Memoir on the Analytical Operations and Projective Sets (II).

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CHAPTER II.

Generalised Souslin’s Operations.

§ 1. Definition and Immediate Consequences.

29. We have studied in the preceding chapter the properties of the $\delta$-operations of Mr. Hausdorff, which are, as we have seen, the most general positive analytical operations exerted upon a countable infinity of sets. There are however certain particular classes of these operations which present a special interest. Such are, for instance, the generalised Souslin’s operations which are defined as follows:

Definition 18$: Let$

\{E_{11, n_2, \ldots, n_k} \} \quad (k = 1, 2, \ldots, n_1 = 1, 2, \ldots, n_2, \ldots)

be a system of sets depending on cortege $(n_1, n_2, \ldots, n_k)$, $N$ a set of irrational numbers, $\mathcal{R} \sim N$ the corresponding set of sequences of positive integers. Then

$$
\Omega_N(\{E_{11, n_2, \ldots, n_k}\}) = \sum_{(n_1, n_2, \ldots) \in \mathcal{R}} \prod_{k} E_{n_1, n_2, \ldots, n_k}.
$$

The corresponding operation upon classes (see def. 4) we shall denote $S_N$, i.e.


$\Omega_N$ is the class of all the sets of the form

$$
\Omega_N(\{E_{n_1, n_2, \ldots, n_k}\})
$$

where all the

$$
E_{n_1, n_2, \ldots, n_k} \in \mathcal{H}.
$$

Remark. In case $N = J$ (i.e. if $\mathcal{R} \sim N$ is the set of all the sequences of natural numbers) the operation $\Omega_N$ becomes the well-known operation (A) of Mr. Souslin

$$
\sum_{(n_1, n_2, \ldots) \in J} \prod_{k} E_{n_1, n_2, \ldots, n_k}.
$$

By Theorem I Cor. (Art. 6) the operation $\Omega_N$ is equivalent with a $\delta$-operation $\Phi_N$. We shall prove now that we can suppose $N'$ homeomorphic to $N$.

Theorem XIX. Let all the cortege $(n_1, n_2, \ldots, n_k)$ be enumerated and let $\nu(n_1, n_2, \ldots, n_k)$ be the natural number corresponding to $(n_1, n_2, \ldots, n_k)$. Then there exists such $N'$ homeomorphic to $N$ that

$$
(29, 2) \quad \Omega_{N'}(\{E_{n_1, n_2, \ldots, n_k}\}) = \Phi_{N'}(\{E_{n_1, n_2, \ldots, n_k}\}).
$$

Proof. Let

$$
\xi = |n_1| + |n_2| + \ldots \in J.
$$

Denote

$$
\varphi_{n_1} = 1 + |n_1| \quad \varphi_{n_2} = 1 + |n_2| + \ldots
$$

Then $N' = \varphi J$ is evidently homeomorphic to $N$ and satisfies (29, 2).

Corollary. For any $\mathcal{H}$ we have

$$
S_{\mathcal{H}}(\mathcal{H}) = \mathcal{H}_{\mathcal{H}}(\mathcal{H}).
$$

30. Here are some simple properties of $\Omega_N$-functions.

1. For any $N$ and $\mathcal{H}$ we have

$$
(30, 1) \quad \mathcal{H}_N(\mathcal{H}) \subset S_N(\mathcal{H}).
$$

Let, in fact, $H \in \mathcal{H}_N(\mathcal{H})$. Then $H = \Phi_N(\{E_n\})$; $E_n \in \mathcal{H}$. 

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Definition 19. A class \( \mathcal{A} \) of sets is a \( \mathcal{d} \)-class if it possesses the following two properties:

1. \( \sum E \in \mathcal{A} \). It is evidently the greatest set in \( \mathcal{A} \).

2. If \( A \in \mathcal{A} \) and \( B \in \mathcal{A} \), then \( AB \in \mathcal{A} \).

The family of all the \( \mathcal{d} \)-classes we shall denote \([d]\).

Definition 20. A class \( \mathcal{A} \) of sets is an \( \mathcal{s} \)-class if it possesses the following properties:

1. \( \Pi E \in \mathcal{A} \). (In all important cases \( \Pi E = 0 \))

2. If \( A \in \mathcal{A} \) and \( B \in \mathcal{A} \), then \( A + B \in \mathcal{A} \).

The family of all the \( \mathcal{s} \)-classes we shall denote \([s]\).

Definition 21. A class \( \mathcal{A} \) of sets is a ring if it is a \( \mathcal{d} \)-class and an \( \mathcal{s} \)-class simultaneously.

The family of all the rings we shall denote \([r]\). (i.e., \([r] = [d] \cap [s]\).

Evidently if a class \( \mathcal{A} \) of subsets of \( R \) is an \( \mathcal{s} \)-class then \( \mathcal{A} \) (see notations) is a \( \mathcal{d} \)-class and vice-versa.

Definition 22. Two systems of sets \( \{E_n, n_1, \ldots, n_k\} \) and \( \{E'_n, n_1, \ldots, n_k\} \) are equivalent if for any \( N \)

\[ \Omega_N(\{E_n, n_1, \ldots, n_k\}) = \Omega_N(\{E'_n, n_1, \ldots, n_k\}) \]

Definition 23. A system \( \{E_n, n_1, \ldots, n_k\} \) is regular if we have always (whatever be \( n \), \( n_1, \ldots, n_k, n_{k+1} \))

\[ E_{n_1, n_2, \ldots, n_k, n_{k+1}} \subseteq E_{n_1, n_2, \ldots, n_k} \]

If \( \mathcal{A} \in [d] \) then every system \( \{E_n, n_1, \ldots, n_k\} \) of sets belonging to \( \mathcal{A} \) may be substituted by an equivalent regular system \( \{E'_n, n_1, \ldots, n_k\} \) of sets belonging to \( \mathcal{A} \). It is sufficient to suppose \( E'_n, n_1, \ldots, n_k \) = \( \{E\} \) \( E_{n_1, n_2, \ldots, n_k} \),

§ 2. Equivalence and Inclusion Theorems.

32. Theorem XX. In order that for every d-class \( \mathcal{A} \) we should have

\[ S_{0}(\mathcal{A}) \subseteq S_{0}(\mathcal{A}) \]

it is necessary and sufficient that \( N \) be a continuous image of \( M \): \( N = \varphi M \).

Proof. A. Sufficiency. Let \( N = \varphi M \) and let \( \varphi \) denote the "inverse function" (see Art. 21 and formula (21, 2a–d)). Denote

\[ \lambda_{n_{1}, n_{2}, \ldots, n_{k}} = \varphi(\delta_{n_{1}, n_{2}, \ldots, n_{k}} \cdot N) \subseteq M \]

We shall now construct a system of open (in \( J \)) sets \( h_{n_{1}, n_{2}, \ldots, n_{k}} \) so that

\[ h_{n_{1}, n_{2}, \ldots, n_{k}} = M = \lambda_{n_{1}, n_{2}, \ldots, n_{k}} \]

(32, 2) \[ h_{n_{1}, n_{2}, \ldots, n_{k}, n_{k+1}} \subseteq \lambda_{n_{1}, n_{2}, \ldots, n_{k}} \]

(32, 3) \[ h_{n_{1}, n_{2}, \ldots, n_{k}} \cdot h_{n_{1}, n_{2}, \ldots, n_{k+1}, n_{k+2}} = 0 \] if \( n_{k} = n_{k+2} \)

Observe first of all that

\[ h_{n_{1}, n_{2}, \ldots, n_{k}} \]

is open and closed in \( M \) simultaneously

(32, 4) \[ \lambda_{n_{1}, n_{2}, \ldots, n_{k}} \cdot \lambda_{n_{1}, n_{2}, \ldots, n_{k+1}} = 0 \] if \( n_{k} \neq n_{k+1} \)

Denote for any \( x \in \lambda_{n_{1}, n_{2}, \ldots, n_{k}} \)

\[ \sigma^{(0)} = \frac{1}{2} \varphi(x, M - \lambda_{n_{1}, n_{2}, \ldots, n_{k}}) > 0 \]

(because \( \lambda_{n_{1}, n_{2}, \ldots, n_{k}} \) is open in \( M \))

\[ S_{0}^{(0)}(x) = (x - \sigma^{(0)}, x + \sigma^{(0)}) \]

\[ h_{n_{1}, n_{2}, \ldots, n_{k}} = \sum_{x \in \lambda_{n_{1}, n_{2}, \ldots, n_{k}}} S_{0}^{(0)} \]

(32, 6)

\[ h_{n_{1}, n_{2}, \ldots, n_{k}} = \prod_{i=k}^{k} h_{n_{i}, n_{i+1}, \ldots, n_{k}} \]

The sets \( h_{n_{1}, n_{2}, \ldots, n_{k}} \) satisfy all our conditions. In fact (32, 2) and (32, 3) are evidently satisfied. (32, 4) is also satisfied: for suppose the contrary i.e. that for certain \( n_{a} \)

\[ h_{n_{1}, n_{2}, \ldots, n_{k-1}, n_{k}, n_{k+1}, \ldots, n_{a}} = 0 \]

Then take \( \xi \in h_{n_{1}, n_{2}, \ldots, n_{k-1}, n_{k}, h_{n_{1}, n_{2}, \ldots, n_{k-1}, n_{k+1}}} \). There exist such

\[ x \in \lambda_{n_{1}, n_{2}, \ldots, n_{k-1}, n_{k}} \quad \text{and} \quad x' \in \lambda_{n_{1}, n_{2}, \ldots, n_{k-1}, n_{k+1}} \]

that

\[ \xi \in \sigma^{0}(0, 0) \]

whence (by 32, 5 and 32, 6)

\[ \varphi(x, \xi) < \frac{1}{2} \varphi(x, M - \lambda_{n_{1}, n_{2}, \ldots, n_{k-1}, n_{k}}) \]

\[ \varphi(x, \xi') < \frac{1}{2} \varphi(x', M - \lambda_{n_{1}, n_{2}, \ldots, n_{k-1}, n_{k}}) \]

Hence follows

\[ \varphi(x, x') \leq \varphi(x, \xi) + \varphi(\xi, \xi') < \frac{1}{2} \varphi(x, x') + \frac{1}{2} \varphi(x, x') \]

which is impossible. This contradiction proves that (32, 4) is satisfied.

