definition 20. A class $\mathcal{K}$ of sets is an *s-class* if it possesses the following properties:

1) $\prod_{E \in \mathcal{K}} E \in \mathcal{K}$. (In all important cases $\prod_{E \in \mathcal{K}} E = 0$)

2) If $A \in \mathcal{K}$ and $B \in \mathcal{K}$ then $A + B \in \mathcal{K}$.

The family of all the s-classes we shall denote $[s]$.

Definition 21. A class $\mathcal{K}$ of sets is a ring if it is a d-class and an s-class simultaneously [1]).

The family of all the rings we shall denote $[r]$ (i. e. $[r] = [s] \cdot [d]$).

Evidently if a class $\mathcal{K}$ of subsets of R is an s-class then $\mathcal{K}_c$ (see notations) is a d-class and *vice-versa*.

Definition 22. Two systems of sets $\{E_{n_1,n_2,\dots,n_k}\}$ and $\{E'_{n_1,n_2,\dots,n_k}\}$ are *equivalent* if for any N

$$\Omega_N(\{E_{n_1,n_2,\dots,n_k}\}) = \Omega_N(\{E'_{n_1,n_2,\dots,n_k}\}).$$

Definition 23. A system $\{E_{n_1,n_2,\dots,n_k}\}$ is *regular* if we have always (whatever be $n_1, n_2, \dots, n_k, n_{k+1}$)

$$E_{n_1,n_2,\dots,n_k,n_{k+1}} \subset E_{n_1,n_2,\dots,n_k}.$$

If $\mathcal{K} \in [d]$ then every system $\{E_{n_1,n_2,\dots,n_k}\}$ of sets belonging to $\mathcal{K}$ may be substituted by an equivalent regular system $\{E'_{n_1,n_2,\dots,n_k}\}$ of sets belonging to $\mathcal{K}$. It is sufficient to suppose $E'_{n_1,n_2,\dots,n_k} = \prod_{i=1}^{k} E_{n_1,n_2,\dots,n_i}$.

[1]) Cp. Hausdorff, p. 77.

§ 2. Equivalence and Inclusion Theorems.

32. Theorem XX. *In order that for every d-class $\mathcal{H}$ we should have.*

$$S_N(\mathcal{H}) \subset S_M(\mathcal{H})$$

it is necessary and sufficient that N be a continuous image of M: $N = \varphi M$.

Proof. A. Sufficiency. Let $N = \varphi M$ and let ψ denote the „inverse function" (see Art. 21 and formula (21, 2a—d)). Denote

(32, 1) $$\lambda_{n_1, n_2, \dots, n_k} = \psi(\delta_{n_1, n_2, \dots, n_k} \cdot N) \subset M$$

We shall now construct a system of open (in J) sets $h_{n_1, n_2, \dots, n_k}$ so that

(32, 2) 1) $h_{n_1, n_2, \dots, n_k} \cdot M = \lambda_{n_1, n_2, \dots, n_k}$

(32, 3) 2) $h_{n_1, n_2, \dots, n_k, n_{k+1}} \subset h_{n_1, n_2, \dots, n_k}$

(32, 4) 3) $h_{n_1, n_2, \dots, n_{k-1}, n_k} \cdot h_{n_1, n_2, \dots, n_{k-1}, n'_k} = 0$ if $n_k \neq n'_k$

Observe first of all that

(32, 2′) 1) $\lambda_{n_1, n_2, \dots, n_k}$ is open and closed in M simultaneously

(32, 3′) 2) $\lambda_{n_1, n_2, \dots, n_k, n_{k+1}} \subset \lambda_{n_1, n_2, \dots, n_k}$

(32, 4′) 3) $\lambda_{n_1, n_2, \dots, n_{k-1}, n_k} \cdot \lambda_{n_1, n_2, \dots, n_{k-1}, n'_k} = 0$ if $n_k \neq n'_k$

Denote for any $x \in \lambda_{n_1, n_2, \dots, n_k}$

(32, 5) $$\sigma_x^{(k)} = \tfrac{1}{2} \varrho(x, M - \lambda_{n_1, n_2, \dots, n_k}) > 0$$

(because $\lambda_{n_1, n_2, \dots, n_k}$ is open in M)

(32, 6) $$\begin{cases} S_x^{(k)} = (x - \sigma_x^{(k)}, \; x + \sigma_x^{(k)}) \\ h'_{n_1, n_2, \dots, n_k} = \sum\limits_{x \in \lambda_{n_1, n_2, \dots, n_k}} S_x^{(k)} \\ h_{n_1, n_2, \dots, n_k} = \prod\limits_{i=1}^{i=k} h'_{n_1, n_2, \dots, n_i}. \end{cases}$$

The sets $h_{n_1, n_2, \dots, n_k}$ satisfy all our conditions. In fact (**32**, 2) and (**32**, 3) are evidently satisfied. (**32**, 4) is also satisfied: for suppose the contrary i. e. that for certain $\{n_i\}$

$$h_{n_1, n_2, \dots, n_{k-1}, n_k} \cdot h_{n_1, n_2, \dots, n_{k-1}, n'_k} \neq 0 \quad (n_k \neq n'_k)$$

Then take $\xi \in h_{n_1, n_2, \dots, n_{k-1}, n_k} \cdot h_{n_1, n_2, \dots, n_{k-1}, n'_k}$. There exist such

$$x \in \lambda_{n_1, n_2, \dots, n_{k-1}, n_k} \quad \text{and} \quad x' \in \lambda_{n_1, n_2, \dots, n_{k-1}, n'_k}$$

that

$$\xi \in S_x^{(k)} \cdot S_{x'}^{(k)}$$

whence (by **32**, 5 and **32**, 6)

$$\varrho(x, \xi) < \tfrac{1}{2} \varrho(x, M - \lambda_{n_1, n_2, \dots, n_{k-1}, n_k}) \leqslant \tfrac{1}{2} \varrho(x, x')$$

$$\varrho(x', \xi) < \tfrac{1}{2} \varrho(x', M - \lambda_{n_1, n_2, \dots, n_{k-1}, n'_k}) \leqslant \tfrac{1}{2} \varrho(x, x').$$

Hence follows

$$\varrho(x, x') \leqslant \varrho(x, \xi) + \varrho(\xi, x') < \tfrac{1}{2} \varrho(x, x') + \tfrac{1}{2} \varrho(x, x')$$

which is impossible. This contradiction proves that (**32**, 4) is satisfied.

Let now $P \in S_N(\mathcal{H})$ i. e.

(32, 7) $$P = \Omega_N(\{E_{n_1, n_2, \dots, n_k}\}); \quad E_{n_1, n_2, \dots, n_k} \in \mathcal{H}.$$

We may suppose moreover that

(32, 8) $$E_{n_1, n_2, \dots, n_k, n_{k+1}} \subset E_{n_1, n_2, \dots, n_k}$$

($\mathcal{H}$ being a d-class; see Art. 31, end).

$\mathcal{H}$ has the greatest element (set) E (Art. **31**, def. 19).

We shall now define a system of sets $\{H_{m_1, m_2, \dots, m_i}\}$ as follows:

1) If $\delta_{m_1, m_2, \dots, m_i}$ is not contained in any $h_{n_1, n_2, \dots, n_k}$ then $H_{m_1, m_2, \dots, m_i} = E$,

2) If $\delta_{m_1, m_2, \dots, m_i}$ is contained in a finite number of h's viz.

$$\delta_{m_1, m_2, \dots, m_i} \subset h_{n_1} \cdot h_{n_1, n_2} \cdot h_{n_1, n_2, \dots, n_k};$$

$$\delta_{m_1, m_2, \dots, m_i} - h_{n_1, n_2, \dots, n_k, j} \neq 0 \quad \text{for any } j$$

then

$$H_{m_1, m_2, \dots, m_i} = E_{n_1, n_2, \dots, n_k}$$

3) If $\delta_{m_1, m_2, \dots, m_i}$ is contained in an infinity of h's viz.

$$\delta_{m_1, m_2, \ldots, m_i} \subset h_{n_1} \cdot h_{n_1, n_2} \cdot \ldots$$ [1])

then

$$H_{m_1, m_2, \ldots, m_i} = E_{n_1, n_2, \ldots, n_i}.$$

Evidently in all cases $H_{m_1, m_2, \ldots, m_i} \in \mathscr{H}$. If we now prove that

$$P = \Omega_M(\{H_{m_2, m_2, \ldots, m_i}\}) = Q$$

then the inclusion $S_N(\mathscr{H}) \subset S_M(\mathscr{H})$ shall be demonstrated.

α) $P \subset Q$. Let

$$x \in \Omega_N(\{E_{n_1, n_2, \ldots, n_k}\}).$$

Then there exists such

$$y = \frac{1|}{|n_1^0} + \frac{1|}{|n_2^0} + \ldots \in N$$

that

$$x \in \prod_k E_{n_1^0, n_2^0, \ldots, n_k^0}.$$

Let $z \in \psi y$ (ψK is non vacuous for any $K \subset N$)

$$z = \frac{1|}{|m_1^0} + \frac{1|}{|m_2^0} + \ldots \in M.$$

Then (for any i and k)

$$z \in \delta_{m_1^0, m_2^0, \ldots, m_i^0} \cdot \lambda_{n_1^0, n_2^0, \ldots, n_k^0} \subset \delta_{m_1^0, m_2^0, \ldots, m_i^0} \cdot h_{n_1^0, n_2^0, \ldots, n_k^0}$$

and consequently (considering **32**, 4) we can have

$$\delta_{m_1^0, m_2^0, \ldots, m_i^0} \subset h_{n_1, n_2, \ldots, n_k}$$

only if $n_1 = n_1^0$; $n_2 = n_2^0; \ldots$; $n_k = n_k^0$. Hence in all 3 cases

$$x \in H_{m_1^0, m_2^0, \ldots, m_i^0}$$

i. e.

$$x \in \prod_i H_{m_1^0, m_2^0, \ldots, m_i^0} \subset \Omega_M(\{H_{m_1, m_2, \ldots, m_i}\})$$

[1]) Evidently (by **32**, 3 and **32**, 4) if $\delta_{m_1, m_2, \ldots, m_i} \subset h_{n_1, n_2, \ldots, n_k}$ and $\delta_{m_1, m_2, \ldots, m_i} \subset h_{n_1', n_2', \ldots, n_{k'}'}$ $(k' \leqslant k)$ then $n_1' = n_1$; $n_2' = n_2; \ldots$; $n_{k'}' = n_{k'}$.

β) $Q \subset P$. Let

$$x \in \Omega_M(\{H_{m_1, m_2, \ldots, m_i}\});$$

then there exists such

$$z = \frac{1|}{|m_1^0} + \frac{1|}{|m_2^0} + \ldots \in M$$

that

(32, 9) $$x \in \prod_i H_{m_1^0, m_2^0, \ldots, m_i^0}$$

let

$$\varphi z = y = \frac{1|}{|n_1^0} + \frac{1|}{|n_2^0} + \ldots \in N.$$

Then for any k: $z \in h_{n_1^0, n_2^0, \ldots, n_k^0}$ and consequently ($h_{n_1^0, n_2^0, \ldots, n_k^0}$ being open) there exists such j that

(32, 10) $$\delta_{m_1^0, m_2^0, \ldots, m_j^0} \subset h_{n_1^0, n_2^0, \ldots, n_k^0}.$$

But then (by the definition of $H_{m_1, m_2, \ldots, m_i}$) we have (see **32**, 8; **32**, 9)

$$x \in H_{m_1^0, m_2^0, \ldots, m_{j+k}^0} = E_{n_1^0, n_2^0, \ldots, n_k^0, n_{k+1}^0, \ldots, n_{k'}^0} \subset E_{n_1^0, n_2^0, \ldots, n_k^0}$$

i. e.

$$x \in \prod_k E_{n_1^0, n_2^0, \ldots, n_k^0} \subset \Omega_N(\{E_{n_1, n_2, \ldots, n_k}\})$$ q. e. d.

33. B. Necessity. Let M and N be two sets of irrational numbers such that for any d-class $\mathscr{H}$ of sets

$$S_N(\mathscr{H}) \subset S_M(\mathscr{H}).$$

Denote $\mathscr{H}_0$ the class consisting of:

1) The whole interval (0, 1) which we shall denote δ^I
2) The vacuous set 0
3) All the sets $\delta^I_{n_1, n_2, \ldots, n_k}$ (see notations, Group C).

