de F_i se trouve située sur la i-ème face ($n-1$-dimensionnelle) du simplexe [1]).

6) Le membre droit de l'égalité (2) est nommé par G. Boole (en Algèbre de la Logique) „*constituant* de l'univers du discours $\mathfrak{D}$ relatif au système $A_0, \ldots, A_n$". A chaque constituant correspond évidemment de façon biunivoque un système d'indices $i_0, \ldots, i_k$ (qui peut d'ailleurs être vide). Le théorème permet d'interpréter *géométriquement* cette correspondance: le système d'indices étant considéré comme simplexe géométrique, la correspondance représente une fonction *continue* satisfaisant à la formule (2) ou, ce qui est équivalent, les simplexes étant disjoints, à l'inclusion

$$f(A_{i_0} \cdot \ldots \cdot A_{i_k} - \sum_{i \neq i_j} A_i) \subset i_0 \ldots i_k. \tag{6}$$

L'hypothèse que l'égalité (1) a lieu, c. à d. que le constituant égal au produit des complémentaires des ensembles A_i est vide — est essentielle, puisqu'à ce constituant correspond le simplexe vide.

7) Nous avons supposé jusqu'à présent que les chiffres $0, \ldots, n$ désignent les sommets d'un simplexe n-dimensionel non-singulier, c. à d. qu'ils ne sont pas situés dans un espace euclidien à $n-1$ dimensions. Si l'on ne fait pas cette restriction, c. à d. *si* $0 \ldots n$ *est un simplexe arbitraire singulier ou non, le théorème reste encore vrai, lorsque l'égalité* (2) *est remplacée par l'inclusion* (6) (cela est indispensable puisque les simplexes ne sont plus nécessairement disjoints). Pour s'en convaincre on répète presque textuellement la démonstration du théorème (on peut aussi, en s'appuyant sur le théorème, appliquer une transformation simpliciale du simplexe ordinaire en le simplexe singulier en question).

[1]) C'est une transformation „duale" à celle du système γ. Elle intervient dans la démonstration du théor. de M. Hurewicz sur le „plongement" d'espaces arbitraires dans des espaces compacts de même dimension. Cf. Mon. f. Math. u. Ph. 37 (1930), p. 202.

On Continuous Curves Irreducible about Subsets.

By

Leo Zippin [1]) (Princeton)

Some years ago, confining himself to the plane and using arguments valid there only, Gehman proved the following theorem: If C is a compact (plane) continuous curve and H a closed subset, then in order that C shall contain an acyclic continuous curve containing H it is necessary and sufficient that H have the following structure: 1) the components of H are acyclic continuous curves or points, 2) not more than a finite number of these components exceed in diameter a preassigned positive number. We have had occasion to require analogous theorems in special applications where the curve C was of rather arbitrary nature but the point set H, on the contrary, sharply delimited: for example, totally disconnected or, again, what we have called a Moore-Kline set. More recently Whyburn has needed a complete extension of this theorem free of the restriction that C be planar. We communicated to him that the result held and inasmuch as he has since used it, crediting it to us [2]), it is proper that a proof now be given. We shall approach a more general problem, which we solve only in part, but whose partial solution includes as very special case the desired extension. This part solution appears to us to include the most likely applications of our „projected" theorem (see I, under section 1). None the less the problem is fairly fundamental in the field of continuous curves and seems to call for a definite answer.

Preliminary: We shall make the definition that a continuous curve C is *irreducible* about a self-compact subset H provided no

[1]) National Research Fellow.

[2]) „*Decomposability*...", Am. Journ. of Math., vol. 54 (1932), p. 173 (footnote).

proper continuous subcurve of C contains H. Now if a continuous curve, even the most general (i. e. one in whose definition completeness is allowed to replace the more usual compactness) contains a self-compact subset H it contains a compact continuous curve containing H (and differing from H, in fact, by a subset of a countable sum of arcs). This is known, and follows readily by an argument which differs but slightly from one given by Whyburn and Ayres in another, related, connection [1]). Then if C is irreducible about H in the sense above, we must suppose it compact. If any point x of $C-H$ fails to separate two points of H, $C-x$ contains a component whose closure is a continuous curve containing H. Then this must coincide with C and x cannot be a cutpoint of C. But in this case there exists a neighborhood of x whose complement is a continuum (Kuratowski) which we may suppose to contain H, since x is not in H. By the theorem of Whyburn and Ayres, above, this must belong to a continuous curve of C not containing x. This again is impossible, and we conclude that every point of $C-H$ is a cutpoint of C and that C is *irreducibly connected* about H. We have further that every cyclic element of C (not a single point) must belong to H. It follows that the components of H are continuous curves (or points) and that not more than a finite number of these exceed in diameter a preassigned positive number [2]).