Let now \( P \in S_{0}(\mathcal{A}) \) i.e.

(32, 7)

\[ P = \varphi(\delta_{n_{1}, n_{2}, \ldots, n_{k}}) \quad ; \quad E_{n_{1}, n_{2}, \ldots, n_{k}} \in \mathcal{A} \]

We may suppose moreover that

(32, 8)

\[ E_{n_{1}, n_{2}, \ldots, n_{k}, n_{k+1}} \subseteq E_{n_{1}, n_{2}, \ldots, n_{k}} \]

(\( \mathcal{A} \) being a d-class; see Art. 31, end).

\( \mathcal{A} \) has the greatest element (set) \( E \) (Art. 31, def. 19).

We shall now define a system of sets \( \{ H_{m_{1}, m_{2}, \ldots, m_{k}} \} \) as follows:

1) If \( \delta_{m_{1}, m_{2}, \ldots, m_{k}} \) is not contained in any \( h_{n_{1}, n_{2}, \ldots, n_{k}} \) then \( H_{m_{1}, m_{2}, \ldots, m_{k}} = E \).

2) If \( \delta_{m_{1}, m_{2}, \ldots, m_{k}} \) is contained in a finite number of \( h \)’s viz.

\[ \delta_{m_{1}, m_{2}, \ldots, m_{k}} \subseteq h_{n_{1}, \ldots, n_{k}} \]

\[ \delta_{m_{j}, m_{j+1}, \ldots, m_{k}} = h_{n_{1}, \ldots, n_{k}, n_{j}} \]

then

\[ H_{m_{1}, m_{2}, \ldots, m_{k}} = E_{n_{1}, n_{2}, \ldots, n_{k}} \]

3) If \( \delta_{m_{1}, m_{2}, \ldots, m_{k}} \) is contained in an infinity of \( h \)’s viz.

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β) \( Q \subseteq P \). Let

\[ x \in \Omega_M(\langle H_{m_1, m_2, \ldots, m_i} \rangle); \]

then there exists such

\[ x = \frac{1}{|m_1|} + \frac{1}{|m_2|} + \ldots + e M \]

that

\[ x \in \prod_k E_{m_i, n_k, n_k^0}; \]

let

\[ q \circ \psi \circ y = \frac{1}{|m_1|} + \frac{1}{|m_2|} + \ldots + e N. \]

Then for any \( k \): \( x \in \delta_{m_i, n_k, n_k^0} \) and consequently \( \delta_{n_k, n_k^0, n_k^0} \) being open there exists such \( j \) that

\[ \delta_{m_i, n_j, n_j^0} \subseteq \delta_{n_k, n_k^0, n_k^0}. \]

But then (by the definition of \( H_{m_1, m_2, \ldots, m_i} \)) we have (see 32.8; 32.9)

\[ x \in H_{m_i, m_2, \ldots, m_{i+k}} = E_{n_i, n_2, \ldots, n_{i+k}} \subseteq E_{n_k, n_k^0, n_k^0} \]

i.e.

\[ x \in \prod_k E_{n_k, n_k^0, n_k^0} \subseteq \Omega_M(\langle H_{m_1, m_2, \ldots, m_i} \rangle) \quad q. \quad e. \]

33. B. Necessity. Let \( M \) and \( N \) be two sets of irrational numbers such that for any \( d \)-class \( \mathcal{A} \) of sets

\[ S_N(\mathcal{A}) \subseteq S_M(\mathcal{A}). \]

Denote \( \mathcal{A}_0 \) the class consisting of:

1) The whole interval \( (0, 1) \) which we shall denote \( \mathcal{A} \).
2) The vacuous set 0.
3) All the sets \( \delta_{n_k, n_k^0, n_k} \) (see notations, Group C).

Evidently \( \mathcal{A} \) is a \( d \)-class; therefore

\[ N = \Omega_N(\langle \delta_{n_k, n_k^0, n_k} \rangle) \subseteq S_N(\mathcal{A}_0) \]

\)[1] This follows immediately from the definitions of \( S_N \) and \( \delta_{n_k, n_k^0, n_k} \).
or

\[(33.1) \quad N = \Omega_m(\{H_{m_1, m_2, \ldots, m_i}\}); \quad H_{m_1, m_2, \ldots, m_i} \in \mathcal{K}_e.\]

Let

\[y = \frac{1}{|m_1|} + \frac{1}{|m_2|} + \ldots \in N.\]

Denote \(\psi y\) the set of all such points

\[\frac{1}{|m_1|} + \frac{1}{|m_2|} + \ldots \in M\]

that

\[y \in \prod_i H_{m_1, m_2, \ldots, m_i}\]

(this set is not vacuous for \(y \in N = \Omega_m(\{H_{m_1, m_2, \ldots, m_i}\}).\)

Denote now \(\psi N = M_i\) (\(\psi K\) denotes \(\Sigma \psi y\)).

We shall prove that \(M_i\) is closed in \(M\).

In fact let

\[(33.2) \quad x = \frac{1}{|m_1|} + \frac{1}{|m_2|} + \ldots \in M - M_i\]

then

\[(33.3) \quad H_{m_1} \cdot H_{m_2} \cdot \ldots = 0\]

because if this set contained a point \(y\) we should have \(x \in \psi y \subseteq M_i\)
in contradiction with (33.2).

Every non vacuous set \(H_{m_1, m_2, \ldots, m_i}\) is of the form

\[H_{m_1, m_2, \ldots, m_i} = \delta_{n_1, n_2, \ldots, n_i}\]

\((n = 1, 2, \ldots; k_i = 0, 1, 2, \ldots; \text{if } k_i = 0 \text{ then } H_{m_1, m_2, \ldots, m_i} = \delta'(0, 1))\)

(33.3) implies that either

1) There exists such \(i\) that \(H_{m_1, m_2, \ldots, m_i} = 0\)

or

2) There exists such natural \(i, j, l\) \((i > j; l < k_i; l < k_j)\) that

\[n'_i \neq n'_j.\]

In both cases (as one may easily see) we have

\[M \cdot \delta_{m_1, m_2, \ldots, m_i} \subseteq M - M_i\]

Thus the set \(M - M_i\) is open in \(M\) and consequently \(M_i\) is closed in \(M\).

We shall prove now the following two properties of \(\psi\)

1) \(\psi y_1 \cdot \psi y_2 = 0; \quad (y_1 = y_2)\)

2) If \(K\) is open in \(N\) then \(\psi K\) is open in \(M_i\)

\[(33.4) \quad 1) \quad \psi y_1 \cdot \psi y_2 = 0 \quad y_1 \neq y_2.\]

Let

\[x = \frac{1}{|m_1|} + \frac{1}{|m_2|} + \ldots \in \psi y_1\]

then \(y_1 \in \prod_i H_{m_1, m_2, \ldots, m_i}\) and we have as above

\[H_{m_1, m_2, \ldots, m_i} = \delta_{n_1, n_2, \ldots, n_i}\]

We can prove that the numbers \(k_i\) are not limited for otherwise they would have the greatest among them (denote it \(k_0\)) and we should have

\[\delta_{n_1, n_2, \ldots, n_0} = \prod_i H_{m_1, m_2, \ldots, m_i} \subseteq N\]

which is impossible, because \(N\) contains no rational numbers.

Now it is evident that

\[y_1 = \frac{1}{|m_1|} + \frac{1}{|m_2|} + \ldots\]

and that

\[\prod_i H_{m_1, m_2, \ldots, m_i} = (y_1)\]

and thus can not contain \(y_2\) so that \(x\) non \(\in \psi y_2\) q. e. d.

2) If a set \(K\) is open in \(N\) then \(\psi K\) is open in \(M_i\).

Let \(K\) be open in \(N\) and let

\[x = \frac{1}{|m_1|} + \frac{1}{|m_2|} + \ldots \in \psi K\]

Then it follows from what just preceded that

\[\prod_i H_{m_1, m_2, \ldots, m_i} = (y) \subseteq K\]
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Therefore $\Phi$ is a continuous function defined in $M$, and having $N$ for its set of values.

But we can now apply a lemma due to Sierpiński¹, which states that if $\Phi$ is a continuous transformation of a set $M$ closed in $X \cap J$ into another set $N$ then there exists a continuous in $M$ function $\Phi^*$ which coincides in $M$ with $\Phi$ and is besides such that

$$\Phi^* M = N$$

i.e., $N$ is a continuous image of $M$. q.e.d.

34. Corollary. In order that two $\Omega$-operations $\Omega_\sigma$ and $\Omega_\delta$ be $r$-equivalent with respect to $[d]$ (i.e., in order that for any $N \in [d]$ we had $S_N(\mathcal{H}) = S_M(\mathcal{H})$; see def. 6 bis and 19) it is necessary and sufficient that each of the sets $N$ and $M$ be a continuous image of the other or, using a notion introduced by Sierpiński⁴, that $M$ and $N$ be of the same type (c).