Evidently $\mathscr{H}$ is a d-class; therefore

$$N = \Omega_N(\{\delta^I_{n_1, n_2, \ldots, n_k}\})$$ [1]) $$\in S_N(\mathscr{H}_0) \subset S_M(\mathscr{H}_0)$$

[1]) This follows immediately from the definitions of Ω_N and $\delta^I_{n_1, n_2, \ldots, n_k}$.

or

(33,1) $$N = \Omega_M(\{H_{m_1,m_2,\dots,m_i}\}); \quad H_{m_1,m_2,\dots,m_i} \in \mathcal{H}_0.$$

Let

$$y = \frac{1|}{|n_1} + \frac{1|}{|n_2} + \dots \in N.$$

Denote ψy the set of all such points

$$\frac{1|}{|m_1} + \frac{1|}{|m_2} + \dots \in M$$

that

$$y \in \prod_i H_{m_1,m_2,\dots,m_i}$$

(this set is not vacuous for $y \in N = \Omega_M(\{H_{m_1,m_2,\dots,m_i}\})$.

Denote now $\psi N = M_1$ (ψK denotes $\sum_{y \in K} \psi y$).

We shall prove that M_1 is closed in M.

In fact let

(33,2) $$x = \frac{1|}{|m_1} + \frac{1|}{|m_2} + \dots \in M - M_1$$

then

(33,3) $$H_{m_1} \cdot H_{m_1,m_2} \cdot \ldots = 0$$

because if this set contained a point y we should have $x \in \psi y \subset M_1$ in contradiction with (33,2).

Every non vacuous set $H_{m_1,m_2,\dots,m_i}$ is of the form

$$H_{m_1,m_2,\dots,m_i} = \delta^{i}_{n_1^i,n_2^i,\dots,n_{k_i}^i}$$

($n_j = 1, 2, \dots$; $k_i = 0, 1, 2, \dots$; if $k_i = 0$ then $H_{m_1,m_2,\dots,m_i} = \delta^i = (0,1)$)

(33,3) implies that eitheir

1) There exists such i that $H_{m_1,m_2,\dots,m_i} = 0$

or

2) There exists such natural i, j, l ($i > j$; $l < k_i$; $l < k_j$) that $n_e^i \neq n_e^j$.

In both cases (as one may easily see) we have

$$M \cdot \delta_{m_1,m_2,\dots,m_i} \subset M - M_1$$

Thus the set $M - M_1$ is open in M and consequently M_1 is closed in M.

We shall prove now the following two properties of ψ

1) $\psi y_1 \cdot \psi y_2 = 0$; ($y_1 \neq y_2$)

2) if K is open in N then ψK is open in M_1

(33,4) 1) $$\psi y_1 \cdot \psi y_2 = 0 \quad y_1 \neq y_2.$$

Let

$$x = \frac{1|}{|m_1} + \frac{1|}{|m_2} + \dots \in \psi y_1$$

then $y_1 \in \prod H_{m_1,m_2,\dots,m_i}$ and we have as above

$$H_{m_1,m_2,\dots,m_i} = \delta^i_{n_1,n_2,\dots,n_{k_i}}.$$

We can prove that the numbers k_i are not limited for otherwise they would have the greatest among them (denote it k_0) and we should have

$$\delta^i_{n_1,n_2,\dots,n_{k_0}} = \prod_i H_{m_1,m_2,\dots,m_i} \subset N$$

which is impossible, because N contains no rational numbers.

Now it is evident that

$$y_1 = \frac{1|}{|n_1} + \frac{1|}{|n_2} + \dots$$

and that

$$\prod_i H_{m_1,m_2,\dots,m_i} = (y_1)$$

and thus can not contain y_2 so that x non $\in \psi y_2$ q. e. d.

2) If a set K is open in N then ψK is open in M_1.

Let K be open in N and let

$$x = \frac{1|}{|m_1} + \frac{1|}{|m_2} + \dots \in \psi K.$$

Then it follows from what just preceded that

$$\prod_i H_{m_2,m_2,\dots,m_i} = (y) \subset K$$

from which follows easily that there exists such i_0 that

$$N \cdot H_{m_1, m_2, \ldots, m_{i_0}} \subset K.$$

Let

$$x' \in \delta_{m_1, m_2, \ldots, m_{i_0}} \cdot M_1$$

we shall prove our proposition if we show that $x' \in \psi K$.

In fact let

$$x' = \frac{1|}{|m_1'} + \frac{1|}{|m_2'} + \ldots$$

We have then $m_1 = m_1'$; $m_2 = m_2'; \ldots; m_{i_0} = m_{i_0}'$ further

$$\prod_i H_{m_1', m_2', \ldots, m_i'} = (y') \subset N$$

but evidently

$$(y') = \prod_i H_{m_1', m_2, \ldots, m_i'} \subset N \cdot H_{m_1', m_2', \ldots, m_{i_0}'} \subset K$$

or $y' \in K$ whence

$$x' \in \psi y' \subset \psi K \qquad \text{q. e. d.}$$

It follows that if K is closed in N then ψK is closed in M_1. In fact $N - K$ is open in N, therefore $\psi(N - K)$ is open in M_1. But it follows from (33, 4) that $\psi(N - K) = M_1 - \psi K$ whence ψK is closed in M_1.

We shall denote $y = \varphi x$ if $x \in \psi y$.

By (33, 4) to every $x \in M_1$ corresponds one and only one point $y = \varphi x$.

Besides φ is continuous in M_1. For let

$$x_0 = \lim x_i \quad (x_i \in M_1;\ i = 0, 1, 2, \ldots).$$

We must prove that $\varphi x_0 = \lim \varphi x_i$.

Suppose the contrary. Then we can find a neighbourhood V of φx_0 and a subsequence $\{\varphi x_{n_i}\}$ of $\{\varphi x_i\}$ such that V contains none of the points φx_{n_i} $[i = 1, 2, \ldots]$. But then $\psi(N \cdot V)$ is an open in M_1 set which contains x_0 but none of the points x_{n_i}. This is in contradiction with our assumption that $x_0 = \lim x_i$. Therefore our supposition that

$$\varphi x_0 \neq \lim \varphi x$$

is false q. e. d.

Therefore φ is a continuous function defined in M_1 and having N for its set of values.

But we can now apply a lemma due to Sierpiński[1]), which states that if φ is a continuous transformation of a set M_1 closed in $M \subset J$ into another set N then there exists a continuous in M function φ^* which coincides in M_1 with φ and is besides such that

$$\varphi^* M = N$$

i. e. N is a continuous image of M. q. e. d.

34. Corrolary. *In order that two Ω-operations Ω_N and Ω_M be r-equivalent with resp. to* $[d]$ (i. e. in order that for any $\mathcal{H} \in [d]$ we had $S_N(\mathcal{H}) = S_M(\mathcal{H})$; see def. 6 bis and 19) *it is necessary and sufficient that each of the sets N and M be a continuous image of the other or, using a notion introduced by Sierpiński[2]), that M and N be of the same type* (c).

This implies some interesting consequences. First of all there is a countable infinity of types (c) of countable closed sets and consequently there is a countable infinity of non equivalent Ω-functions having for their base a countable closed sets. These types we shall call for the moment „inferior types" and the corresponding Ω-functions — „inferior functions". Outside these inferior types there are only five types (c) of (A)-sets of irrational numbers[2]). These are: 1) the type of a countable non-closed set, 2) the type of a perfect set, 3) the type of a sum of two sets belonging to preceding types, 4) the type of an $\mathcal{F}_\sigma$ not belonging to 3), 5) the type of J. Consequently there exist five $S_N(\mathcal{H})$ classes $(\mathcal{H} \in [d])$ whose bases are (A)-set viz.:

$$1)\ \mathcal{H}_{\delta\sigma} \quad 2)\ \mathcal{H}_{s\delta} \quad 3)\ \mathcal{H}_{\delta\sigma+s\delta} \quad 4)\ \mathcal{H}_{s\delta\sigma} \quad 5)\ A(\mathcal{H})$$

where the meaning of δ, σ and s see in notations group B; $\mathcal{H}_{\delta\sigma+s\delta}$ means the class of all the sets of the form $A + B$ where $A \in \mathcal{H}_{\delta\sigma}$ and $B \in \mathcal{H}_{s\delta}$; $A(\mathcal{H})$ is the class of all the results of operation A effected upon sets belonging to $\mathcal{H}$.

As is easily seen, for rings these five Ω-functions and all the inferior Ω-functions are reduced to three, viz.: $\mathcal{H}_\delta$, $\mathcal{H}_{\delta\sigma}$ and $A(\mathcal{H})$.

[1]) W. Sierpiński, Fund. Math. t. XI, p. 118.
[2]) Fund. Math. t. XIV, p. 345.

These are the only $S_N(\mathcal{H})$-classes whose base is an (A)-set. If we now remember that

$$N = \Omega_N(\{\delta_{n_1, n_2, \dots, n_k}\}) = \Omega_N(\{\delta'_{n_1, n_2, \dots, n_k}\}) = \Omega_N(\{\bar{\delta}'_{n_1, n_2, \dots, n_k}\})$$

we can say that: If $\mathcal{H}$ is a Borelian class ($\mathcal{F}$, $\mathcal{G}$, $\mathcal{F}_\sigma$, $\mathcal{G}_\delta$ etc.) in J or in I then the only Borelian classes that can be represented in the form $S_N(\mathcal{H})$ are $\mathcal{H}_\delta$ and $\mathcal{H}_{\delta\sigma}$ (e. g. if $\mathcal{H} = \mathcal{F}$ then they are $\mathcal{F}$ and $\mathcal{F}_\sigma$ if $\mathcal{H} = \mathcal{G}$ they are $\mathcal{G}_\delta$ and $\mathcal{G}_{\delta\sigma}$). So e. g. $\mathcal{F}_{\sigma\delta}$ is not an $S_N(\mathcal{F})$-class though $\mathcal{F}_\sigma$ is one, and $\mathcal{F}_{\sigma\delta}$ is an $S_N(\mathcal{F}_\sigma)$ class. This shows that there exists no theorem corresponding to Theorem VI (i. e. $S_{N'}(S_{N''}(\mathcal{H}))$ is not always an $S_N(\mathcal{H})$).

35. Theorem XXI. *In order that for every ring $\mathcal{H}$ be $S_N(\mathcal{H}) \subset S_M(\mathcal{H})$ it is necessary and sufficient that N be a continuous image of $M \times \Delta$.*

Proof. A. Sufficiency. Let

$$N = \varphi(M \times \Delta).$$

Denote M_0 the set of all such points

$$\frac{1|}{|m_1} + \frac{1|}{|m_2} + \frac{1|}{|m_3} + \frac{1|}{|m_4} + \dots$$

that

1) $\frac{1|}{|m_1} + \frac{1|}{|m_3} + \dots \in M$

2) $m_{2i} \leqslant 2 \quad [i = 1, 2, \dots]$

then evidently M_0 is homeomorphic to $M \times \Delta$ (to any point

$$\frac{1|}{|m_1} + \frac{1|}{|m_2} + \dots$$

of M_0 we may correlate the point

$$\left(\frac{1|}{|m_1} + \frac{1|}{|m_3} + \dots;\ 0, m'_2, m'_4, \dots\right) \in M \times \Delta \quad \text{where} \quad m'_{2i} = m_{2i} - 1)$$

and therefore N is a continuous image of M_0.

It follows from Theorem XX that

$$S_N(\mathcal{H}) \subset S_{M_0}(\mathcal{H})$$

for any $\mathcal{H} \in [d]$. We have only to prove that for any $\mathcal{H} \in [r]$:

(35, 1) $$S_{M_0}(\mathcal{H}) \subset S_M(\mathcal{H}).$$

(*Remark.* It is evident that $S_M(\mathcal{H}) \subset S_{M_0}(\mathcal{H})$ so that we shall have $S_{M_0}(\mathcal{H}) = S_M(\mathcal{H})$).

Let $P \in S_{M_0}(\mathcal{H})$ i. e.

$$P = \Omega_{M_0}(\{E_{m_1, m_2, \dots, m_i}\})\ ^{1)}; \quad E_{m_1, \dots, m_i} \in \mathcal{H}.$$

Denote

$$\sum_{m_2=1}^{2} \cdots \sum_{m_{2i}=1}^{2} E_{m_1, m_2, \dots, m_{2i}} = H_{m_1, m_3, \dots, m_{2i-1}}.$$

This set being the finite sum of sets belonging to $\mathcal{H}(\mathcal{H} \in [r])$ belongs to $\mathcal{H}$ itself. Let now

$$Q = \Omega_M(\{H_{p_1, p_2, \dots, p_i}\}).$$

Evidently $Q \in S_M(\mathcal{H})$; we shall prove that $Q = P$.

α) $Q \supset P$. Let $x \in P$ then there exists such $\frac{1|}{|m_1^0} + \frac{1|}{|m_2^0} + \dots \in M_0$ that $x \in \prod_i E_{m_1^0, m_2^0, \dots, m_i^0}$.