Definition: A self-compact point set H having the properties α) its components are continuous curves or points, β) not more than a finite number of these exceed in diameter a preassigned positive number, shall be called a *curve-set.*

1): We are interested in the following proposition:

I: Every curve-set of a continuous curve C belongs to a continuous subcurve G of C which is irreducible about it.

We do not intend here to settle this proposition completely. We propose to effect a considerable simplification of it and finally to prove it in case the curve-set is of dimension one (Menger-Urysohn). This case embraces curve-sets whose components are acyclic, regular, perfect, or rational and in each instance the „irreducible" continuous curve G is of the same character. It is these continuous curves principally which have been the objects of highly specialised study.

Definition: A curve-set which consists of a countable set of components, F, F_1, $F_2, \ldots$, such that if (x_n) is a sequence of points of the set converging to a limit point x *then* either x belongs to F (which we shall call the principal curve of the set) *or* all but a finite number of the points of (x_n) belong to some one of the other continuous curves F_i we shall call a *restricted curve-set.*

2) We show first that I is implied by

II: Every restricted curve-set of a continuous curve C belongs to a continuous subcurve G of C which is irreducible about it.

For, let H be an arbitrary curve-set of the continuous curve C and let F_n be a countable set of components of H which is dense in H and includes all components not single points. Let $S_n = S(H, 1/n)$ and $E_n = C - S_n$ [1]). Let K_n be the set of points of H which cannot be $(1/n)$-chained to a point of F_1 but which can be $(1/n-1)$-chained [2]). The set K_n is open and closed in H [3]). We can define a countable set of mutually exclusive continuous curves T_i such that $K_n \subset \Sigma_{(i_{n-1}+1)}^{i_n} T_i \subset S_n$, where $i_0 = 0$. It is readily seen that $H' = F_1 + \Sigma T_i$ is a restricted curve-set with F_1 as principal curve. Then, by II, C contains a continuous curve C_1 irreducible about H'. C_1 contains H.

Definition: A continuum G will be said to be *perfect* about a point set K provided every subcontinuum of G which contains K has every point of K for regular point (i. e. for a point at which it is locally connected).

It is clear that C_1 is perfect about F_1 and perfect about every point of $C_1 \cdot E_1$. We can continue inductively on the curves F_n and establish the existence of a monotonic decreasing sequence of continuous curves C_n such that C_n is perfect about F_n (and about F_i, $i < n$, as subcontinuum of C_i) and perfect about every point of $C_n \cdot E_n$, and contains H. The infinite product of these curves C_n

[1]) „*On continuous curves in n dimensions*", Bulletin Am. Math. Soc. (1928) Theorem I, page 350.

[2]) We are assuming an acquaintance with the subject of Cyclic elements (Whyburn).

[1]) S_n is the set of points of C whose distance from some point of H is less than $1/n$.

[2]) We use this in its customary sense.

[3]) The proof of this is quite simple, and we suppose it to exist some where.

contains a continuum G irreducible about H (Wilson) and G must be perfect about every point of $G-H$ and about every component F_n of H. Now if G could fail to be a continuous curve it would have to contain a continuum of irregular points (Mazurkiewicz). This continuum would have to be a subset of H, and therefore of some component F_n of H. This is impossible, and we have proven that II implies I.

3) **Definition**: A restricted curve-set all but one of whose components are single points we shall call a *special curve-set.*

We prove that II is implied by

III: A special curve-set of a continuous curve C belongs to a continuous subcurve G irreducible about it.