This implies some interesting consequences. First of all there is a countable infinity of types (c) of countable closed sets and consequently there is a countable infinity of non equivalent $\Omega$-functions having for their base a countable closed sets. These types we shall call for the moment "inferior types" and the corresponding $\Omega$-functions — "inferior functions". Outside these inferior types there are only five types (c) of (A)-sets of irrational numbers. These are:

1) the type of a countable non-closed set, 2) the type of a perfect set, 3) the type of a sum of two sets belonging to preceding types, 4) the type of an $A$ not belonging to $J$, 5) the type of $J$. Consequently there exist five $S_N(\mathcal{H})$ classes ($\mathcal{H} \in [d]$) whose bases are (A)-set viz.:

1) $\mathcal{H}_0$
2) $\mathcal{H}_0$
3) $\mathcal{H}_0 + \delta$
4) $\mathcal{H}_\sigma$
5) $A(\mathcal{H})$

where the meaning of $\delta$, $\sigma$ and $s$ see in notations group B; $\mathcal{H}_0 + \delta$ means the class of all the sets of the form $A + B$ where $A \in \mathcal{H}_0$ and $B \in \mathcal{H}_\sigma$; $A(\mathcal{H})$ is the class of all the results of operation $A$ effected upon sets belonging to $\mathcal{H}$.

As is easily seen, for rings these five $\Omega$-functions and all the inferior $\Omega$-functions are reduced to three, viz.: $\mathcal{H}_0$, $\mathcal{H}_\sigma$ and $A(\mathcal{H})$.

³ Fundamenta Mathematicae. T. XX.
These are the only $S_N(\mathcal{A})$-classes whose base is an \(\mathcal{A}\)-set. If we now remember that
\[
N = \Omega_N((\beta_{n_1, n_2, \ldots, n_k}) = \Omega_N((\beta_{n_1, n_2, \ldots, n_k}))
\]
we can say that: If \(\mathcal{A}\) is a Borelian class (\(\mathcal{F}, \mathcal{B}, \mathcal{F}, \mathcal{B}\) etc.) in \(J\) or in I then the only Borelian classes that can be represented in the form $S_N(\mathcal{A})$ are \(\mathcal{A}_0\) and \(\partial_{\mathcal{A}_0}\) (e.g. if \(\mathcal{A} = \mathcal{B}\) then they are \(\mathcal{A}\) and \(\mathcal{F}\) where \(\mathcal{F}\) is an \(\mathcal{A}\)-class). This shows that there exists no theorem corresponding to Theorem VI (i.e. $S_N(\mathcal{A})$ is not always an $S_N(\mathcal{A})$).

35. Theorem XXI. In order that for every ring \(\mathcal{A}\) be $S_N(\mathcal{A}) \subset \subset S_N(\mathcal{A})$ it is necessary and sufficient that \(N\) be a continuous image of \(M \times \Delta\).

**Proof.** A. Sufficiency. Let
\[
N = \varphi(M \times \Delta).
\]
Denote $M_0$ the set of all such points
\[
\left\{ \frac{1}{|m_1|} + \frac{1}{|m_2|} + \frac{1}{|m_3|} + \ldots \right\}
\]
that
1) \(\frac{1}{|m_1|} + \frac{1}{|m_2|} + \ldots \in M\)
2) \(m_i \leq 2\) \((i = 1, 2, \ldots)\)
then evidently $M_0$ is homeomorphic to $M \times \Delta$ (to any point
\[
\left\{ \frac{1}{|m_1|} + \frac{1}{|m_2|} + \ldots \right\}
\]
of $M_0$ we may correlate the point
\[
\left( \frac{1}{|m_1|} + \frac{1}{|m_2|} + \ldots; 0, m_i', m_i'\ldots \right) \in M \times \Delta \text{ where } m_i' = m_i - 1)
\]
and therefore $N$ is a continuous image of $M_0$.

It follows from Theorem XX that
\[
S_N(\mathcal{A}) \subset S_M(\mathcal{A})
\]
for any \(\mathcal{A} \in [d]\). We have only to prove that for any \(\mathcal{A} \in [r]\):
\[
35, 1)
\]
\[
S_M(\mathcal{A}) \subset S_M(\mathcal{A})
\]
(Remark. It is evident that $S_M(\mathcal{A}) \subset S_M(\mathcal{A})$ so that we shall have $S_M(\mathcal{A}) = S_M(\mathcal{A})$).

Let $P \in S_M(\mathcal{A})$ and $E_{m_1, m_2, \ldots, m_k} \in \mathcal{A}$.

Denote
\[
\sum_{m_1}^{2} \ldots \sum_{m_k}^{2} E_{m_1, m_2, \ldots, m_k} = H_{m_1, m_2, \ldots, m_k}
\]
This set being the finite sum of sets belonging to $S_M(\mathcal{A})$ belongs to $\mathcal{A}$ itself. Let now
\[
Q = \Omega_M((H_{P_1, P_2, \ldots, P_k})
\]
Evidently $Q \in S_M(\mathcal{A})$; we shall prove that $Q = P$.

a) $Q \supset P$. Let $x \in P$ then there exists such
\[
\left\{ \frac{1}{|m_1|} + \frac{1}{|m_2|} + \ldots \in M_0\right\}
\]
that $x \in \bigcap_{i} E_{m_1, m_2, \ldots, m_k}$
We have by the definition of $M_0$
\[
\left\{ \frac{1}{|m_1|} + \frac{1}{|m_2|} + \ldots \in M; \; m_i \leq 2 \text{ for any } k\right\}
\]
Hence
\[
x \in \bigcap_{i} \sum_{m_1}^{2} \ldots \sum_{m_k}^{2} E_{m_1, m_2, \ldots, m_k} = \bigcap_{i} H_{m_1, m_2, \ldots, m_k} = \bigcap_{i} Q
\]

\[\beta\) $Q \supset P$. Let $x \in Q$ then there exists such
\[
\left\{ \frac{1}{|m_1|} + \frac{1}{|m_2|} + \ldots \in M\right\}
\]
that $x \in H_{P_1, P_2, \ldots, P_k}$ for any \(i\).
Consequently for any \(i\) there exists such a torgue of range \(i\)
\[
(r_1', r_2', \ldots, r_i'); \; r_i' = 1, 2
\]
\(1\) We shall suppose the system \(\{E_{m_1, m_2, \ldots, m_k}\}\) regular (see def. 25 Art. 31).
that
\[(35, 2) \quad x \in E_{p_1}^{p_1}, p_2^{p_2}, \ldots, p_n^{p_n} \subseteq \Omega_{m_1}(E_{m_1}, m_2, \ldots, m_n) \quad (\nu' = \iota).\]

We can now apply the following lemma due to Denes König 1). If we have a set \( S \) of corseges \( (r_1, r_2, \ldots, r_n) \) possessing the following three properties:
1) there exists such sequence of natural numbers \( \nu_1, \nu_2, \ldots, \nu_n \), that for any corseg \( (r_1, r_2, \ldots, r_n) \in S \) we have \( r_{\nu_i} \leq \nu_{\nu_i} \),
2) if \( (r_1, r_{i_1}, \ldots, r_{i_j}) \in S \) then all its segments also belong to \( S \):
   \( (r_1, r_{i_1}, \ldots, r_{i_j}) \in S \) for any \( i' \leq i \),
3) the range of the corsegs belonging to \( S \) is not limited;
then there exists a sequence \( q_1, q_2, \ldots \), such that all its segments belong to \( S \) : \( (q_1, q_2, \ldots, q_i) \in S \) for any \( i \).

Denote now \( S \) the set of all the corsegs \( (q_1, q_2, \ldots, q_i) \) (see above).
This set \( S \) evidently satisfies all the conditions of the lemma (the sequence \( \nu_1, \nu_2, \ldots \) demanded by condition 1) being 2, 2, 2, \ldots). Therefore its conclusion must be also true, i.e. there exists such sequence \( q_1, q_2, \ldots \) that whatever be \( i' \) we may find such \( i \geq i' \) that
\[r_{q_i} = r_{q_i}'; \quad r_{q_i} = r_{q_i}'; \ldots ; \quad r_{q_i} = r_{q_i}'.\]

It follows then from (35, 2) that
\[x \in \Pi_E p_1^{p_1}, p_2^{p_2}, \ldots, p_n^{p_n} \subseteq \Omega_{m_1}(E_{m_1}, m_2, \ldots, m_n) = P \quad q. e. d.\]

The relation (35, 1) is now wholly demonstrated.

36. B. Necessity 1). We shall begin with the following

Lemma. If to every corseg \( (n_1, n_2, \ldots, n_i) \) we correlate an integer
\[\pi(n_1, n_2, \ldots, n_i) \geq 2\]
and if \( M_0 \) is the set of all the numbers
\[\frac{1}{n_1} + \frac{1}{n_2} + \ldots \]

1) We must not forget that the system \( E_{m_1}, m_2, \ldots, m_n \) is regular.

2) Fund. Math. VIII, p. 120. (Sur les correspondances multivocues des ensembles, th. 2). The lemma is stated there in a somewhat different form but the essence is the same.

3) Necessity follows also from Theorem XXIII below (p. 72). We give here this direct proof because it possesses some interest of its own.

such that
1) \( \frac{1}{m_1} + \frac{1}{m_2} + \ldots \in M \)
2) \( m_{g_i} \leq \pi(m_1, m_2, \ldots, m_{g_i-1}) \)

then \( M_0 \) is homeomorphic to \( M \times A \).