We have by the definition of M_0.

$$\frac{1|}{|m_1^0} + \frac{1|}{|m_3^0} + \dots \in M; \quad m_{2k} \leqslant 2 \text{ for any } k.$$

Hence

$$x \in \prod_i \sum_{m_2=1}^{2} \cdots \sum_{m_{2i}=1}^{2} E_{m_1^0, m_2, m_3^0, \dots, m_{2i}} = \prod_i H_{m_1^0, m_3^0, \dots, m_{2i-1}^0} \subset Q \quad \text{q. e. d.}$$

β) $Q \subset P$. Let $x \in Q$ then there exists such $\frac{1|}{|p_1^0} + \frac{1|}{|p_2^0} + \dots \in M$ that $x \in H_{p_1^0, p_2^0, \dots, p_i^0}$ for any i.

Consequently for any i there exists such a cortege of rang i

$$(r_1^i, r_2^i, \dots, r_i^i); \quad r_s^i = 1, 2$$

[1]) We shall suppose the system $\{E_{m_1, m_2, \dots, m_i}\}$ regular (see def. 23 Art. 31).

that

$$(35,2) \quad x \in E_{p_1^0, r_1^i, p_2^0, r_2^i, \dots, p_i^0, r_i^i} \subset E_{p_1^0, r_1^i, \dots, p_{i'}^0, r_{i'}^i} \quad (i' \leqslant i)$$ [1].

We can now apply the following lemma due to Denes König[2]). If we have a set S of corteges $(r_1, r_2, \dots, r_k)$ possessing the following three properties:

1) there exists such sequence of natural numbers $\pi_1, \pi_2, \dots,$ that for any cortege $(r_1, r_2, \dots, r_k) \in S$ we have $r_i \leqslant \pi_i$,

2) if $(r_1, r_2, \dots, r_i) \in S$ then all its segments also belong to S: $(r_1, r_2, \dots, r_{i'}) \in S$ for any $i' \leqslant i$,

3) the rang of the corteges belonging to S is not limited;

then there exists a sequence $r_1^0, r_2^0, \dots,$ such that all its segments belong to S: $(r_1^0, r_2^0, \dots, r_i^0) \in S$ for any i.

Denote now S the set of all the corteges $(r_1^i, r_2^i, \dots, r_{i'}^i)$ (see above). This set S evidently satisfies all the conditions of the lemma (the sequence $\pi_1, \pi_2, \dots$ demanded by condition 1) being 2, 2, 2,...). Therefore its conclusion must be also true, i. e. there exists such sequence $r_1^0, r_2^0, \dots$ that whatever be i' we may find such $i \geqslant i'$ that

$$r_1^0 = r_1^i; \quad r_2^0 = r_2^i; \dots; \quad r_{i'}^0 = r_{i'}^i.$$

It follows then from (35, 2) that

$$x \in \prod_{i'} E_{p_1^0, r_1^0, \dots, p_{i'}^0, r_{i'}^0} \subset \Omega_{M_0}(\{E_{m_1, m_2, \dots, m_i}\}) = P \quad \text{q. e. d.}$$

The relation (35, 1) is now wholy demonstrated.

36. B. Necessity[3]). We shall begin with the following

Lemma. If to every cortege $(n_1, n_2, \dots, n_i)$ *we correlate an integer*

$$\pi(n_1, n_2, \dots, n_i) \geqslant 2$$

and if M_0' *is the set of all the numbers*

$$\frac{1|}{|m_1} + \frac{1|}{|m_2} + \dots$$

such that

1) $\dfrac{1|}{|m_1} + \dfrac{1|}{|m_3} + \dots \in M$

2) $m_{2i} \leqslant \pi(m_1, m_3, \dots, m_{2i-1})$

then M_0' *is homeomorphic to* $M \times \Delta$.

Proof. To every point

$$z = \frac{1|}{|m_1} + \frac{1|}{|m_2} + \dots \in M_0'$$

we shall correlate a point

$$\vartheta z = (x, y) \in M \times \Delta$$

where

$$x = \frac{1|}{|m_1} + \frac{1|}{|m_3} + \dots; \quad y = 0, q_1 q_2 \dots$$

where $q_i = 0$ except when

$$i = \sum_{j=1}^{k} \min(m_{2j}, \pi(m_1, m_3, \dots, m_{2j-1}) - 1) \text{ and } m_{2k} < \pi(m_1, m_3, \dots, m_{2k-1})$$

in which case $q_i = 1$.

We must prove that ϑ is a homeomorphic transformation. Denote ϑ' the inverse transformation i. e. if $(x, y) = \vartheta z$ then $z = \vartheta'(x, y)$. We shall prove the following properties of ϑ'.

1) To every point $z \in M_0'$ corresponds one and only one point ϑz. This is evident.

2) To every point $(x, y) \in M \times \Delta$ corresponds one and only one point $z = \vartheta'(x, y)$.

In fact let

$$x = \frac{1|}{|m_1} + \frac{1|}{|m_3} + \dots; \quad y = 0, q_1 q_2 \dots$$

We shall now define the numbers $m_2, m_4, \dots$ by induction.

Let all the numbers m_{2k} $(k < k_0)$ be defined and let i_0 be the least index i greater than

$$\sum_{j < k_0} \min(m_{2j}, \pi(m_1, m_3, \dots, m_{2j-1}) - 1) = t_{k_0}$$

such that $q_i = 1$. Then

$$m_{2k_0} = \min(i_0 - t_{k_0}, \pi(m_1, m_3, \dots, m_{2k_0-1})).$$

[1]) We must not forget that the system $\{E_{m_1, m_2, \dots, m_i}\}$ is regular.

[2]) Fund. Math. VIII, p. 120. (Sur les coorespondences multivoques des ensembles, th. E.). The lemma is stated there in a somewhat different form but the es sens is the same.

[3]) Necessity follows also from Theorem XXIII below (p. 77). We give here this direct proof because it possesses some interest of its own.

One may easily see that

$$z = \frac{1|}{|m_1} + \frac{1|}{|m_2} + \dots \in M'_0$$

is the only point such that $\vartheta z = (x, y)$.

3) The transformations ϑ and ϑ' are continuous.

In fact if

$$z \in \Delta_{m_1, m_2, \dots, m_{2k}}$$

then

$$\vartheta z \in \delta_{m_1, m_3, \dots, m_{2k-1}} \times \Delta_{q_1, q_2, \dots, q_i}$$

(where $i = \sum_{j=1}^{k} \min(m_{2j}, \pi(m_1, m_3, \dots, m_{2j-1}) - 1)$ and $q_1, q_2, \dots$ are defined as above) and *vice-versâ*.

These properties show that ϑ is a homeomorphy, q. e. d.

Proof of the theorem (necessity): Let $S_N(\mathscr{H}) \subset S_M(\mathscr{H})$ for any ring $\mathscr{H}$. Denote $\mathscr{H}_{0s}$ the class of all the finite sums of $\delta'_{n_1, n_2, \dots, n_k}$ (including I and 0 = vacuous set). $\mathscr{H}_{0s}$ being a ring we have

$$N \in S_N(\mathscr{H}_{0s}) \subset S_M(\mathscr{H}_{0s})$$

i. e.

$$N = \Omega_M(\{H_{n_1, n_2, \dots, n_l}\})$$

where any non vacuous

$$H_{n_1, n_2, \dots, n_l} = \sum_{p=1}^{\pi(n_1, n_2, \dots, n_l)} \delta'_{\eta_1^p, \eta_2^p, \dots, \eta_{k_p}^p} \tag{36, 1}$$

η_j^p and k_p depending also of $n_1, n_2, \dots, n_l$

(in case $k_p = 0$ we have $H_{n_1, n_2, \dots, n_l} = \delta' = I$)

denote

$$E_{m_1, m_2, \dots, m_{2j}} = E_{m_1, m_2, \dots, m_{2j+1}} = \delta'_{\eta_1^{m_{2j}}, \dots, \eta_{k_{m_2j}}^{m_{2j}}}$$

where $\eta_s^{m_{2j}}$ correspond to $n_1 = m_1$; $n_2 = m_3, \dots, n_l = m_{2j-1}$ (see **36**, 1) if

$$m_{2k} \leqslant \pi(m_1, m_3, \dots, m_{2j-1})$$

and

$$E_{m_1, m_2, \dots, m_{2j}} = 0$$

otherwise (i. e. if $m_{2j} > \pi(m_1, m_3, \dots, m_{2j-1})$ or if $H_{m_1, m_3, \dots, m_{2j-1}} = 0$).

Evidently

$$N = \sum_M \prod_i H_{n_1, n_2, \dots, n_i} = \sum_{M'_0} \prod_i E_{m_1, m_2, \dots, m_{2i-1}, m_{2i}}$$

where M'_0 denotes (as in the preceding lemma) the set of all the points

$$\frac{1|}{|m_1} + \frac{1|}{|m_2} + \dots$$

such that

$$\frac{1|}{|m_1} + \frac{1|}{|m_3} + \dots \in M$$

and $m_{2i} \leqslant \pi(m_1, m_3, \dots, m_{2i-1})$. We may besides suppose $\pi(m_1, m_3, \dots, m_{2i-1}) \geqslant 2$.

Thus $N \in S_{M'_0}(\mathscr{H}_0)$ from which follows as in Th. XX that N is a continuous image of M'_0 or, which is the same (see lemma), of $M \times \Delta$, q. e. d.

Corrolary. *In order that two operations Ω_N and Ω_M be (r)-equivalent with resp. to [r]* (see def. 6bis and 21) *it is necessary and sufficient that $N \times \Delta$ and $M \times \Delta$ be of the same type* (c).

In fact if $N \times \Delta$ and $M \times \Delta$ are of the same type (c) then $N \times \Delta$ is a continuous image of $M \times \Delta$ and consequently N (which is a continuous image of $N \times \Delta$) is a continuous image of $M \times \Delta$. The same for M and $N \times \Delta$. If on the other hand N is a continuous image of $M \times \Delta$ then $N \times \Delta$ is a continuous image of $(M \times \Delta) \times \Delta$ (homeomorphic to $M \times \Delta$). The same for M and $N \times \Delta$. Hence the corrolary.

§ 3. The $S_N(\mathscr{F})$-Classes in a Compact Metric Space.

37. Of all the $S_N(\mathscr{H})$-classes the most interesting are $S_N(\mathscr{F})$. We shall begin their theory with the following

Theorem XXII. *Every set closed in $R \times J$ is the scheme for a certain system $\{F_{n_1, n_2, \dots, n_k}\}$ of sets, closed in R* [1]) (see def. 10bis).

Proof. Let $P \subset R^{(J)} = R \times J$ be a set closed in $R \times J$. Denote

$$F_{n_1, n_2, \dots, n_k} = \overline{Pr_R[P \cdot (R \times \delta_{n_1, n_2, \dots, n_k})]}$$

$F_{n_1, n_2, \dots, n_k}$ are closed in R.

It remains to prove that

$$P = S(\{F_{n_1, n_2, \dots, n_k}\}) = \prod_k \sum_{(n_1, n_2, \dots, n_k)} F_{n_1, n_2, \dots, n_k} \times \delta_{n_1, n_2, \dots, n_k}$$

[1]) The converse is also true. See def. 10, remark (Fund. Math. t. XVIII, p. 237).

$$\alpha)\quad \prod_k \sum_{(n_1,n_2,\dots,n_k)} F_{n_1,n_2,\dots,n_k} \times \delta_{n_1,n_2,\dots,n_k} \supset P$$

as one may easily see.

$$\beta)\quad \prod_k \sum_{(n_1,n_2,\dots,n_k)} F_{n_1,n_2,\dots,n_k} \times \delta_{n_1,n_2,\dots,n_k} \subset P$$

because if

$$(x, y) \in \prod_k \sum_{(n_1,n_2,\dots,n_k)} F_{n_1,n_2,\dots,n_k} \times \delta_{n_1,n_2,\dots,n_k}$$

and

$$y = \frac{1|}{|n_1^0} + \frac{1|}{|n_2^0} + \dots + \frac{1|}{|n_k^0} + \dots$$

then for any k

$$x \in F_{n_1^0, n_2^0, \dots, n_k^0} = \overline{Pr_R[P(R \times \delta_{n_1^0, n_2^0, \dots, n_k^0})]}$$

i. e. whatever be the neighbourhood $V_x^{(\alpha)}$ of x there exists such

$$x_k^{(\alpha)} \in V_x^{(\alpha)} \quad \text{and} \quad y_k^{(\alpha)} \in \delta_{n_1^0, n_2^0, \dots, n_k^0}$$

that

$$(x_k^{(\alpha)}, y_k^{(\alpha)}) \in P.$$

But the point (x, y) is a point of accumulation of the set of all the points $(x_k^{(\alpha)}, y_k^{(\alpha)})$ whence, P being closed,

$$(x, y) \in P \qquad \text{q. e. d.}$$

From α) and β) follows that

$$P = S(\{F_{n_1, n_2, \dots, n_k}\}) \qquad \text{q. e. d.}$$

Corrolary 1. *$S_N(\mathcal{F})$ is identical with the class $\mathcal{P}$ of projections of sets closed in $R \times N$.*

Corrolary 2. (*in a compact space*). *The class $S_N(\mathcal{F})$ possesses the following property: if $P \in S_N(\mathcal{F})$ and Q is a continuous image of P $(Q = \varphi P)$ then $Q \in S_N(\mathcal{F})$.*

In fact, as we already know, P is the projection of a set closed in $R \times N = R^{(0)}$

$$P = Pr_R P^{(0)}; \quad P^{(0)} \in \mathcal{F}^{(0)}.$$

Denote $\varphi(x, y) = (\varphi x, y)$ and $Q^{(0)} = \varphi P^{(0)}$.

Then (1) $Q^{(0)} \in \mathcal{F}^{(0)}$. We shall suppose that the pseudo character of R is $= \aleph_0$ [1]) i. e. every point of R is the commun part of a countable infinity of open sets. The case when it is greater than $\aleph_0$ vill be treated in an Appendix.

Let then $(x, y) \in \overline{Q}^{(0)}$ and

$$V_1 \supset V_2 \supset \dots$$

be a sequence of open sets such that

$$\prod_k V_k = (x)$$

and let

$$y = \frac{1|}{|n_1^0} + \frac{1|}{|n_2^0} + \dots$$

Then in any set

$$V_k \times \delta_{n_1^0, n_2^0, \dots, n_k^0}$$

there exists a point

$$(x_k, y_k) \in Q^0;$$

denote $\overline{x}_k$ a point such that $\varphi(\overline{x}_k, y_k) = (x_k, y_k)$ and let $\overline{x}$ be an accumulation point of the set of all the $\overline{x}_k$. Then, taking into account that $(\overline{x}_k, y_k) \in P^{(0)}$ and that $P^{(0)}$ is closed, we have $(\overline{x}, y) \in P_0$. But it is easily proved that $\varphi(\overline{x}, y)$ must necessarily be (x, y) whence $(x, y) \in Q_0$, q. e. d.

$$(2) \qquad Pr\, Q_0 = Q$$

this is evident.

From (1) and (2) (and Cor. 1) follows that $Q \in S_N(\mathcal{F})$.

38. Theorem XXIII. *If R is a compact metric space then $S_N(\mathcal{F})$ is identical with the class $\mathcal{P}$ of continuous images of the set $N \times \Delta$.*

P r o o f. a) $\mathcal{P} \subset S_N(\mathcal{F})$.

Let $P \in \mathcal{P}$ i. e. $P = \varphi(N \times \Delta)$.

Denote

$$F_{n_1, n_2, \dots, n_k} = \overline{\varphi(N \cdot \delta_{n_1, n_2, \dots, n_k} \times \Delta)} \in \mathcal{F}$$

and

$$Q = \Omega_N(\{F_{n_1, n_2, \dots, n_k}\}) \in S_N(\mathcal{F}).$$

We shall prove that $P = Q$.

[1]) P. Alexandroff. Memoire sur les espaces topologiques compacts.

α) $P \subset Q$. Let $z \in P$ i. e. $z = \varphi(x, y)$ where

$$x = \frac{1|}{|n_1^0} + \frac{1|}{|n_2^0} + \ldots \in N \quad \text{and} \quad y \in \Delta$$

then

$$z \in \varphi(N \cdot \delta_{n_1^0, n_2^0, \ldots, n_k^0} \times \Delta)$$

for any k, whence $z \in Q$, q. e. d.

β) $Q \subset P$. Let $z \in Q$; then there exists such

$$x = \frac{1|}{|n_1^0} + \frac{1|}{|n_2^0} + \ldots \in N$$

that

$$z \in F_{n_1^0, n_2^0, \ldots, n_k^0} = \overline{\varphi(N \cdot \delta_{n_1^0, n_2^0, \ldots, n_k^0} \times \Delta)} \text{ for any } k;$$

i. e. there exists for any k such

$$z_k \in \varphi(N \cdot \delta_{n_1^0, n_2^0, \ldots, n_k^0} \times \Delta)$$

that $\varrho(z, z_k) < \frac{1}{k}$

$$z_k = \varphi(x_k, y_k); \quad x_k \in N \cdot \delta_{n_1^0, n_2^0, \ldots, n_k^0}; \quad y_k \in \Delta.$$

Let $\{y_{k_l}\}$ be a convergent subsequence of $\{y_k\}$ and

$$y = \lim_{l \to \infty} y_{k_l} \in \Delta$$

then

$$(x, y) = \lim_{l = \infty} (x_{k_l}, y_{k_l}) \quad \text{i. e.}$$

$$z = \lim z_{k_l} = \lim \varphi(x_{k_l}, y_{k_l}) = \varphi \lim (x_{k_l}, y_{k_l}) = \varphi(x, y) \in P \quad \text{q. e. d.}$$

b) $S_N(\mathcal{F}) \subset P$. Let $Q \in S_N(\mathcal{F})$ i. e.

$$Q = \Omega_N(\{F_{n_1, n_2, \ldots, n_k}\}); \quad F_{n_1, n_2, \ldots, n_k} \in \mathcal{F}.$$

R being a compact metric space, there exists a continuous transformation φ^* of Δ into R [1]) $[R = \varphi^* \Delta]$.

Denote ψz $(z \in Q)$ the set of all such points $(x, y) \in N \times \Delta$ that $z = \varphi^* y$ and

$$z \in \prod_k F_{n_1, n_2, \ldots, n_k} \quad \text{where} \quad \frac{1|}{|n_1} + \frac{1|}{|n_2} + \ldots = x.$$

[1]) Hausdorff, p. 197.

Then to every point $(x, y) \in N \times \Delta$ corresponds not more than one such point $z \in Q$ that

$$(x, y) \in \psi z.$$

We shall denote such point z (if it exists) $\varphi(x, y)$.

(A) φ is defined in a closed subset M of $N \times \Delta$ and is continuous in M. In fact let

$$(x, y) = \lim (x_i, y_i); \quad (x_i, y_i) \in M.$$

Denote $z_i = \varphi(x_i, y_i)$.

Then:

1) $z_i = \varphi^* y_i$; hence

$$(38,1) \qquad z = \lim z_i = \lim \varphi^* y_i = \varphi^* \lim y_i = \varphi^* y$$

2) $z_i \in \prod_k F_{n_1^i, n_2^i, \ldots, n_k^i}$ where $x_i = \frac{1|}{|n_1^i} + \frac{1|}{|n_2^i} + \ldots$

But whatever be k we have for i great enough $(i > i_k)$ $n_1^i = n_1$, $n_2^i = n_2, \ldots, n_k^i = n_k$; whence

$$z_i \in F_{n_1, n_2, \ldots, n_k} \ (i > i_k) \quad \text{or}$$

$$z = \lim z_i \in F_{n_1, n_2, \ldots, n_k} \quad \text{for any } k;$$

therefore

$$(38,2) \qquad z \in \prod_i F_{n_1, n_2, \ldots, n_k}; \quad x = \frac{1|}{|n_1} + \frac{1|}{|n_2} + \ldots$$

(38,1) and (38,2) show that $z = \varphi(x, y)$ which proves the assumption (A).

Now φ being defined and continuous in a closed subset M of $N \times \Delta$ there exists [1]) a function φ_0 defined and continuous in $N \times \Delta$ and such that

$$\varphi_0(N \times \Delta) = \varphi M = Q \qquad \text{q. e. d.}$$

39. Remark. It is easy to see from the proof given above that if

$$x = \frac{1|}{|n_1} + \frac{1|}{|n_2} + \ldots \in N$$

[1]) This follows easily from a lemma of Sierpiński cited on p. 65.

and if

$$\prod_k F_{n_1,n_2,\dots,n_k} \neq 0$$

then

$$\varphi(M\cdot((x)\times\Delta)) = \prod_k F_{n_1,n_2,\dots,n_k}.$$

We may deduce whence easily the following:

Theorem of Sierpiński[1]: *Denote (in a compact space) S'_N the class of all the sets which can be represented in the form:*

$$\Omega_N(\{F_{n_1,n_2,\dots,n_k}\})$$

where $F_{n_1,n_2,\dots,n_k} \in \mathcal{F}$ possess the following property:

For any

$$\frac{1|}{|n_1} + \frac{1|}{|n_2} + \dots \in N$$

$\prod_k F_{n_1,n_2,\dots,n_k}$ consists of not more than one point.

The class S'_N is the class of all the continuous images of N.

Proof. If

$$P \in S'_N \subset S_N(\mathcal{F})$$

then by Theorem XXIII and remark, P is a continuous image of $N\times\Delta$ ($P=\varphi(N\times\Delta)$) and we have

$$\varphi(M\cdot((x)\times\Delta)) = \prod_k F_{n_1,n_2,\dots,n_k}$$

which consists of not more that one point.

Denote M_0 the set of all the points $x = \frac{1|}{|n_1} + \frac{1|}{|n_2} + \dots$ such that $\prod_k F_{n_1,n_2,\dots,n_k} \neq 0$; evidently M_0 is a relatively closed subset of N homeomorphic to M and $\varphi M = P$. It follows that there exists such continuous (in N) function φ_0 that $\varphi_0 N = P$ i. e.

P is a continuous image of N, q. e. d.

If on the other hand $P=\varphi N$ (φ being a continuous transformation) then denote $\varphi_1(x,y)=\varphi x$ for any $y\in\Delta$; then we have $P=\varphi_1(N\times\Delta)$ and $\varphi_1((x)\times\Delta)$ consists of a single point. But denoting

$$F_{n_1,n_2,\dots,n_k} = \overline{\varphi(N\cdot\delta_{n_1,n_2,\dots,n_k})} = \overline{\varphi_1(N\cdot\delta_{n_1,n_2,\dots,n_k}\times\Delta)}$$

[1]) Sierpiński VI.

we have (see the beginning of the proof of Th. XXIII, art. 38)

$$P = \Omega_N(\{F_{n_1,n_2,\dots,n_k}\}); \quad F_{n_1,n_2,\dots,n_k} \in \mathcal{F}$$

and $\prod_k F_{n_1,n_2,\dots,n_k}$ consists of the single point

$$z = \varphi\left(\frac{1|}{|n_1} + \frac{1|}{|n_2} + \dots\right)$$

because

$$\prod_k \delta_{n_1,n_2,\dots,n_k} = \left(\frac{1|}{|n_1} + \frac{1|}{|n_2} + \dots\right)$$

and therefore

$$P \in S'_N.$$

Proof of Necessity of the Conditions of Theoreme XXI. (See p. 68).

If for any ringe $\mathcal{H}$

$$S_N(\mathcal{H}) \subset S_M(\mathcal{H})$$

then in particular $S_N(\mathcal{F}) \subset S_M(\mathcal{F})$ (where $\mathcal{F}$ is the class of sets closed in I). But

$$N = \Omega_N(\{\delta_{n_1,n_2,\dots,n_k}\}) \in S_N(\mathcal{F}) \subset S_M(\mathcal{F})$$

therefore (by Theorem XXIII) N is a continuous image of $M\times\Delta$, q. e. d.