Proof. To every point
\[z = \frac{1}{m_1} + \frac{1}{m_2} + \ldots \in M_0\]
we shall correlate a point
\[\vartheta z = (x, y) \in M \times A\]
where
\[z = \frac{1}{m_1} + \frac{1}{m_2} + \ldots ; \quad y = 0, g_1, g_2, \ldots \]

where \( g_i = 0 \) except when
\[i = \sum_{j=1}^{\infty} \min(m_{g_j}, \pi(m_1, m_2, \ldots, m_{g_j-1}) - 1) \quad \text{and} \quad m_{g_i} < \pi(m_1, m_2, \ldots, m_{g_i-1}) \]
in which case \( g_i = 1 \).

We must prove that \( \vartheta \) is a homeomorphic transformation. Denote \( \vartheta' \) the inverse transformation i.e. if \( (x, y) = \vartheta z \) then \( z = \vartheta'(x, y) \).
We shall prove the following properties of \( \vartheta' \):
1) To every point \( z \in M_0 \) corresponds one and only one point \( \vartheta z \).
2) To every point \( (x, y) \in M \times A \) corresponds one and only one point \( z = \vartheta'(x, y) \).

In fact let
\[z = \frac{1}{m_1} + \frac{1}{m_2} + \ldots ; \quad y = 0, g_1, g_2, \ldots \]

We shall now define the numbers \( m_1, m_2, \ldots \) by induction.
Let all the numbers \( m_{g_k} (k < g_k) \) be defined and let \( g_k \) be the least index \( i \) greater than
\[\sum_{j<k} \min(m_{g_j}, \pi(m_1, m_2, \ldots, m_{g_j-1}) - 1) = t_k\]
such that \( g_k = 1 \). Then
\[m_{g_k} = \min(t_k - t_{g_k}, \pi(m_1, m_2, \ldots, m_{g_k-1})).\]
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where \( M' \) denotes (as in the preceding lemma) the set of all the points

\[
\frac{1}{m_1} + \frac{1}{m_2} + \ldots
\]

such that

\[
\frac{1}{m_1} + \frac{1}{m_2} + \ldots \in M
\]

and \( m_\infty \leq \pi(m_1, m_2, \ldots, m_\infty) \). We may besides suppose \( \pi(m_1, m_2, \ldots, m_\infty) \geq 2 \).

Thus \( N \in \hat{S}'(\mathcal{H}_0) \) from which follows as in Th. XX that \( N \) is a continuous image of \( M' \) or, which is the same (see lemma), of \( M \times \Delta \), q. e. d.

**Corollary.** In order that two operations \( \Omega_N \) and \( \Omega_M \) be (r)-equivalent with resp. to \([\mathcal{R}] \) (see def. 6 bis and 21) it is necessary and sufficient that \( N \times \Delta \) and \( M \times \Delta \) be of the same type (c).

In fact if \( N \times \Delta \) and \( M \times \Delta \) are of the same type (c) then \( N \times \Delta \) is a continuous image of \( M \times \Delta \) and consequently \( N \) (which is a continuous image of \( N \times \Delta \)) is a continuous image of \( M \times \Delta \). The same for \( M \) and \( N \times \Delta \). If on the other hand \( N \) is a continuous image of \( M \times \Delta \) then \( N \times \Delta \) is a continuous image of \( (M \times \Delta) \times \Delta \) (homeomorphic to \( M \times \Delta \)). The same for \( M \) and \( N \times \Delta \). Hence the corollary.

§ 3. The \( S_N(\mathcal{H}) \)-Classes in a Compact Metric Space.

37. Of all the \( S_N(\mathcal{H}) \)-classes the most interesting are \( S_N(\mathcal{H}) \). We shall begin their theory with the following

**Theorem XXII**. Every set closed in \( R \times J \) is the scheme for a certain system \( \{F_{n_1, n_2, \ldots, n_k}\} \) of sets, closed in \( R' \) (see def. 10 bis).

**Proof.** Let \( P \subset R^J \) be a set closed in \( R \times J \). Denote

\[
F_{n_1, n_2, \ldots, n_k} = \mathcal{P}_R(P \times \mathcal{H}_{n_1, n_2, \ldots, n_k})
\]

where \( F_{n_1, n_2, \ldots, n_k} \) are closed in \( R \).

It remains to prove that

\[
P = S(\{F_{n_1, n_2, \ldots, n_k}\}) = \prod_{k=1}^{\infty} \sum_{(n_1, n_2, \ldots, n_k)} F_{n_1, n_2, \ldots, n_k} \times \mathcal{H}_{n_1, n_2, \ldots, n_k}
\]

1) The converse is also true. See def. 10, remark (Fund. Math. t. XVIII, p. 237).
Denote \( \varphi(x, y) = (\varphi x, y) \) and \( Q^{(0)} = \varphi P^{(0)} \).
Then (1) \( Q^{(0)} \in \mathcal{F}^{(0)} \). We shall suppose that the pseudo character of \( R \) is \( \kappa_a \) i.e. every point of \( R \) is the common part of a countable infinity of open sets. The case when it is greater than \( \kappa_a \) will be treated in an Appendix.

Let then \((x, y) \in Q^{(0)} \) and
\[
V_1 \supset V_2 \supset \ldots
\]
be a sequence of open sets such that
\[
\prod V_k = (x)
\]
and let
\[
y = \frac{1}{|\nu^1_k|} + \frac{1}{|\nu^2_k|} + \ldots
\]
Then in any set
\[
V_k \times \delta \nu^1_k, \delta \nu^2_k, \delta \nu^3_k
\]
there exists a point
\[
(x, y) \in Q^{(0)}
\]
denote \( \exists \), a point such that \( \varphi (\exists, y) = (x, y) \) and let \( \exists \) be an accumulation point of the set of all the \( \exists \). Then, taking into account that \( (\exists, y) \in P^{(0)} \) and that \( P^{(0)} \) is closed, we have \( (\exists, y) \in P_{(0)} \). But it is easily proved that \( \varphi (\exists, y) \) must necessarily be \((x, y)\) whence \((x, y) \in Q_{(0)}\), q.e.d.

(2)

This is evident.

From (1) and (2) (and Cor. 1) follows that \( Q \in S^{N}(\mathcal{F}) \).

38. Theorem XXIII. If \( R \) is a compact metric space then \( S^{N}(\mathcal{F}) \) is identical with the class \( \mathcal{S} \) of continuous images of the set \( N \times A \).

Proof. a) \( \mathcal{S} \subset S^{N}(\mathcal{F}) \). Let \( P \in \mathcal{S} \) i.e. \( P = \varphi(N \times A) \).
Denote
\[
F_{n_1, n_2, \ldots, n_k} = \varphi(N \times \delta _{n_1, n_2, \ldots, n_k} \times A) \in \mathcal{F}
\]
and
\[
Q = \Omega(N (F_{n_1, n_2, \ldots, n_k})) \in S^{N}(\mathcal{F})
\]
We shall prove that \( P = Q \).

1) P. Alexandroff, Memoire sur les espaces topologiques compacts.
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Then to every point \((x, y) \in N \times \Delta\) corresponds not more than one such point \(z \in Q\) that
\[(x, y) \in \varphi(z).
\]

We shall denote such point \(z\) (if it exists) \(\varphi(x, y)\).

(\(\Delta\) \(\varphi\) is defined in a closed subset \(M\) of \(N \times \Delta\) and is continuous in \(M\). In fact let
\[(x, y) = \lim (x_i, y_i); \quad (x_i, y_i) \in M.
\]

Denote \(z_i = \varphi(x_i, y_i)\).
Then:

1) \(z_i = \varphi^* y_i\); hence

\[(38,1) \quad z = \lim z_i = \lim \varphi^* y_i = \varphi^* \lim y_i = \varphi^* y\]

2) \(z \in \bigcap_{a} F_{n_1, n_2, \ldots, n_k}\) where \(x_i = \frac{1}{|n_1|} + \frac{1}{|n_2|} + \ldots\)

But whatever be \(k\) we have for \(i\) great enough \((i > i_k)\) \(n_i' = n_i;\)
\(n_i' = n_2, \ldots, n_k' = n_k;\) hence
\[z_i \in F_{n_1, n_2, \ldots, n_k} \quad (i > i_k)\] or
\[z = \lim z_i \in F_{n_1, n_2, \ldots, n_k} \quad \text{for any } k;
\]

therefore

\[(38,2) \quad z \in \bigcap_{a} F_{n_1, n_2, \ldots, n_k}; \quad x = \frac{1}{|n_1|} + \frac{1}{|n_2|} + \ldots\]

(38,1) and (38,2) show that \(z = \varphi(x, y)\) which proves the assumption (A).

Now \(\varphi\) being defined and continuous in a closed subset \(M\) of \(N \times \Delta\) there exists 1) a function \(\varphi_0\) defined and continuous in \(N \times \Delta\) and such that
\[\varphi_0(N \times \Delta) = \varphi M = Q\] q. e. d.

39. Remark. It is easy to see from the proof given above that if
\[x = \frac{1}{|n_1|} + \frac{1}{|n_2|} + \ldots \in N\]

1) This follows easily from a lemma of Sierpiński cited on p. 65.
and if

\[ \prod_a F_{n_1, n_2, \ldots, n_k} = 0 \]

then

\[ \varphi(M \cdot ((x) \times \Delta)) = \prod_a F_{n_1, n_2, \ldots, n_k} n_k. \]

We may deduce whence easily the following:

**Theorem of Sierpiński**: Denote (in a compact space) \( S_N \) the class of all the sets which can be represented in the form:

\[ \Omega_N((F_{n_1, n_2, \ldots, n_k}) \}

where \( F_{n_1, n_2, \ldots, n_k} \in \mathcal{F} \) possess the following property:

For any

\[ \frac{1}{|n_1|} + \frac{1}{|n_2|} + \ldots \in \mathbb{N} \]

\( \prod_a F_{n_1, n_2, \ldots, n_k} \) consists of not more than one point.

The class \( S_N \) is the class of all the continuous images of \( N \).

**Proof.** If

\[ P \in S_N \subset S_N(\mathcal{F}) \]

then by Theorem XXIII and remark, \( P \) is a continuous image of \( N \times \Delta \) \((P = \varphi(N \times \Delta)) \) and we have

\[ \varphi(M \cdot ((x) \times \Delta)) = \prod_a F_{n_1, n_2, \ldots, n_k} n_k \]

which consists of not more than one point.

Denote \( M_n \) the set of all the points \( x = \frac{1}{|n_1|} + \frac{1}{|n_2|} + \ldots \) such that

\[ \prod_a F_{n_1, n_2, \ldots, n_k} = 0 \]; evidently \( M_n \) is a relatively closed subset of \( N \) homeomorphic to \( M \) and \( \varphi M = P \). It follows that there exists such continuous (in \( N \)) function \( \varphi_0 \) that \( \varphi_0 N = P \), i.e.,

\[ P \] is a continuous image of \( N \), q.e.d.

If on the other hand \( P = \varphi N \) (\( \varphi \) being a continuous transformation) then denote \( \varphi_1(x, y) = \varphi x \) for any \( y \in \Delta \); then we have \( P = \varphi_1(N \times \Delta) \) and \( \varphi_1(\Delta \times \Delta) \) consists of a single point. But denoting

\[ F_{n_1, n_2, \ldots, n_k} = \varphi(N \cdot \delta_{n_1, n_2, \ldots, n_k}) = \varphi_1(N \cdot \delta_{n_1, n_2, \ldots, n_k} \times \Delta) \]

\[ 1) \text{Sierpiński VI.} \]
and
\[ \sum \phi_n \left( \left| \sum \mathcal{E}_{n}^{P_{1}, P_{2}, \ldots, P_{i}} \right| \right) \]
where \( \mathcal{E}_{n}^{P_{1}, P_{2}, \ldots, P_{i}} \in \mathcal{H} \), \( \mathcal{H} \) being a ring, are the same \( \mathcal{S}_{n}(\mathcal{H}) \) class.

Proof. We may evidently suppose that \( N \) is in its reduced completed form (def. 9, art. 7). Let now \( \mathcal{S} \) and \( \mathcal{S}^{*} \) be the classes defined by the operations resp. \((40,1)\) and \((40,1^{*})\) and let \( \mathcal{S}_{n} \) and \( \mathcal{S}_{n}^{*} \) be the classes defined by the operations:
\[ \sum_{(g_{1}, g_{2}, \ldots, g_{k})} Q_{N} \left( \prod_{j} H_{g_{1}^{j}, g_{2}^{j}, \ldots, g_{k}^{j}} \right) \]
and
\[ \sum_{(g_{1}, g_{2}, \ldots, g_{k})} Q_{N} \left( \prod_{j} H_{g_{1}^{j}, g_{2}^{j}, \ldots, g_{k}^{j}} \right) \]
where
\[ H_{g_{1}^{j}, g_{2}^{j}, \ldots, g_{k}^{j}} \in \mathcal{H}. \]

Our theorem will be proved if we prove that:
A. \( \mathcal{S} = \mathcal{S}_{n} \); \( \mathcal{S}^{*} = \mathcal{S}_{n}^{*} \)
B. \( \mathcal{S}_{n} = \mathcal{S}_{n}^{*} \)
C. \( \mathcal{S}_{n} \) is an \( \mathcal{S}_{n}(\mathcal{H}) \)-class.

A) The inclusions \( \mathcal{S} \subseteq \mathcal{S}_{n} \) and \( \mathcal{S}^{*} \subseteq \mathcal{S}_{n}^{*} \) are evident.
We have only to set
\[ H_{g_{1}^{j}, g_{2}^{j}, \ldots, g_{k}^{j}} = \mathcal{E}_{n}^{g_{1}^{j}, g_{2}^{j}, \ldots, g_{k}^{j}} \]
b) \( \mathcal{S}_{n} \subseteq \mathcal{S} \). Let \( P \in \mathcal{S}_{n} \), i.e.
\[ P = \sum_{(g_{1}, g_{2}, \ldots, g_{k})} Q_{N} \left( \prod_{j} H_{g_{1}^{j}, g_{2}^{j}, \ldots, g_{k}^{j}} \right) \]
where \( \mathcal{H} \) being a ring we may evidently suppose
\[ H_{g_{1}^{j}, g_{2}^{j}, \ldots, g_{k}^{j}} \subset H_{g_{1}^{j}, g_{2}^{j}, \ldots, g_{k}^{j}} \]
for any \( k, j, j', j'' \) satisfying the relations \( k' \leq k, j' \leq j \).

We shall define a system \( \{E_{n}^{P_{1}, P_{2}, \ldots, P_{i}}\} \) of sets as follows:

1) if \( i < 2n \) then \( E_{n}^{P_{1}, P_{2}, \ldots, P_{i}} = R \)
2) if \( i \geq 2n \) and \( P_{x_{i}} > 1 \) then \( E_{n}^{P_{1}, P_{2}, \ldots, P_{i}} = 0 \)
3) let \( i \geq 2n \) and \( P_{x_{i}} = 1 \). Denote
\[ x_{1}, x_{2}, \ldots, x_{k} \quad (x_{1} < x_{2} < \ldots < x_{k} = n) \]
all the positive integers \( \leq n \) such that \( P_{x_{2}} = 1 \) and let \( g_{1}, g_{2}, \ldots, g_{j} \) be the sequence 1)
\[ P_{1}, P_{2} - 1, P_{3}, P_{4} - 1, \ldots, P_{k} - 1, P_{k+1} - 1, P_{k+2} - 1, P_{k+3} - 1, \ldots, P_{2n-k-1}, P_{2n-k} - 1, P_{2n-k-2} - 1, P_{2n-k-3} - 1. \]

Now denote
\[ E_{n}^{P_{1}, P_{2}, \ldots, P_{i}} = H_{g_{1}, g_{2}, \ldots, g_{j}} \]
and
\[ Q = \sum_{(P_{1}, P_{2}, \ldots, P_{i})} Q_{N} \left( \prod_{j} E_{n}^{P_{1}, P_{2}, \ldots, P_{i}} \right) \in \mathcal{S}. \]
We shall prove that \( P = Q \).
Denote the number of integers \( 2x_{k} < x \) by \( k(x) \)
a) \( P \subseteq Q \). Let \( x \in P \). Then there exists
\[ \frac{1}{x_{1}} + \frac{1}{x_{2}} + \ldots + \frac{1}{x_{k}} \in N \]
and such \( g_{1}, g_{2}, \ldots, g_{j} \) that
\[ x \in \prod_{j, k} H_{g_{1}^{j}, g_{2}^{j}, \ldots, g_{k}^{j}} \]
Define \( P_{x} \) as follows:
1) \( P_{x_{2}} = 1 \) (for any \( x \))
2) \( P_{x} = 2_{2n-(x_{k}+1)} - 1 \) in all other cases
3) \( P_{x_{k}} = 2_{2n-4(x_{k}+1)} + 1 \) for any \( x \).
Then we have
\[ E_{n}^{P_{1}, P_{2}, \ldots, P_{i}} = H_{g_{1}, g_{2}, \ldots, g_{j}} \quad \text{if} \quad i \geq 2n \]
\[ E_{n}^{P_{1}, P_{2}, \ldots, P_{i}} = R \quad \text{if} \quad i < 2n. \]

1) In strict conformity with our notations we should use the word "corresponding" instead of "sequence".
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By L. Kantorovitch and E. Livenson:

In both cases

\[ x \in \mathcal{P}_{\mathcal{Q}} \]

So that we have

\[ x \in \prod_{i, k} \mathcal{E}_{\mathcal{Q} \mathcal{P}_{\mathcal{Q}}} \subset \mathcal{Q} \quad \text{q. e. d.} \]

2) \( \mathcal{Q} \subset \mathcal{P} \). Let \( x \in \mathcal{Q} \). Then there exists such

\[ \frac{1}{\nu_1} + \frac{1}{\nu_2} + \ldots \in \mathbb{N} \]

and such sequence \( \mathcal{P}_1, \mathcal{P}_2, \ldots \), that

\[ x \in \prod_{i, k} \mathcal{E}_{\mathcal{Q} \mathcal{P}_{\mathcal{Q}}} \]

It follows that for any \( k \), \( \mathcal{P}_{2k} = 1 \) for otherwise \( \mathcal{E}_{\mathcal{Q} \mathcal{P}_{\mathcal{Q}}} \) would be vacuous for \( i \geq 2 \), \( n_k \). Denote \( \nu_1, \nu_2, \ldots \) the sequence of all the numbers such that \( \nu_{2k} = 1 \) and let \( \gamma_1, \gamma_2, \ldots \) be the sequence

\[ \mathcal{P}_1, \mathcal{P}_2 = 1, \mathcal{P}_{2k-1}, \mathcal{P}_{2k+1}, \mathcal{P}_{2k+2}, \mathcal{P}_{2k+3}, \ldots \]

then

(1) \( \frac{1}{\nu_1} + \frac{1}{\nu_2} + \ldots \in \mathbb{N} \)

because the sequence

\[ \nu_1, \nu_2, \ldots \quad [\nu_1 < \nu_2 < \ldots] \]

contains all the numbers \( n_k \) and \( \mathbb{N} \) is in its reduced completed form.