40. We shall conclude this chapter with the following theorem (which shall be used in the second part of this memoir).

Theorem XXIV. *If N contains only such numbers*

$$x = \frac{1|}{|n_1} + \frac{1|}{|n_2} + \dots$$

that the sequence $(n_1, n_2, \dots)$ contains an infinity of different elements, then the classes defined by the operations[1])

$$(40,1) \qquad \sum_{(p_1,p_2,\dots)} \Phi_N\left(\left\{\prod_i E_n^{p_1,p_2,\dots,p_i}\right\}\right)$$

[1]) The operations (**40**,1) and (**40**,1*) are the operations considered in § 3 of Chapter I (See art. 22, p. 265). Therefore from theorem **XXIV** follows that if N satisfies the condition of this theorem then the classes of projections of sets belonging to $\mathcal{H}_N(\mathcal{F})$ and $\mathcal{H}_N(\mathcal{G})$ are classes $S_M(\mathcal{F})$ and $S_M(\mathcal{G})$.

and

$$(40,1^*)\qquad \sum_{(p_1,p_2,\dots)} \Phi_N \left(\left\{\sum_i E_n^{p_1,p_2,\dots,p_i}\right\}\right)$$

where $E_n^{p_1,p_2,\dots,p_i} \in \mathcal{H}$; $\mathcal{H}$ *being a ring, are the same* $S_M(\mathcal{H})$ *class.*

Proof. We may evidently suppose that N is in its »reduced completed« form (def. 9, art. 7). Let now $\mathcal{P}$ and $\mathcal{P}^*$ be the classes defined by the operations resp. $(40,1)$ and $(40,1^*)$ and let $\mathcal{P}_\omega$ and $\mathcal{P}_\omega^*$ be the classes defined by the operations:

$$(40,1_\omega)\qquad \sum_{(q_1,q_2,\dots)} \Omega_{N\,\nu_1,\nu_2,\dots,\nu_k} \left(\left\{\prod_j H_{\nu_1,\nu_2,\dots,\nu_k}^{q_1,q_2,\dots,q_j}\right\}\right)$$

and

$$(40,1_\omega^*)\qquad \sum_{(q_1,q_2,\dots)} \Omega_{N\,\nu_1,\nu_2,\dots,\nu_k} \left(\left\{\sum_j H_{\nu_1,\nu_2,\dots,\nu_k}^{q_1,q_2,\dots,q_j}\right\}\right)$$

where

$$H_{\nu_1,\nu_2,\dots,\nu_k}^{q_1,q_2,\dots,q_j} \in \mathcal{H}.$$

Our theorem will be proved if we prove that:

A. $\mathcal{P} = \mathcal{P}_\omega$; $\mathcal{P}^* = \mathcal{P}_\omega^*$
B. $\mathcal{P}_\omega = \mathcal{P}_\omega^*$
C. $\mathcal{P}_\omega$ is an $S_M(\mathcal{H})$-class.

A a) The inclusions $\mathcal{P} \subset \mathcal{P}_\omega$ and $\mathcal{P}^* \subset \mathcal{P}_\omega^*$ are evident. We have only to set

$$H_{\nu_1,\nu_2,\dots,\nu_k}^{q_1,q_2,\dots,q_j} = E_{\nu_k}^{q_1,q_2,\dots,q_j}$$

b) $\mathcal{P}_\omega \subset \mathcal{P}$. Let $P \in \mathcal{P}_\omega$ i. e.

$$P = \sum_{(q_1,q_2,\dots)} \Omega_{N\,\nu_1,\nu_2,\dots,\nu_k} \left(\left\{\prod_j H_{\nu_1,\nu_2,\dots,\nu_k}^{q_1,q_2,\dots,q_j}\right\}\right); \quad H_{\nu_1,\nu_2,\dots,\nu_k}^{q_1,q_2,\dots,q_j} \in \mathcal{H}.$$

$\mathcal{H}$ being a ring we may evidently suppose

$$H_{\nu_1,\nu_2,\dots,\nu_k}^{q_1,q_2,\dots,q_j} \subset H_{\nu_1,\nu_2,\dots,\nu_{k'}}^{q_1,q_2,\dots,q_{j'}}$$

for any k, j, k', j' satisfying the relations $k' \leqslant k$, $j' \leqslant j$.

We shall define a system $\{E_n^{p_1,p_2,\dots,p_i}\}$ of sets as follows:

1) if $i < 2n$ then $E_n^{p_1,p_2,\dots,p_i} = R$

2) if $i \geqslant 2n$ and $p_{2n} > 1$ then $E_n^{p_1,p_2,\dots,p_i} = 0$

3) let $i \geqslant 2n$ and $p_{2n} = 1$. Denote

$$\nu_1, \nu_2, \dots, \nu_k \quad (\nu_1 < \nu_2 < \dots < \nu_k = n)$$

all the positive integers $\leqslant n$ such that $p_{2\nu_\lambda} = 1$ and let $q_1, q_2, \dots, q_j$ be the sequence [1])

$$p_1,\ p_2 - 1,\ p_3,\ p_4 - 1, \dots,\ p_{2s-1},\ p_{2s} - 1, \dots,\ p_{2\nu_1 - 1},\ p_{2\nu_1+1}, \dots,$$
$$p_{2\nu_\lambda - 1},\ p_{2\nu_\lambda+1}, \dots,\ p_{2n-3},\ p_{2n-2} - 1,\ p_{2n-1}.$$

Now denote

$$E_n^{p_1,p_2,\dots,p_i} = H_{\nu_1,\nu_2,\dots,\nu_k}^{q_1,q_2,\dots,q_j}$$

and

$$Q = \sum_{(p_1,p_2,\dots)} \Phi_N \left(\left\{\prod_i E_n^{p_1,p_2,\dots,p_i}\right\}\right) \in \mathcal{P}.$$

We shall prove that $P = Q$.

Denote the number of integers $2\nu_\lambda < \tau$ by $k(\tau)$

α) $P \subset Q$. Let $x \in P$. Then there exists such

$$\frac{1|}{|\nu_1} + \frac{1|}{|\nu_2} + \dots \in N$$

and such $q_1, q_2, \dots$ that

$$x \in \prod_{j,k} H_{\nu_1,\nu_2,\dots,\nu_k}^{q_1,q_2,\dots,q_j}.$$

Define p_s as follows:

1) $p_{2\nu_\lambda} = 1$ (for any λ)
2) $p_{2s} = q_{2s-k(2s)} + 1$ in all other cases
3) $p_{2s+1} = q_{2s+1-k(2s+1)}$ for any s.

Then we have

$$E_{\nu_k}^{p_1,p_2,\dots,p_i} = H_{\nu_1,\nu_2,\dots,\nu_k}^{q_1,q_2,\dots,q_j} \text{ if } i \geqslant 2\nu_k$$

$$E_{\nu_k}^{p_1,p_2,\dots,p_i} = R \text{ if } i < 2\nu_k.$$

[1]) In strict conformity with our notations we should use the word „cortege" instead of „sequence".

In both cases

$$x \in E^{p_1, p_2, \dots, p_i}_{\nu_k}.$$

So that we have

$$x \in \prod_{i,k} E^{p_1, p_2, \dots, p_i}_{\nu_k} \subset Q \qquad \text{q. e. d.}$$

β) $Q \subset P$. Let $x \in Q$. Then there exists such

$$\frac{1|}{|n_1} + \frac{1|}{|n_2} + \dots \in N$$

and such sequence $p_1, p_2, \dots$ that

$$x \in \prod_{i,k} E^{p_1, p_2, \dots, p_i}_{n_k}.$$

It follows that for any k, $p_{2n_k} = 1$ for otherwise $E^{p_1, p_2, \dots, p_i}_{n_k}$ would be vacuous for $i \geqslant 2n_k$. Denote $\nu_1, \nu_2, \dots$ the sequence of all the numbers such that $p_{2\nu_\lambda} = 1$ and let $q_1, q_2, \dots$ be the sequence

$$p_1,\ p_2 - 1,\ p_3, \dots,\ p_{2\nu_1 - 1},\ p_{2\nu_1+1},\ p_{2\nu_1+2} - 1,\ p_{2\nu_1+3}, \dots,\ p_{2\nu_2-1},\ p_{2\nu_2+1}, \dots$$

then

(1) $\dfrac{1|}{|\nu_1} + \dfrac{1|}{|\nu_2} + \dots \in N$

because the sequence

$$\nu_1,\ \nu_2, \dots\ [\nu_1 < \nu_2 < \dots]$$

contains all the numbers n_k and N is in its reduced completed form;

(2) $x \in H^{q_1, q_2, \dots, q_j}_{\nu_1, \nu_2, \dots, \nu_k}$ for any k and j.

In fact there exists an $n_s = \nu_{k'} > j + \nu_k$. Let

$$i > 2\nu_{k'}$$

then

$$x \in E^{p_1, p_2, \dots, p_i}_{n_s} = H^{q_1, q_2, \dots, q_{j'}}_{\nu_1, \nu_2, \dots, \nu_{k'}} \subset H^{q_1, q_2, \dots, q_j}_{\nu_1, \nu_2, \dots, \nu_k}.$$

So that

$$x \in \prod_{k,j} H^{q_1, q_2, \dots, q_j}_{\nu_1, \nu_2, \dots, \nu_k} \subset P \qquad \text{q. e. d.}$$

b*) $\mathscr{P}^*_\omega \subset \mathscr{P}^*$ can be proved in the same way. Only:
1) Instead of the inclusion

$$H^{q_1, q_2, \dots, q_j}_{\nu_1, \nu_2, \dots, \nu_k} \subset H^{q_1, q_1, \dots, q_{j'}}_{\nu_1, \nu_2, \dots, \nu_{k'}} \quad \text{for } j \geqslant j',\ k \geqslant k'$$

we have the following

$$H^{q_1, q_2, \dots, q_j}_{\nu_1, \nu_2, \dots, \nu_k} \subset H^{q_1, q_2, \dots, q_{j'}}_{\nu_1, \nu_1, \dots, \nu_{k'}} \quad \text{for } j \leqslant j',\ k \geqslant k'$$

2) $E^{p_1, p_2, \dots, p_i}_{n} = 0$ in case 1) as well as in case 2).

B. $\mathscr{P}_\omega = \mathscr{P}^*_\omega$, a) $\mathscr{P}_\omega \subset \mathscr{P}^*_\omega$. Let $P \in \mathscr{P}_\omega$ then

$$P = \sum_{q_1, q_2, \dots} \sum_{\frac{1|}{|\nu_1} + \frac{1|}{|\nu_2} + \dots \in N} \prod_k \prod_j H^{q_1, q_2, \dots, q_j}_{\nu_1, \nu_2, \dots, \nu_k} =$$

$$= \sum_{q_1, q_2, \dots} \sum_{\frac{1|}{|\nu_1} + \frac{1|}{|\nu_2} + \dots \in N} \prod_k H^{q_1, q_2, \dots, q_k}_{\nu_1, \nu_2, \dots, \nu_k} =$$

$$= \sum_{q_1, q_2, \dots} \sum_{\frac{1|}{|\nu_1} + \frac{1|}{|\nu_2} + \dots \in N} \prod_k \sum_j \tilde{H}^{q_1, q_2, \dots, q_j}_{\nu_1, \nu_2, \dots, \nu_k} \in P^*_\omega$$

where

$$\tilde{H}^{q_1, q_2, \dots, q_j}_{\nu_1, \nu_1, \dots, \nu_k} = 0 \quad \text{if } j < k$$

and

$$\tilde{H}^{q_1, q_2, \dots, q_j}_{\nu_1, \nu_2, \dots, \nu_k} = H^{q_1, q_2, \dots, q_k}_{\nu_1, \nu_2, \dots, \nu_k} \quad \text{if } j \geqslant k.$$

b) $\mathscr{P}^*_\omega \subset \mathscr{P}_\omega$. Let $P \in P^*_\omega$, then

$$P = \sum_{q_1, q_2, \dots} \sum_{\frac{1|}{|\nu_1} + \frac{1|}{|\nu_2} + \dots \in N} \prod_k \sum_j H^{q_1, q_2, \dots, q_j}_{\nu_1, \nu_2, \dots, \nu_k} =$$

$$= \sum_{q_1, q_2, \dots} \sum_{j_1, j_2, \dots} \sum_{\frac{1|}{|\nu_1} + \frac{1|}{|\nu_2} + \dots \in N} \prod_k H^{q_1, q_2, \dots, q_{j_k}}_{\nu_1, \nu_2, \dots, \nu_k} =$$

$$= \sum_{\tau_1, \tau_2, \dots} \sum_{\frac{1|}{|\nu_1} + \frac{1|}{|\nu_2} + \dots \in N} \prod_k \prod_s \tilde{H}^{\tau_1, \tau_2, \dots, \tau_s}_{\nu_1, \nu_2, \dots, \nu_k} \in \mathscr{P}_\omega$$

where

$$H^{\tau_1, \tau_2, \ldots, \tau_s}_{\nu_1, \nu_2, \ldots, \nu_k} = R,$$

if $s < 2k-1$ or $s \geqslant 2k-1$ and $s < 2\tau_{2k-1}$, and

$$\hat{H}^{\tau_1, \tau_2, \ldots, \tau_s}_{\nu_1, \nu_1, \ldots, \nu_k} = H^{\tau_2, \tau_4, \ldots, \tau_{2\tau_{2k-1}}}_{\nu_1, \nu_2, \ldots, \nu_k},$$

if $s \geqslant 2k-1$ and $s \geqslant 2\tau_{2k-1}$.

C. $\mathcal{F}_\omega$ is an $S_M(\mathcal{H})$-class. This is almost evident.

We have only to suppose M to be the set if all such points

$$\xi = \frac{1|}{|n_1} + \frac{1|}{|q_1} + \frac{1|}{|n_2} + \frac{1|}{|q_2} + \ldots$$

that

$$\frac{1|}{|n_1} + \frac{1|}{|n_2} + \ldots \in N$$

and $q_1, q_2, \ldots$ are arbitrary.

Then evidently $\mathcal{F}_\omega = S_M(\mathcal{H})$, q. e. d.

CHAPTER III.

The Operations of Mr. Kolmogoroff[1]).

41. The present chapter is consecrated to one particular kind of operations introduced by Mr. A. Kolmogoroff. These operations will be found useful in the theory of projective sets.

We shall begin with the following remarks:

Let $\Phi(\{E_\xi\})$ be a positive analytical operation effected upon a system $\{E_\xi\}$ of sets, depending of a certain set $\Xi = \{\xi\}$ of indices. Then the operation Φ is wholly determined by the set X of all such subsets Λ of Ξ (these subsets we shall call „chains") that for any system $\{E_\xi\}$ we have

$$\prod_{\xi \in \Lambda} E_\xi \subset \Phi(\{E_\xi\}).$$

In fact from the definition of a positive analytical operation it follows that if

$$x \in \Phi(\{E_\xi\})$$

then, denoting Λ the set of all the indices $\{\xi\}$ such that $x \in E_\xi$ we have $\Lambda \in X$ because otherwise there would exist a system $\{E'_\xi\}$ of arguments and a point x' such that

$$x' \in \prod_{\xi \in \Lambda} E'_\xi$$

$$x' \text{ non } \in \Phi(\{E'_\xi\})$$

which contradicts the definition of a positive analytical operation (see def. 2, art. 1).

Thus we have

$$(41,1) \qquad \Phi(\{E_\xi\}) = \sum_{\Lambda \in X} \prod_{\xi \in \Lambda} E_\xi$$

and the operation Φ is wholly defined by the set X.

42. Let now Φ be a positive analytical operation effected upon a system $\{E_\xi\}$ of arguments depending of a certain set Ξ of indices and let X be the corresponding (see the foregoing art.) set of „chains" (i. e. of subsets of Ξ).

We shall define a new operation

$$R\,\Phi(\{E_{\xi_1, \xi_2, \ldots, \xi_n}\})$$

effected upon a system $\{E_{\xi_1, \xi_1, \ldots, \xi_n}\}$ of sets depending of the set Ξ_0 of corteges (see notations) group D) of indices $\xi \in \Xi$.

Instead of defining directly the operation $R\,\Phi$ we shall define the corresponding set of chains as follows:

Definition 24. If Φ is determined by the set X of chains of indices $\xi \in \Xi$ then $R\,\Phi$ is the positive analytical operation which is defined by the set X_0 of all the chains Λ_0 of corteges of ξ, which satisfy the following conditions:

1) if a cortege $(\xi_1, \xi_2, \ldots, \xi_n)$ belongs to Λ_0 then all its segments also belong to Λ_0 (i. e. $(\xi_1, \xi_2, \ldots, \xi_{n'}) \in \Lambda_0$ for any $n' \leqslant n$);

2) if a cortege $(\xi_1, \xi_2, \ldots, \xi_n) \in \Lambda_0$ then the set $\Lambda_{\xi_1, \xi_2, \ldots, \xi_n}$ of all the indices ξ such that $(\xi_1, \xi_2, \ldots, \xi_n, \xi) \in \Lambda_0$, belongs to X.

[1]) Most of the results of this chapter belong to Mr. Kolmogoroff and are printed here with the kind leave of the author. In particular to Mr. Kolmogoroff belongs the definition of $R\,\Phi_N$-operations and theorems XXV, XXVII, XXIX.

In the same manner the set Λ of all the ξ such that $(\xi) \in \Lambda_0$ belongs to X.

We may write

$$(42,1) \qquad R\,\Phi(\{E_{\xi_1, \xi_2, \ldots, \xi_n}\}) = \sum_{\Lambda_0 \in X_0} \prod_{(\xi_1, \xi_2, \ldots, \xi_n) \in \Lambda_0} E_{\xi_1, \xi_2, \ldots, \xi_n}.$$

In the case most important to us when Φ is a δs-function Φ_N we may define $R\,\Phi_N$ as follows:

Definition 24 bis.

$$(42,2) \qquad R\,\Phi_N(\{E_{n_1, n_2, \ldots, n_k}\}) = \sum_{\eta \in \Theta} \prod_{(i_1, i_2, \ldots, i_k)} E_{n_{i_1}, n_{i_1, i_2}, \ldots, n_{i_1, i_2, \ldots, i_k}}$$

where Θ is the set of all such systems η of corteges

$$\{(n_{i_1}, n_{i_1, i_2}, \ldots, n_{i_1, i_2, \ldots, i_k})\}$$

that for any $i_1, i_2, \ldots, i_k$ we have

$$(42,3) \qquad \frac{1}{|n_{i_1, i_2, \ldots, i_k, 1}} + \frac{1}{|n_{i_1, i_2, \ldots, i_k, 2}} + \ldots \in \breve{N}\,^{1)}$$

and in the same manner

$$(42,3') \qquad \frac{1}{|n_1} + \frac{1}{|n_2} + \ldots \in \breve{N}.$$

43. *Examples of $R\,\Phi_N$ operations*:

1) Let N consist of the single point

$$\frac{1}{|1} + \frac{1}{|2} + \frac{1}{|3} + \ldots$$

Then, as we already know,

$$\Phi_N(\{E_n\}) = \prod_n E_n.$$

The set Θ of the preceding article consists in this case of all such systems of corteges which contain all the corteges and we have

$$R\,\Phi_N(\{E_{n_1, n_2, \ldots, n_k}\}) = \prod_{n_1, n_2, \ldots, n_k} E_{n_1, n_2, \ldots, n_k}.$$

1) $\breve{N}$ denotes the *completed form* of a set N; see def. 8, Fund. Math. t. XVIII, p. 233.

2) If $\Phi_N(\{E_n\}) = \sum_n E_n$ then N (supposed complete) coincides with J. The set Θ in this case contains all the systems of the type

$$[(n_1), (n_1, n_2), (n_1, n_2, n_3), \ldots]$$

(„A-chains"). On the other hand evidently every system contained in Θ, contains an A-chain.

Hence follows

$$R\,\Phi_N(\{E_{n_1, n_2, \ldots, n_k}\}) = \sum_{n_1, n_2, \ldots} \prod_k E_{n_1, n_2, \ldots, n_k}$$

i. e. the operation $R\,\Sigma$ is A-operation of Mr. Souslin.

44. Definition 25. If $\mathcal{H}$ is an operation on classes of sets corresponding to a positive analytical operation Φ then $R\,\mathcal{H}$ denotes the operation coresponding to $R\,\Phi$.

45. We shall prove now some theorems relating to $R\,\Phi$ operations. We begin with the following remark:

In def. 28 bis we can substitue conditions (42, 3) and (42, 3′) by the following:

$$(45,1) \qquad \frac{1}{|n_{i_1, i_2, \ldots, i_k, 1}} + \frac{1}{|n_{i_1, i_2, \ldots, i_k, 2}} + \ldots \in N$$

and

$$(45,1') \qquad \frac{1}{|n_1} + \frac{1}{|n_2} + \ldots \in N.$$

Theorem XXV. *If Ψ is a positive analytical operation, $\mathcal{H}$ the corresponding operation on classes of sets, then we have for any class $\mathcal{K}$*

$$(45,2) \qquad \mathcal{H}(\mathcal{K}) \subset R\,\mathcal{H}(\mathcal{K}).$$

Proof. Let $P \in \mathcal{H}(\mathcal{K})$ i. e.

$$P = \Psi(\{E_\xi\}); \quad E_\xi \in \mathcal{K}.$$

Denote

$$E_{\xi_1, \xi_2, \ldots, \xi_k} = E_{\xi_1} \quad (\in \mathcal{K}).$$

We have evidently

$$P = R\,\Psi(\{E_{\xi_1, \xi_2, \ldots, \xi_k}\}) \in R\,\mathcal{H}(\mathcal{K}).$$

Theorem XXVI. *If $\Psi^{(1)}$ and $\Psi^{(2)}$ are positive analytical operations on a countable*[1]) *infinity of sets, $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ are the corresponding operations upon classes of sets and if*

$$\mathcal{H}^{(1)}(\mathfrak{D}) \subset \mathcal{H}^{(2)}(\mathfrak{D}) \quad \text{(see def. 5, art. 5)}$$

then for any d-class $\mathfrak{L}$

$$R\mathcal{H}_1(\mathfrak{L}) \subset R\mathcal{H}_2(\mathfrak{L}).$$

Proof. Let $\Psi^{(1)}$ depend of a system of arguments $\{E_\xi\}$ depending of a countable set $\Xi = \{\xi\}$ of indices and in the same manner let $\Psi^{(2)}$ depend of $\{E_\eta\}$ depending of a set $\Theta = \{\eta\}$ of indices. Enumerate all the indices $\{\xi\}$:

$$\xi^{(1)},\ \xi^{(2)}, \dots$$

and also the indices $\{\eta\}$

$$\eta^{(1)},\ \eta^{(2)}, \dots$$

Then, as we already know (Theorem I), there exist two δs-functions

$$\Phi_{N_1}_k(\{E_{\xi^{(k)}}\}) = \Psi^{(1)}_\xi(\{E_\xi\})$$

and

$$\Phi_{N_2}_k(\{E_{\eta^{(k)}}\}) = \Psi^{(2)}_\eta(\{E_\eta\}).$$

Evidently

$$R\,\Phi_{N_1}(\{E_{\xi^{(i_1)}, \xi^{(i_2)}, \dots, \xi^{(i_k)}}\}) = R\,\Psi^{(1)}(\{E_{\xi_1, \xi_2, \dots, \xi_k}\})$$

$$R\,\Phi_{N_2}(\{E_{\eta^{(i_1)}, \eta^{(i_2)}, \dots, \eta^{(i_k)}}\}) = R\,\Psi^{(2)}(\{E_{\eta_1, \eta_2, \dots, \eta_k}\}).$$

Consequently for any $\mathfrak{L}$

$$(45,3)\quad \left\{ \begin{array}{lll} R\,\mathcal{H}_{N_1}(\mathfrak{L}) = R\,\mathcal{H}^{(1)}(\mathfrak{L}) & \text{and} & R\,\mathcal{H}_{N_2}(\mathfrak{L}) = R\,\mathcal{H}^{(2)}(\mathfrak{L}) \\ \mathcal{H}_{N_1}(\mathfrak{L}) = \mathcal{H}^{(1)}(\mathfrak{L}) & \text{and} & \mathcal{H}_{N_2}(\mathfrak{L}) = \mathcal{H}^{(2)}(\mathfrak{L}). \end{array} \right.$$

We shall suppose N_2 complete ($N_2 = \tilde{N}_2$). Therefore, by Theorem IV (art. 8.) there exist a function $k(n)$ (n and $k(n)$ being positive integers) and a set N_1' equivalent to N_1, such that

$$\frac{1}{|n_1} + \frac{1}{|n_2} + \dots \in N_1'$$

[1]) The analytical operations on an uncountable infinity of sets shall be considered in an appendix.

if and only if there exist such m_i that

$$1)\quad n_i = k(m_i) \quad \text{for any } i \quad \text{and}$$

$$(45,4)\quad 2)\quad \frac{1}{|m_1} + \frac{1}{|m_2} + \dots \in N_2.$$

Let now $P \in R\,\mathcal{H}_{N_1}(\mathfrak{L})$; $\mathfrak{L}$ being a d-class, i. e.

$$P = R\,\Phi_{N_1'}(\{E_{n_1, n_2, \dots, n_k}\}); \quad E_{n_1, n_2, \dots, n_k} \in \mathfrak{L}$$

$$E_{n_1, n_2, \dots, n_k, n_{k+1}} \subset E_{n_1, n_2, \dots, n_k}.$$

Denote

$$(45,5)\qquad K_{m_1, m_2, \dots, m_k} = E_{k(m_1), k(m_2), \dots, k(m_k)}$$

and

$$Q = R\,\Phi_{N_2}(\{K_{m_1, m_2, \dots, m_k}\}) \in R\,\mathcal{H}_{N_2}(\mathfrak{L})$$

we shall prove that $P = Q$.