2) \( x \in H_{\nu_1, \ldots, \nu_k} \mathcal{Q}_i \) for any \( k \) and \( j \).

In fact there exists an \( \nu_i = \nu_j > 2 \nu_k \). Let

\[ i > 2 \nu_k \]

then

\[ x \in \mathcal{P}_{\mathcal{Q} \mathcal{P}_{\mathcal{Q}}} \]

So that

\[ x \in \prod_{k, j} H_{\gamma_1, \ldots, \gamma_j} \mathcal{Q}_i \subset \mathcal{P} \quad \text{q. e. d.} \]

b) \( \mathcal{Q} \subset \mathcal{P} \) can be proved in the same way. Only:

1) Instead of the inclusion

\[ H_{\gamma_1, \ldots, \gamma_j} \mathcal{Q}_i \subset H_{\gamma_1, \ldots, \gamma_j} \mathcal{Q}_i \mathcal{P}_{\mathcal{Q}} \]

we have the following

\[ H_{\gamma_1, \ldots, \gamma_j} \mathcal{Q}_i \subset H_{\gamma_1, \ldots, \gamma_j} \mathcal{P}_{\mathcal{Q}} \]

2) \( \mathcal{P} = 0 \) in case 1 as well as in case 2).

3) \( \mathcal{P} = 0 \). Let \( P \in \mathcal{P} \), then

\[ P = \sum \sum \prod \mathcal{H}_{\gamma_1, \ldots, \gamma_j} \mathcal{Q}_i \mathcal{P}_{\mathcal{Q}} = \sum \sum \prod \mathcal{H}_{\gamma_1, \ldots, \gamma_j} \mathcal{Q}_i \mathcal{P}_{\mathcal{Q}} \]

where

\[ \mathcal{H}_{\gamma_1, \ldots, \gamma_j} \mathcal{Q}_i = 0 \quad \text{if} \quad i < k \]

and

\[ \mathcal{H}_{\gamma_1, \ldots, \gamma_j} \mathcal{Q}_i = \mathcal{H}_{\gamma_1, \ldots, \gamma_j} \mathcal{Q}_i \mathcal{P}_{\mathcal{Q}} \quad \text{if} \quad i > k \]

3) \( \mathcal{Q} \subset \mathcal{P} \). Let \( P \in \mathcal{P} \), then

\[ P = \sum \sum \prod \mathcal{H}_{\gamma_1, \ldots, \gamma_j} \mathcal{Q}_i \mathcal{P}_{\mathcal{Q}} = \sum \sum \prod \mathcal{H}_{\gamma_1, \ldots, \gamma_j} \mathcal{Q}_i \mathcal{P}_{\mathcal{Q}} \]

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then, denoting $A$ the set of all the indices $\{\xi\}$ such that $x \in E_i$ we have $A \in X$ because otherwise there would exist a system $\{E_i\}$ of arguments and a point $x'$ such that

$$x' \in \prod_{\xi \in A} E_i$$

$$x' \notin \phi(\{E_i\})$$

which contradicts the definition of a positive analytical operation (see def. 2, art. 1).

Thus we have

$$(41,1) \quad \phi(\{E_i\}) = \sum_{\alpha \in X} \prod_{\xi \in \Lambda} E_i$$

and the operation $\phi$ is wholly defined by the set $X$.

42. Let now $\phi$ be a positive analytical operation effected upon a system $\{E_i\}$ of arguments depending of a certain set $\Sigma$ of indices and let $X$ be the corresponding (see the foregoing art.) set of "chains" (i.e., of subsets of $\Sigma$).

We shall define a new operation

$$R \phi(\{E_{1, 2, \ldots, \xi, \ldots}\})$$

effected upon a system $\{E_{\xi, 2, \ldots, \xi, \ldots}\}$ of sets depending of the set $\Sigma$ of corteges (see notations) group D) of indices $\xi \in \Sigma$.

Instead of defining directly the operation $R \phi$ we shall define the corresponding set of chains as follows:

**Definition 24.** If $\phi$ is determined by the set $X$ of chains of indices $\xi \in \Sigma$ then $R \phi$ is the positive analytical operation which is defined by the set $X_0$ of all the chains $A_0$ of corteges of $\xi$, which satisfy the following conditions:

1) if a cortege $(\xi_1, \xi_2, \ldots, \xi_n)$ belongs to $A_0$ then all its segments also belong to $A_0$ (i.e. $(\xi_n, \xi_{n-1}, \ldots, \xi_1) \in A_0$ for any $n' \leq n$);

2) if a cortege $(\xi_1, \xi_2, \ldots, \xi_n) \in A_0$ then the set $A_0$ is $\xi_{n-1}, \xi_n$ of all the indices $\xi$ such that $(\xi_1, \xi_2, \ldots, \xi_{n'} \sqrt{\xi}) \in A_0$, belongs to $X$.
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2) If \( \Phi_N((E_n)) = \sum \Pi E_{n_1,n_2,\ldots,n_k} \) then \( N \) (supposed complete) coincides with \( J \). The set \( \Theta \) in this case contains all the systems of the type

\[ \{(i_1, i_2, \ldots, i_k)\} \]

\((A\text{-chains}^4)\). On the other hand evidently every system contained in \( \Theta \) contains an \( A \)-chain.

Hence follows

\[ R \Phi_N((E_n)) = \sum \Pi E_{n_1,n_2,\ldots,n_k} \]

i.e. the operation \( R \Sigma \) is \( A \)-operation of Mr. Souslin.

44. Definition 25. If \( \mathcal{A} \) is a \( \mathcal{H} \)-operation on classes of sets corresponding to a positive analytical operation \( \Phi \) then \( R \mathcal{A} \) denotes the operation corresponding to \( R \Phi \).

45. We shall prove now some theorems relating to \( R \Phi \) operations. We begin with the following remark:

In def. 28 bis we can substitute conditions (42,3) and (42,3)' by the following:

\[ \frac{1}{|n_1|} + \frac{1}{|n_2|} + \ldots \in \tilde{N}. \]

(45,1)

\[ \frac{1}{|n_1|} + \frac{1}{|n_2|} + \ldots \in N. \]

(45,1')

Theorem XXV. If \( \Psi \) is a positive analytical operation, \( \mathcal{A} \) the corresponding operation on classes of sets, then we have for any class \( \mathcal{A} \)

\[ \mathcal{A}(\mathcal{A}) \subset R \mathcal{A}(\mathcal{A}). \]

Proof. Let \( P \in \mathcal{A}(\mathcal{A}) \) i.e.

\[ P = \Psi(E_1); \quad E_1 \in \mathcal{A}. \]

Denote

\[ E_{\tilde{E}_1, \tilde{E}_2, \ldots, \tilde{E}_k} = E_{\tilde{E}_1} \ (\varepsilon \mathcal{A}). \]

We have evidently

\[ P = R \Psi((E_{\tilde{E}_1, \tilde{E}_2, \ldots, \tilde{E}_k}) \in R \mathcal{A}(\mathcal{A}). \]
Theorem XXVI. If \( \mathcal{V}^{(1)} \) and \( \mathcal{V}^{(0)} \) are positive analytical operations on a countable \(^{1)}\) infinity of sets, \( \mathcal{A}^{(1)} \) and \( \mathcal{A}^{(0)} \) are the corresponding operations upon classes of sets and if
\[
\mathcal{A}^{(1)}(\mathcal{E}) \subset \mathcal{A}^{(0)}(\mathcal{E}) \quad \text{(see def. 5, art. 5)}
\]
then for any d-class \( \mathcal{E} \)
\[
R \mathcal{A}_1(\mathcal{E}) \subset R \mathcal{A}_0(\mathcal{E}).
\]

Proof. Let \( \mathcal{V}^{(1)} \) depend on a system of arguments \( \{E_i\} \) depending on a countable set \( \mathcal{E} = \{e\} \) of indices and in the same manner let \( \mathcal{V}^{(0)} \) depend on \( \{E_j\} \) depending on a set \( \Theta = \{\eta\} \) of indices. Enumerate all the indices \( \{e\} \):
\[
e_i, \eta^n, \ldots
\]
and also the indices \( \{\eta\} \):
\[
\eta^n, \eta^{(n)}, \ldots
\]
Then, as we already know (Theorem I), there exist two \( \delta \)-functions
\[
\Phi_1(\mathcal{E}^1(\eta)) = \mathcal{V}^{(1)}(\mathcal{E}^1)
\]
and
\[
\Phi_0(\mathcal{E}^0(\eta)) = \mathcal{V}^{(0)}(\mathcal{E}^0).
\]
Evidently
\[
R \mathcal{A}_1(\{E_i(\eta)\}, \{E_j(\eta)\}) = R \mathcal{A}_0(\{E_i\}, \{E_j\})
\]
and
\[
R \mathcal{A}_0(\{E_i(\eta^n), \eta^{(n)}\}, \{E_j\}) = R \mathcal{A}_0(\{E_i\}, \{\eta^n, \eta^{(n)}\}).
\]
Consequently for any \( \mathcal{E} \)
\[
R \mathcal{A}_1(\mathcal{E}) = R \mathcal{A}_1(\mathcal{E}) \quad \text{and} \quad R \mathcal{A}_0(\mathcal{E}) = R \mathcal{A}_0(\mathcal{E}).
\]

We shall suppose \( N_0 \) complete \( (N_0 = \mathcal{N}_0) \). Therefore, by Theorem IV (art. 8) there exist a function \( k(n) \) \( (n \text{ and } k(n) \text{ being positive integers}) \) and a set \( N_1 \) equivalent to \( N_0 \), such that
\[
\frac{1}{m_1} + \frac{1}{m_2} + \ldots \in N_1.
\]

\(^{1)}\) The analytical operations on an uncountable infinity of sets shall be considered in an appendix.
Analytical Operations and Projective Sets

Remark. If the condition that \( \mathcal{L} \) contains the sum of all its elements is not fulfilled then the theorem sometimes ceases to be true. Let, e. g. \( \mathcal{L} \) be the class of all the sets of diameter not greater than 1 (in Euclidean space) and \( N \) the set consisting of two numbers only:

\[
1|1 + 1|1 + \ldots \quad \text{and} \quad 1|2 + 1|2 + \ldots
\]

Then, as we may easily see, \( \mathcal{R}_N(\mathcal{L}) \) is the class of all the sets which are the sums of two sets each of the diameter not greater than 1. While \( \mathcal{R}_N(\mathcal{R}_N(\mathcal{L})) \) is the class of all the sums of four sets each of the diameter not greater than 1. Thus we have

\[
\mathcal{R}_N(\mathcal{R}_N(\mathcal{L})) \neq \mathcal{R}_N(\mathcal{L}).
\]

However our theorem remains true even without the condition that \( \mathcal{L} \) contains the sum of all its elements, provided that the operation \( \Phi_N \) be normal.

Corollary. \( \mathcal{R}_N(\mathcal{L}) \supset \mathcal{R}_N(\mathcal{R}_N(\mathcal{L})). \)

The above remark may be applied to this corollary.

46. Theorem XXVIII. The operation \( R \ R \Phi_N \) is equivalent with the operation \( \Phi_N \).

Proof. We must prove that for any class \( \mathcal{L} \) of sets

\[
(46,1) \quad R \ R \Phi_N(\mathcal{L}) = R \mathcal{R}_N(\mathcal{L}).
\]

As we already know (Theorem XXV)

\[
R \mathcal{R}_N(\mathcal{L}) \supset R \ R \mathcal{R}_N(\mathcal{L}).
\]

If we now prove that

\[
(46,2) \quad R \ R \Phi_N(\mathcal{L}) \supset R \mathcal{R}_N(\mathcal{L})
\]

then Theorem XXVIII will be completely demonstrated.

Let

\[
P = R \ R \Phi_N(\mathcal{L})
\]

\( i. e. \)

\[
P = R \ R \Phi_N((E_n)) \quad E_n \in \mathcal{L}
\]

\( \{E_n\} \) being a system of sets depending on the set of all the corteges of the second order (doublet corteges) i. e. corteges having for their elements the corteges of natural numbers. (In fact \( \Phi_N \) is
the function of a sequence of sets, \( R \mathcal{N} \) is the function of a system of sets depending of the corteges of the first order (see notations group D, 2) so \( R R \mathcal{N} \) must be the function of a system of sets depending of the corteges of corteges i.e. of the double corteges).

We shall now define a system of sets
\[
\{K_{v_1}, v_{2}, ..., v_{i} \}; \quad i = 1, 2, ..., v_{i} = 1, 2, ...
\]
as follows: let
\[
(46,3) \quad i = 2^{r_1} + r_2 + ... + r_{i-1} - 1 + 2^{r_1} + r_2 + ... + r_{i-1} - 1 + ... + 2^{r_{i-1}} - 1 = \mu(v_1, r_2, ..., r_{i-1})
\]
and denote by \( \sigma(v_1, v_{2}, ..., v_{i}) \) the following double cortege
\[
(46,4) \quad \sigma(v_1, v_{2}, ..., v_{i}) = \begin{pmatrix}
\nu(1), \nu(2), & \cdots & \nu(i) \\
\nu(1), \nu(2), & \cdots & \nu(i)
\end{pmatrix}
\]
Then we
\[
(46,5) \quad K_{v_1}, v_{2}, ..., v_{i} = E_{\sigma(v_1, v_{2}, ..., v_{i})}
\]
We shall prove that
\[
(46,6) \quad Q = R \mathcal{N}(K_{v_1}, v_{2}, ..., v_{i}) = P.
\]
We shall suppose that \( N = N'. \)

a) \( Q \subseteq P. \) Let \( x \in Q. \) Then there exists an \( R \mathcal{N} \)-chain
\[
T = \{(v_{s_1}, v_{s_2}, ..., v_{s_{i-1}}, v_{s_i}) \}
\]
of corteges, such that
\[
x = \prod K_{v_{s_1}, v_{s_2}, ..., v_{s_{i-1}}, v_{s_i}} \quad (s_1, s_2, ..., s_{i})
\]
i.e.
\[
x = \prod E_{\sigma(v_{s_1}, v_{s_2}, ..., v_{s_{i-1}}, v_{s_i})} \quad (s_1, s_2, ..., s_{i})
\]
If we now prove that the set of all the
\[
\sigma(v_{s_1}, v_{s_2}, ..., v_{s_{i-1}}, v_{s_i})
\]
constitutes an \( R R \mathcal{N} \)-chain then the inclusion \( Q \subseteq P \) will be proved.

Analytical Operations and Projective Sets

The set
\[
\mathcal{T} = \{ \sigma(v_{s_1}, v_{s_2}, ..., v_{s_{i-1}}, v_{s_i}) \}
\]
satisfies all the conditions which define an \( R R \mathcal{N} \)-chain viz.

1) if \( \sigma' \) is a segment of \( \sigma(v_{s_1}, v_{s_2}, ..., v_{s_{i-1}}, v_{s_i}) \in \mathcal{T} \) then \( \sigma' \in \mathcal{T}. \) In fact if \( \sigma' \) is a segment \( \sigma' = \sigma(v_{s_1}, v_{s_2}, ..., v_{s_{i-1}}, v_{s_i}) \) then \( \sigma' = \sigma(v_{s_1}, v_{s_2}, ..., v_{s_{i-1}}, v_{s_i}) \) (\( i' < i \)) (though the converse is not true: \( \sigma(v_{s_1}, v_{s_2}, ..., v_{s_{i-1}}, v_{s_i}) \) is not generally speaking, a segment of \( \sigma(v_{s_1}, v_{s_2}, ..., v_{s_i}) \)). But if \( \sigma(v_{s_1}, v_{s_2}, ..., v_{s_i}) \in \mathcal{T} \) then \( \sigma(v_{s_1}, v_{s_2}, ..., v_{s_i}) \in T \) whence \( \sigma(v_{s_1}, v_{s_2}, ..., v_{s_i}) \in \mathcal{T} \) and consequently
\[
\sigma' = \sigma(v_{s_1}, v_{s_2}, ..., v_{s_i}) \in \mathcal{T}.
\]

2) if \( \sigma_{0} \in \mathcal{T} \) and if \( \sigma_{0}, \sigma_{1}, ..., \sigma_{n} \) is the sequence of all the double corteges which: a) belong to \( \mathcal{T}; \) b) have the same rank \( k \) (i.e. number of elements) \( \gamma \) as \( \sigma_{0} \); \( \gamma \) have all their elements except the last identical with the corresponding elements of \( \sigma_{0} \); then, denoting \( \mathcal{W} \), the last element of \( \sigma_{0} \) (i.e. \( i = 0, 1, 2, ... \)), the set of all \( \mathcal{W} \)'s is an \( R \mathcal{N} \)-chain.

In fact let
\[
\sigma_{0} = (o', \mathcal{W})
\]
where \( o' \) is a double cortege of the rang \( k - 1 \) if \( k > 1 \) and is vacuous if \( k = 1 \). Then for any \( t \)
\[
\sigma_{0} = (o', \mathcal{W}).
\]
To prove that \( \{ \mathcal{W} \} \) is an \( R \mathcal{N} \)-chain we must demonstrate
a) that if \( \mathcal{W}_{0} \in \{ \mathcal{W} \} \) then any segment of \( \mathcal{W}_{0} \) also belongs to \( \{ \mathcal{W} \} \);

b) that if \( \mathcal{W} \) belongs to \( \{ \mathcal{W} \} \) (or is vacuous) and \( (n) \) is the sequence of all such numbers that \( (\mathcal{W}, n) \in \mathcal{W} \) then
\[
\left\lfloor \frac{1}{n_1} \right\rfloor + \left\lfloor \frac{1}{n_2} \right\rfloor + \left\lfloor \frac{1}{n_3} \right\rfloor + ... \in \mathcal{N}.
\]

a) If \( \mathcal{W} = (\mu_1, \mu_2, ..., \mu_{r}) \) and if \( \mathcal{W}' = (\mu_1, \mu_2, ..., \mu_{r}) \) is a segment of \( \mathcal{W} \), then we have \( (o', \mathcal{W}) \in \mathcal{T} \) i.e.
\[
(\sigma'; \mathcal{W}) = \sigma(v_{s_1}, v_{s_2}, ..., v_{s_i}), \quad (v_{s_1}, v_{s_2}, ..., v_{s_i}) \in \mathcal{T}.
\]

1) A segment (of rang \( k \)) of a double cortege \( \sigma \) is another double cortege consisting of the first \( k \) corteges of \( \sigma \).