α) $Q \subset P$. Let $x \in Q$. Then there exists a system of natural numbers $\{m_{i_1, i_2, \dots, i_k}\}$ such that for any $i_1, i_2, \dots, i_k$

$$\frac{1}{|m_{i_1, i_2, \dots, i_k, 1}} + \frac{1}{|m_{i_1, i_2, \dots, i_k, 2}} + \dots \in N_2$$

and that $x \in K_{m_{i_1}, m_{i_1, i_2}, \dots, m_{i_1, i_2, \dots, i_k}}$ for any k; $i_1, i_2, \dots, i_k$

We have thus

$$x \in \prod_{i_1, i_2, \dots, i_k} K_{m_{i_1}, m_{i_1, i_2}, \dots, m_{i_1, i_2, \dots, i_k}} = \prod_{i_1, i_2, \dots, i_k} E_{k(m_{i_1}), k(m_{i_1, i_2}), \dots, k(m_{i_1, i_2, \dots, i_k})}.$$

But for any $i_1, i_2, \dots, i_k$

$$\frac{1}{|k(m_{i_1, i_2, \dots, i_k, 1})} + \frac{1}{|k(m_{i_1, i_2, \dots, i_k, 2})} + \dots \in N_1'$$

Whence follows that $x \in P$.

β) $P \subset Q$. Let $x \in P$. The there exists a system of natural numbers $\{n_{i_1, i_2, \dots, i_k}\}$ such that for any $i_1, i_2, \dots, i_k$

$$(45,6)\qquad \frac{1}{|n_{i_1, i_2, \dots, i_k, 1}} + \frac{1}{|n_{i_1, i_2, \dots, i_k, 2}} + \dots \in N_1'$$

and that $x \in E_{n_1, n_2, \dots, n_k}$ for any k; $n_1, n_2, \dots, n_k$.

By (45,4) there exists such system $\{m_{i_1, i_2,\dots, i_k}\}$ of natural numbers that for any $i_1, i_2,\dots, i_k$

1) $k(m_{i_1, i_2,\dots, i_k}, 1) = n_{i_1, i_2,\dots, i_k}$

2) $\dfrac{1}{|m_{i_1,i_2,\dots,i_k,1}} + \dfrac{1}{|m_{i_1,i_2,\dots,i_k,2}} + \dots \in N_2$.

It follows from 1) and (45,5) that

$$E_{n_{i_1}, n_{i_1,i_2},\dots, n_{i_1,i_2,\dots,i_k}} = K_{m_{i_1}, m_{i_1,i_2},\dots, m_{i_1,i_2,\dots,i_k}}.$$

Whence $x \in Q$. So we have proved that

$$P = Q \in R\mathcal{H}_{N_2}(\mathfrak{L})$$

if $P \in R\mathcal{H}_{N_1}(\mathfrak{L})$.

Thus $R\mathcal{H}_{N_1}(\mathfrak{L}) \subset R\mathcal{H}_{N_2}(\mathfrak{L})$ and then from (45,3) follows

$$R\mathcal{H}^{(1)}(\mathfrak{L}) \subset R\mathcal{H}^{(2)}(\mathfrak{L}) \quad \text{q. e. d.}$$

Theorem XXVII. *For any N and $\mathfrak{L}$ containing the greatest set $K^{(0)}$*

$$\mathcal{H}_N(R\mathcal{H}_N(\mathfrak{L})) = R\mathcal{H}_N(\mathfrak{L}).$$

Proof. It is evident that

$$R\mathcal{H}_N(\mathfrak{L}) \subset \mathcal{H}_N(R\mathcal{H}_N(\mathfrak{L})).$$

So we have only to prove that

$$\mathcal{H}_N(R\mathcal{H}_N(\mathfrak{L})) \subset R\mathcal{H}_N(\mathfrak{L}).$$

Let $P \in \mathcal{H}_N(R\mathcal{H}_N(\mathfrak{L}))$ then

$$P = \underset{n}{\Phi_N}(\{\underset{n_1, n_2,\dots, n_k}{R\quad \Phi_N}(\{E^{n}_{n_1, n_2,\dots, n_k}\})\}).$$

Denote

$$H_{n_1, n_2,\dots, n_k} = E^{n_1}_{n_2,\dots, n_k} \quad (k > 1)$$

and

$$H_{n_1} = K^{(0)} \quad (\text{for any } n_1).$$

Then it follows immediately from the definition of R that

$$P = \underset{n_1, n_2,\dots, n_k}{R \quad \Phi_N}(\{H_{n_1, n_2,\dots, n_k}\}) \in R\mathcal{H}_N(\mathcal{J}).$$

Remark. If the condition that $\mathfrak{L}$ contains the sum of all its elements is not fulfilled then the theorem sometimes ceases to be true. Let, e. g. $\mathfrak{L}$ be the class of all the sets of diameter not greater than 1 (in Euclidean space) and N the set consisting of two numbers only:

$$\frac{1|}{|1} + \frac{1|}{|1} + \dots \quad \text{and} \quad \frac{1|}{|2} + \frac{1|}{|2} + \dots$$

Then, as we may easily see, $R\mathcal{H}_N(\mathfrak{L})$ is the class of all the sets which are the sums of two sets each of the diameter not greater than 1. Whille $\mathcal{H}_N(R\mathcal{H}_N(\mathfrak{L}))$ is the class of all the sums of four sets each of the diameter not greater than 1. Thus we have

$$\mathcal{H}_N(R\mathcal{H}_N(\mathfrak{L})) \neq R\mathcal{H}_N(\mathfrak{L}).$$

However our theorem remains true even without the condition that $\mathfrak{L}$ contains the sum of all its elements, provided that the operation Φ_N be normal.

Corollary. $R\mathcal{H}_N(\mathfrak{L}) \supset \mathcal{H}_N(\mathcal{H}_N(\mathfrak{L}))$.

The above remark may be applied to this corollary.

46. Theorem XXVIII. *The operation $RR\Phi_N$ is equivalent with the operation $R\Phi_N$.*

Proof. We must prove that for any class $\mathfrak{L}$ of sets

(46,1) $$RR\mathcal{H}_N(\mathfrak{L}) = R\mathcal{H}_N(\mathfrak{L}).$$

As we already know (Theorem XXV)

$$R\mathcal{H}_N(\mathfrak{L}) \subset RR\mathcal{H}_N(\mathfrak{L}).$$

If we now prove that

(46,2) $$RR\mathcal{H}_N(\mathfrak{L}) \subset R\mathcal{H}_N(\mathfrak{L})$$

then Theorem XXVIII will be completely demonstrated.

Let

$$P \in RR\mathcal{H}_N(\mathfrak{L})$$

i. e.

$$P = RR\Phi_N(\{E_\sigma\}); \quad E_\sigma \in \mathfrak{L}$$

$\{E_\sigma\}$ being a system of sets depending of the set of all the corteges of the second order (doubles corteges) i. e. corteges having for their elements the corteges of natural numbers. (In fact Φ_N is the function of a sequence of sets, $R\,\Phi_N$ is the function of a system of sets depending of the corteges of the first order (see notations group D, 2) so $R\,R\,\Phi_N$ must be the function of a system of sets depending of the corteges of corteges i. e. of the double corteges).

We shall now define a system of sets

$$\{K_{\nu_1, \nu_2, \dots, \nu_i}\}; \quad i = 1, 2, \dots; \quad \nu_j = 1, 2, \dots$$

as follows: let

$$(46,3) \quad i = 2^{r_1 + r_2 + \dots + r_\varkappa - 1} + 2^{r_2 + r_3 + \dots + r_\varkappa - 1} + \dots + 2^{r_\varkappa - 1} = u(r_1, r_2, \dots, r_\varkappa)$$

and denote by $\sigma(\nu_1, \nu_2, \dots, \nu_i)$ the following double cortege

$$(46,4) \quad \sigma(\nu_1, \nu_2, \dots, \nu_i) = \begin{pmatrix} \nu_{u(1)}, \nu_{u(2)}, & \dots & \nu_{u(r_1)} \\ \nu_{u(r_1, 1)}, \nu_{u(r_1, 2)}, & \dots & \nu_{u(r_1, r_2)} \\ \dots & \dots & \dots \\ \nu_{u(r_1, r_2, \dots, r_{\varkappa-1}, 1)}, \dots, & & \nu_{u(r_1, r_2, \dots, r_\varkappa)} \end{pmatrix}$$

Then we

$$(46,5) \qquad K_{\nu_1, \nu_2, \dots, \nu_i} = E\sigma(\nu_1, \nu_2, \dots, \nu_i).$$

We shall prove that

$$(46,6) \qquad Q \equiv R\,\Phi_N(\{K_{\nu_1, \nu_2, \dots, \nu_i}\}) = P.$$

We shall suppose that $N = \tilde{N}$.

α) $Q \subset P$. Let $x \in Q$. Then there exists an $R\,\Phi_N$-chain

$$T = \{(\nu_{s_1}, \nu_{s_1, s_2}, \dots, \nu_{s_1, s_2, \dots, s_i})\}$$

of corteges, such that

$$x \in \prod_{(s_1, s_2, \dots, s_i)} K_{\nu_{s_1}, \nu_{s_1, s_2}, \dots, \nu_{s_1, s_2, \dots, s_i}}$$

i. e.

$$x \in \prod_{(s_1, s_2, \dots, s_i)} E\sigma(\nu_{s_1}, \nu_{s_1, s_2}, \dots, \nu_{s_1, s_2, \dots, s_i}) \quad \text{(see (46,5))}.$$

If we now prove that the set of all the

$$\sigma(\nu_{s_1}, \nu_{s_1, s_2}, \dots, \nu_{s_1, s_2, \dots, s_i})$$

constitutes an $R\,R\,\Phi_N$-chain then the inclusion $Q \subset P$ will be proved.

The set

$$\mathfrak{T} = \{\sigma(\nu_{s_1}, \nu_{s_1, s_2}, \dots, \nu_{s_1, s_2, \dots, s_i})\}$$

satisfies all the conditions which definie an $R\,R\,\Phi_N$-chain viz.:

1) if σ' is a segment of $\sigma(\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(i)}) \in \mathfrak{T}$ then $\sigma' \in \mathfrak{T}$. In fact if σ' is a segment [1]) of $\sigma(\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(i)})$ then $\sigma' = \sigma(\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(i')})$ $(i' \leqslant i)$ (though the converse is not true: $\sigma(\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(i')})$ is not, generally speaking, a segment of $\sigma(\nu^{(1)}, \nu^{(2)}, \nu^{(i)})$). But if $\sigma(\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(i)}) \in \mathfrak{T}$ then $(\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(i)}) \in T$ whence $(\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(i')}) \in T$ and consequently

$$\sigma' = \sigma(\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(i')}) \in \mathfrak{T}.$$

2) if $\sigma^{(0)} \in \mathfrak{T}$ and if $\sigma^{(1)}, \sigma^{(2)}, \dots$ is the sequence of *all* the double corteges which: α) belong to $\mathfrak{T}$; β) have the same rang k (i. e. number of elements) [2]) as $\sigma^{(0)}$; γ) have all their elements except the last identical with the corresponding elements of $\sigma^{(0)}$; then, denoting $\mathfrak{w}_i$ the last element of $\sigma^{(i)}$ $(i = 0, 1, 2, \dots)$, the set of all $\mathfrak{w}_i$'s is an $R\,\Phi_N$-chain.

In fact let

$$\sigma^{(0)} = (\sigma'; \mathfrak{w}_0)$$

where σ' is a double cortege of the rang $k-1$ if $k > 1$ and is vacuous if $k = 1$. Then for any t

$$\sigma^{(t)} = (\sigma'; \mathfrak{w}_t).$$

To prove that $\{\mathfrak{w}_i\}$ is an $R\,\Phi_N$ chain we must demonstrate a) that if $\mathfrak{w}^{(0)} \in \{\mathfrak{w}_i\}$ then any segment of $\mathfrak{w}^{(0)}$ also belongs to $\{\mathfrak{w}_i\}$; b) that if $\mathfrak{w}'$ belongs to $\{\mathfrak{w}_i\}$ (or is vacuous) and $\{n_j\}$ is the sequence of *all* such numbers that $(\mathfrak{w}', n_j) \in \{\mathfrak{w}_i\}$ then

$$\frac{1|}{|n_1} + \frac{1|}{|n_2} + \frac{1|}{|n_3} + \dots \in N.$$

a) If $\mathfrak{w}_t = (\mu_1, \mu_2, \dots, \mu_\tau)$ and if $\mathfrak{w}' = (\mu_1, \mu_2, \dots, \mu_{\tau'})$ is a segment of $\mathfrak{w}_t$ then we have $(\sigma'; \mathfrak{w}_t) \in \mathfrak{T}$. i. e.

$$(\sigma'; \mathfrak{w}_t) = \sigma(\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(l)}); \quad (\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(l)}) \in T.$$

[1]) A segment (of rang k) of a double cortege σ is another double cortege consisting of the first k corteges of σ.

[2]) We must not forget that the elements of a double cortege are corteges (of the first order), not numbers.

but then evidently

$$(\sigma'; \mathfrak{w}') = \sigma(\nu^{(1)}; \nu^{(2)}, \dots, \nu^{(i')}); \quad (i' \leqslant i)$$

or

$$(\sigma'; \mathfrak{w}') \in \mathfrak{T}, \quad \text{because} \quad (\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(i')}) \in T.$$

Therefore $\mathfrak{w}' = \mathfrak{w}_{t'}$ and we have proved that every segment of a cortege belonging to $\{\mathfrak{w}_t\}$ also belongs to $\{\mathfrak{w}_t\}$.

b) Let $\mathfrak{w} = \mathfrak{w}_{t_0}$ and consider the sequence $\mathfrak{w}_{t_1}, \mathfrak{w}_{t_2}, \dots$ of all the corteges $\mathfrak{w}_t$ which do not differ from $\mathfrak{w}_{t_0}$ except in their last element; and let n_j be the last element of $\mathfrak{w}_{t_j}$ $(j = 0, 1, 2, \dots)$. Then if $\mathfrak{w}_{t_0} = (\mathfrak{w}', n_0)$ and consequently $\mathfrak{w}_{t_j} = (\mathfrak{w}', n_j)$.

We have $(\sigma'; \mathfrak{w}_{t_0}) \in \mathfrak{T}$ or

$$(\sigma'; \mathfrak{w}_{t_0}) = \sigma(\nu^{)}, \nu^{(2)}, \dots, \nu^{(i)}); \quad (\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(i)}) \in T$$

whence

$$(\sigma'; \mathfrak{w}') = \sigma(\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(i')}) = \sigma(\nu_{s_1}, \nu_{s_1,s_2}, \dots, \nu_{s_1,s_2,\dots,s_{i'}}) \quad (i' \leqslant i)$$

if we now denote $s_{i'+1} = s_{i'+2} = \dots = s_{2i'-1} = 1$ then (by the def. of $\sigma(\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(i)})$, **(46, 4)** and **(46, 3)**)

$$(\sigma'; \mathfrak{w}', \nu_{s_1,s_2,\dots,s_{2i'-1},s}) = \sigma(\nu_{s_1}, \nu_{s_1,s_2}, \dots, \nu_{s_1,s_2,\dots,s_{2i'-1},s}) \in \mathfrak{T}$$

for any s. Hence and from the definition of the numbers n_j follows that all the numbers $\nu_{s_1,s_2,\dots,s_{2i'-1},s}$ are found among the numbers n_j i. e.

$$\nu_{s_1,s_2,\dots,s_{2i'-1},s} = n_{j_s}$$

whence by the definition of the $R\,\Phi_N$-chain $\{(\nu_{s_1}, \nu_{s_1,s_2}, \dots, \nu_{s_1,s_2,\dots,s_i})\}$

$$\frac{1|}{|n_{j_1}} + \frac{1|}{|n_{j_2}} + \dots \in N$$

or, N being complete,

$$\frac{1|}{|n_0} + \frac{1|}{|n_1} + \frac{1|}{|n_2} + \dots \in N.$$

b) is proved and with it is proved that $\{\mathfrak{w}_t\}$ is an $R\,\Phi_N$-chain i. e. we have proved the property 2). But the properties 1) and 2) constitute the definition of an $R\,R\,\Phi_N$-chain, and thus inclusion α) is proved.

47. β) $P \subset Q$. Let $x \in P$. Then there exists such $R\,R\,\Phi_N$-chain $\mathfrak{T}$ that

$$x \in \prod_{\sigma \in \mathfrak{T}} E_\sigma.$$

Denote T the set of all the corteges $(\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(i)})$ such that

(47, 1) $$\sigma(\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(i')}) \in \mathfrak{T}$$

for any $i' \leqslant i$.

Evidently

$$x \in \prod_{(\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(i)}) \in T} K_{\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(i}}$$

We shall prove that T is an $R\,\Phi_N$-chain.

In fact: 1) evidently if $\mathfrak{w} \in T$ then any segment of $\mathfrak{w}$ also belongs to T.

2) Let

$$(\nu_1^{(0)}, \nu_2^{(0)}, \dots, \nu_i^{(0)}) \in T;$$

$$i + 1 = u(r_1, r_2, \dots, r_\varkappa) \quad (\text{See } \mathbf{(46, 3)}).$$

Denote

$$i_0 = 2^{r_1 + r_2 + \dots + r_\varkappa - 2} + \dots + 2^{r_\varkappa - 2} = u(r_1, r_2, \dots, r_{\varkappa-1}, r_\varkappa - 1).$$

if $r_\varkappa > 1$ and

$$i_0 = u(r_1, r_2, \dots, r_{\varkappa-1})$$

if $r_\varkappa = 1$.

Evidently $i_0 \leqslant i$. Consequently

$$(\nu_1^{(0)}, \nu_2^{(0)}, \dots, \nu_{i_0}^{(0)}) \in T$$

and therefore

$$\sigma(\nu_1^{(0)}, \nu_2^{(0)}, \dots, \nu_{i_0}^{(0)}) \in \mathfrak{T}.$$

but **(46, 4)**

$$\sigma(\nu_1^{(0)}, \nu_2^{(0)}, \dots, \nu_{i_0}^{(0)}) = \begin{pmatrix} \nu_{u(1)}^{(0)}, & \nu_{u(2)}^{(0)}, & \dots & \dots & \nu_{u(r_1)}^{(0)} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ (\nu_{u(r_1,\dots,r_{\varkappa-1},1}^{(0)}, & \dots, & & & \nu_{u(r_1,\dots,r_{\varkappa-1},r_\varkappa-1)}^{(0)}). \end{pmatrix}$$

$\mathfrak{T}$ being an $R\,R\,\Phi_N$-chain there exists a number

$$\xi = \frac{1|}{|\nu_{i+1}^1} + \frac{1|}{|\nu_{i+1}^2} + \dots \in N$$

such that

$$\sigma^{(\lambda)} = \begin{pmatrix} \nu^{(0)}_{u(1)}, & \nu^{(0)}_{u(2)}, & \dots & \dots & \nu^{(0)}_{u(r_1)} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \nu^{(0)}_{u(r_1, r_2, \dots, r_{\varkappa-1}, 1)}, & \dots, & \nu^{(0)}_{u(r_1, \dots, r_{\varkappa-1}, r_\varkappa - 1)}, & \nu^{\lambda}_{i+1} \end{pmatrix} \in \mathfrak{T}$$

for any λ.

But one may easily see that

$$\sigma^{(\lambda)} = \sigma(\nu_1^{(0)}, \nu_2^{(0)}, \dots, \nu_i^{(0)}, \nu_{i+1}^{\lambda})$$

$$(\nu_1^{(0)}, \nu_2^{(0)}, \dots, \nu_i^{(0)}, \nu_{i+1}^{\lambda}) \in T,$$

because $\sigma^{(\lambda)}$ and $\sigma(\nu_1^{(0)}, \nu_2^{(0)}, \dots, \nu_{i'}^{(0)})$ $(i' \leqslant i)$ belong to $\mathfrak{T}$. So that

W h a t e v e r b e c o r t e g e

$$(\nu_1^0, \nu_2^0, \dots, \nu_i^0) \in T$$

t h e r e e x i s t s s u c h n u m b e r

$$\frac{1|}{|\nu^1_{i+1}} + \frac{1|}{|\nu^2_{i+1}} + \dots \in N$$

that

$$(\nu_1^0, \nu_2^0, \dots, \nu_i^0, \nu_{i+1}^{\lambda}) \in T \quad \text{for any } \lambda.$$

In the same way we could prove that there exists such number

$$\frac{1|}{|\nu_1^1} + \frac{1|}{|\nu_1^2} + \dots \in N$$

that $(\nu_1^{\lambda}) \in T$ for any λ.

These properties taken together with 1) prove that T is an $R\,\Phi_N$-chain, q. e. d.

Theorem XXVIII is now completely demonstrated.

48. Theorem XXIX. *The operation $R\,\Phi_N$ is r-normal with resp. to* $[d]$ (see def. 12*bis* art. 15 and def. 19 art. 31).

P r o o f. If $\mathfrak{L} \in [d]$ then

$$R\,\mathcal{H}_N(\mathfrak{L}) = R\,R\,\mathcal{H}_N(\mathfrak{L}) \supset R\,\mathcal{H}_N(R\,\mathcal{H}_N(\mathfrak{L})) \supset R\,\mathcal{H}_N(\mathfrak{L})$$

(Theorems XXVIII, XXVII) and therefore

$$R\,\mathcal{H}_N(\mathfrak{L}) = R\,\mathcal{H}_N(R\,\mathcal{H}_N(\mathfrak{L})) \quad \text{q. e. d.}$$

Remark. If $\mathfrak{L}$ non $\in [d]$ then $R\,\mathcal{H}_N(R\,\mathcal{H}_N(\mathfrak{L}))$ does not always coincide with $R\,\mathcal{H}_N(\mathfrak{L})$. See remark to Theorem XXVII.

49. Theorem XXX. *$R\,\Phi_N$-operation is equivalent with a δs-operation $\Phi_{N'}$, the base N' of which can be obtained from N and J by the operations of multiplication, homeomorphic transformations and countable intersection.*

P r o o f. Denote $N^*_{i_1, i_2, \dots, i_k}$ the set of all the numbers

$$\xi = \frac{1|}{|p_1} + \frac{1|}{|p_2} + \dots$$

such that $p_{u(i_1, i_2, \dots, i_k)}$ can be represented in the form

$$p_{u(i_1, i_2, \dots, i_k)} = u(m_1, m_2, \dots, m_k) \quad (\text{See } \mathbf{46}, 3)$$

and

$$p_{u(i_1, i_2, \dots, i_k, s)} = u(m_1, m_2, \dots, m_k, n_s^*)$$

$$\frac{1|}{|n_1^*} + \frac{1|}{|n_2^*} + \dots \in N.$$

One may easily verify that $N^*_{i_1, i_2, \dots, i_k}$ is homeomorphic with $N \times J$. In the same manner we denote N^* the set of all the numbers

$$\xi = \frac{1|}{|p_1} + \frac{1|}{|p_2} + \dots$$

such that

$$p_{u(s)} = u(n_s^*), \qquad \frac{1|}{|n_1^*} + \frac{1|}{|n_2^*} + \dots \in N$$

N^* is also homeomorphic with $N \times J$.

Denote now

$$N' = N^* \prod_{(i_1, i_2, \dots, i_k)} N^*_{i_1, i_2, \dots, i_k}$$

we must prove that $\Phi_{N'}$ is equivalent with $R\,\Phi_N$. In fact it follows from the definition of N' that N' is the set of all such numbers

$$\xi = \frac{1|}{|p_1} + \frac{1|}{|p_2} + \dots$$

that

$$p_{u(i_1, i_2, \dots, i_k)} = u(n_{i_1}, n_{i_1, i_2}, \dots, n_{i_1, i_2, \dots, i_k})$$

and for any $i_1, i_2, \dots, i_k$

$$\frac{1}{|n_{i_1, i_2, \dots, i_k, 1}} + \frac{1}{|n_{i_1, i_2, \dots, i_k, 2}} + \dots \in N;$$

also

$$\frac{1}{|n_1} + \frac{1}{|n_2} + \dots \in N.$$

Now if

(49,1) $$P = R\,\Phi_N(\{E_{n_1, n_2, \dots, n_k}\})$$

then denoting

(49,2) $$H_{u(n_1, n_2, \dots, n_k)} = E_{n_1, n_2, \dots, n_k}$$

we shall have evidently (see **42**,2)

(49,3) $$P = \Phi_{N'}(\{H_i\}).$$

If on the other hand

(49,3) $$P = \Phi_{N'}(\{H_i\})$$

then denoting

(49,2) $$E_{n_1, n_2, \dots, n_k} = H_{u(n_1, n_2, \dots, n_k)}$$

we shall have evidently

(49,1) $$P = R\,\Phi_N(\{E_{n_1, n_2, \dots, n_k}\}).$$

Both these formulae are too evident to require any proof.

But then it follows from the equivalence of (49,1) and (49,3) that for any $\mathfrak{L}$

$$R\,\mathscr{H}_N(\mathfrak{L}) = \mathscr{H}_{N'}(\mathfrak{L})$$

whence $R\,\Phi_N$ and $\Phi_{N'}$ are equivalent, q. e. d.

50. This chapter stands somewhat apart from the other two chapters of this part because it deals with operations upon operations and not operations upon systems of sets. Of course the operation which we considered in this chapter is only one particular operation of many, and it might seem strange why we considered it at all in this work. But we have put it here because we shall need it in the second part of this work and because in the construction of this first part we were guided chiefly by the considerations concerning the second part. It seems however (though we do not know

it for sure) that R is only the first of a sequence of operations (upon operations) possessing some very remarkable properties which can not be explained without mentioning projective classes.

Contents of the First Part.