1) We must not forget that the elements of a double cortege are corteges (of the first order), not numbers.
but then evidently
\[ (\sigma'; \mathcal{W}') = \sigma(x^{(i)}, y^{(i)}; \ldots, y^{(i)}); \quad (i' \leq i) \]
or
\[ (\sigma'; \mathcal{W}') \in \mathcal{T}, \quad \text{because} \quad (x^{(i)}, y^{(i)}; \ldots, y^{(i)}) \in \mathcal{T}. \]

Therefore \( \mathcal{W}' = \mathcal{W}_F \), and we have proved that every segment of a cortedge belonging to \( \{\mathcal{W}_i\} \) also belongs to \( \{\mathcal{W}_i\} \).

b) Let \( \mathcal{W} = \mathcal{W}_F \) and consider the sequence \( \mathcal{W}_{i_1}, \mathcal{W}_{i_2}, \ldots \) of all the cortedges \( \mathcal{W}_i \) which do not differ from \( \mathcal{W}_F \) except in their last element; and let \( n_j \) be the last element of \( \mathcal{W}_{i_j} \) (\( j = 0, 1, 2, \ldots \)). Then if \( n_j = (\mathcal{W}_i, n_j) \) and consequently \( \mathcal{W}_{i_j} = (\mathcal{W}_i, n_j) \).

We have \( (\sigma'; \mathcal{W}_F) \in \mathcal{T} \) or
\[ (\sigma'; \mathcal{W}_F) = \sigma(x^{(i)}, y^{(i)}; \ldots, y^{(i)}), \quad (x^{(j)}, y^{(j)}; \ldots, y^{(j)}) \in \mathcal{T} \]
whence
\[ (\sigma'; \mathcal{W}_F) = \sigma(x^{(i)}, y^{(i)}; \ldots, y^{(i)}) = \sigma(n_j, x^{(i)}, y^{(i)}; \ldots, y^{(i)}) \quad (i' \leq i) \]
if we now denote \( x_{j+1} = n_{j+1} = \ldots = n_{j+1} = 1 \) then (by the def. of \( \sigma(x^{(i)}, y^{(i)}; \ldots, y^{(i)}) \), (46, 4) and (46, 3))
\[ (\sigma'; \mathcal{W}', \mathcal{V}_{i_1}, \mathcal{V}_{i_2}, \ldots, \mathcal{V}_{i_{n-1}, s}) = \sigma(x, y_{i_1}, x_{i_2}, \ldots, x_{i_{n-1}, s}) \in \mathcal{T} \]
for any \( s \). Hence and from the definition of the numbers \( n_j \) follows that all the numbers \( x_{i_1}, x_{i_2}, \ldots, x_{i_{n-1}, s} \) are found among the numbers \( n_j \) i.e.
\[ x_{i_1}, x_{i_2}, \ldots, x_{i_{n-1}, s} = n_{i_0} \]
whence by the definition of the \( \mathcal{R} \mathcal{D}_N \)-chain \( \{x_{i_1}, x_{i_2}, \ldots, x_{i_{n-1}, s}\} \)
\[ \frac{1}{|n_j|} + \frac{1}{|n_k|} + \ldots \in N \]
or, \( N \) being complete,
\[ \frac{1}{|n_j|} + \frac{1}{|n_k|} + \ldots \in N \]
\[ b) \] is proved and with it is proved that \( \{\mathcal{W}_i\} \) is an \( \mathcal{R} \mathcal{D}_N \)-chain i.e. we have proved the property 2). But the properties 1) and 2) constitute the definition of an \( \mathcal{R} \mathcal{R} \mathcal{D}_N \)-chain, and thus inclusion \( a) \) is proved.

---

\[ 47. \beta) P \subseteq Q. \text{ Let } x \in P. \text{ Then there exists such } \mathcal{R} \mathcal{R} \mathcal{D}_N \text{-chain } \mathcal{T} \text{ that } \]
\[ x \in \bigcap_{\sigma \in \mathcal{T}} E_{\sigma}. \]

Denote \( T \) the set of all the cortedges \( (x^{(i)}, y^{(i)}; \ldots, y^{(i)}) \) such that
\[ (\sigma^{(i)}, x^{(i)}; \ldots, x^{(i)}) \in \mathcal{T} \]
for any \( i' \leq i \).

Evidently
\[ x \in \bigcap_{\sigma \in \mathcal{T}} K_{x^{(i)}, y^{(i)}; \ldots, y^{(i)}} \]
\[ (x^{(i)}, y^{(i)}; \ldots, y^{(i)}) \in \mathcal{T} \]
We shall prove that \( T \) is an \( \mathcal{R} \mathcal{D}_N \)-chain.

In fact: 1) evidently if \( \mathcal{W} \in T \) then any segment of \( \mathcal{W} \) also belongs to \( T \).

2) Let
\[ x, y, \ldots, y^{(i)} \in \mathcal{T}; \quad i + 1 = u(r_1, r_2, \ldots, r_s) \] (See (46, 3)).

Denote
\[ i_0 = x^{(i)}, y^{(i)}; \ldots, y^{(i)} \in \mathcal{T}; \]
\[ i + 1 = u(r_1, r_2, \ldots, r_s) \] (See (46, 3)).

Denote
\[ i_0 = x^{(i)}, y^{(i)}; \ldots, y^{(i)} \in \mathcal{T}; \]
\[ i + 1 = u(r_1, r_2, \ldots, r_s) \]
if \( r_1 = 1 \) and
\[ i_0 = u(r_1, r_2, \ldots, r_s) \]
if \( r_1 = 1 \).

Evidently \( i_0 = i \). Consequently
\[ \mathcal{R}, \mathcal{D}_N \)-chain there exists the number
\[ i = \frac{1}{|x^{(i)}|} + \frac{1}{|x^{(i)}|} + \ldots \in N \]
49. Theorem XXX. \( R \mathcal{O}_N \)-operation is equivalent with a \( d \)-operation \( \mathcal{O}_N \), the base \( N' \) of which can be obtained from \( N \) and \( J \) by the operations of multiplicity, homeomorphic transformations and countable intersection.

Proof. Denote \( N^*_i, i_2, \ldots, i_k \) the set of all the numbers

\[
\xi = \frac{1}{|P_1|} + \frac{1}{|P_2|} + \ldots
\]

such that \( P_u(i_1, i_2, \ldots, i_k) \) can be represented in the form

\[
P_u(i_1, i_2, \ldots, i_k) = u(m_1, m_2, \ldots, m_k) \quad \text{ (See 46, 3)}
\]

and

\[
P_u(i_1, i_2, \ldots, i_k, \xi) = u(m_1, m_2, \ldots, m_k, \xi^*)
\]

\[
\frac{1}{|n_1^*|} + \frac{1}{|n_2^*|} + \ldots + \epsilon N
\]

One may easily verify that \( N^*_i, i_2, \ldots, i_k \) is homeomorphic with \( N \times J \). In the same manner we denote \( N^* \) the set of all the numbers

\[
\xi = \frac{1}{|P_1|} + \frac{1}{|P_2|} + \ldots
\]

such that

\[
P_u(\xi) = u(n_1^*), \quad \frac{1}{|n_1^*|} + \frac{1}{|n_2^*|} + \ldots + \epsilon N
\]

\( N^* \) is also homeomorphic with \( N \times J \).

Denote now

\[
N' = N^* \prod_{(i_1, i_2, \ldots, i_k)} N^*_i
\]

we must prove that \( \mathcal{O}_{N'} \) is equivalent with \( R \mathcal{O}_N \). In fact it follows from the definition of \( N' \) that \( N' \) is the set of all such numbers

\[
\xi = \frac{1}{|P_1|} + \frac{1}{|P_2|} + \ldots
\]

that

\[
P_u(i_1, i_2, \ldots, i_k) = u(n_1^*, n_1^*, n_2^*, \ldots, n_1^*, i_2^*, \ldots, i_k^*)
\]

and for any \( i_1, i_2, \ldots, i_k \)
Analytical Operations and Projective Sets

L. Kantorovitch and E. Livenson:

\[ \frac{1}{n_1, n_2, \ldots, n_k, 1} + \frac{1}{n_1, n_2, \ldots, n_k, 2} + \cdots \in N; \]
also

\[ \frac{1}{n_1} + \frac{1}{n_2} + \cdots \in N. \]

Now if

\[ P = R \Phi_N(\langle E_{n_1, n_2, \ldots, n_k} \rangle) \]
then denoting

\[ H_{\alpha}(n_1, n_2, \ldots, n_k) = E_{n_1, n_2, \ldots, n_k} \]
we shall have evidently (see 42, 2)

\[ P = \Phi_N(\langle H \rangle). \]

If on the other hand

\[ P = \Phi_N(\langle H \rangle) \]
then denoting

\[ E_{n_1, n_2, \ldots, n_k} = H_{\alpha}(n_1, n_2, \ldots, n_k) \]
we shall have evidently

\[ P = R \Phi_N(\langle E_{n_1, n_2, \ldots, n_k} \rangle). \]

Both these formulae are too evident to require any proof.

But then it follows from the equivalence of (49, 1) and (49, 3) that for any \( \mathcal{L} \)

\[ R \mathcal{H}_N(\mathcal{L}) = \mathcal{H}_N(\mathcal{L}) \]
whence \( R \Phi_N \) and \( \Phi_N \) are equivalent, q. e. d.

50. This chapter stands somewhat apart from the other two chapters of this part because it deals with operations upon operations and not operations upon systems of sets. Of course the operation which we considered in this chapter is only one particular operation of many, and it might seem strange why we considered it at all in this work. But we have put it here because we shall need it in the second part of this work and because in the construction of this first part we were guided chiefly by the considerations concerning the second part. It seems however (though we do not know