For, let H be a restricted curve-set and F its principal curve, $F_1, F_2, \dots,$ its other components. We subject the curve C (which we may suppose compact, as always) to an upper semicontinuous decomposition in which every component F_n becomes a single point f_n while all other points of C (including the points of F) remain points [1]). Then under this transformation C becomes a continuous curve C' and the set H a special curve-set H' of C'. By III C' contains a continuous curve G' irreducible about H'. Each point f_n of H' is an isolated point of H' and in its „neighborhood" G' is a finite acyclic continuous curve. Let x_n' be a finite set of points of G' which irreducibly separate f_n from every other point of H' and which, further, $(1/n)$-separate f_n (Urysohn). Let Q_n be the component of $G'-X_n'$ which contains f_n and let K_n' be a continuous curve of C' *which has f_n for inner point*, whose diameter does not exceed $1/n$, and such that $K_n' \cdot G' = \overline{Q}_n$. We may suppose, proceeding inductively, that the curves K_n' are mutually exclusive. Now $G' + \Sigma K_n'$ is a continuous curve containing H' and irreducible about F' (the transform of F) plus $\Sigma K_n'$. If G is the image of G'' under the decomposition of C, it is not difficult to verify that G contains H, that it is a continuous curve, that it is perfect about F, and that it is irreducible about $F + \Sigma K_n$ (these being the image sets of the K_n' of C') [2]). Nor is it difficult to see that if in G (in C) we replace each continuous curve K_n by a subcurve M_n irreducible about $F_n + X_n$ (the image in C of the point set X_n' of C': this image is uniquely and continuously defined for every point x' of C' which is not a point of H') there results a continuous curve $\hat{G}$ which is irreducible about H.

[1]) Our formulation, while not the most rigorous, should be understood.

[2]) We are dispensing with necessary but not doubtful details.

We now abandon the general proposition I, and prove the following lemma: If, *in a one dimensional continuous curve C*, H is a special curve-set with F as its principal curve and P the countable set of its remaining point-components, *then* C contains a finite set of continuous curves $M_1, M_2, \dots, M_n$ such that

1) the diameter of M_i is less than a preassigned positive number, say e,

2) $M_i \cdot F$ is a continuous curve,

3) $M_i \cdot M_j$ is contained in F, $i \neq j$,

4) all but a finite number of the points of P belong to ΣM_i.

We know that we can express F as the sum of a finite set of continuous curves $A_1, \dots, A_k$ *such that the product of any two of these is totally disconnected* and such that we may associate with each of these a set D_i open in C, containing A_i, and of diameter less than e (Vanek). These sets A_i have the first three properties of our lemma, but certainly not the fourth: no point of P belongs to any of them. It is not difficult to show (we skip the slight argument) that there exists in C a finite set of continuous curves $B_1, \dots, B_k$ such that: 1) $A_i \subset B_i \subset D_i$, 2) *if x is an inner point of A_i relative to F, then it is an inner point of B_i* (*in C*), 3) $B_i \cdot B_j = A_i \cdot A_j$, 4) the boundary of B_i is totally disconnected. Now the boundary of $B = \Sigma B_i$ is totally disconnected. Therefore $C - B$ is the sum of a countable set of components (*whose boundaries are totally disconnected*) the set of whose diameters converges to zero, and such that each has at least one point of some B_i on its boundary. We are interested, of course, only in those components which contain points of P. We observe that of the components of $C - B$ there can be an at most finite number which do not have this property that they belong to some D_i and have as a boundary point at least one point of the corresponding B_i. Let us designate these exceptional ones by $T_1, \dots, T_r$, and for the moment disregard them.

The remaining components we shall separate conveniently into two sets, a set (t_n) such that no one of these has any point of F on its boundary, and a set (t_n') such that each of these does have some point of F on its boundary. We shall treat them separately, the latter set (with some modifications) we shall add back to F for a „fresh start". Let $P_n = P \cdot t_n$. It is clear that P_n is a finite point set. Let for a given t_n let i be the least integer such that t_n belongs to D_n and has some point b_n of B_i on its boundary. Clearly b_n is arcwise accessible from t_n and $t_n + b_n$ contains a continuous curve g_n^i which contains $P_n + b_n$. Now g_n^i has no point in common with any B_j excepting B_i. We may suppose that with each t_n we have associated a g_n^i as above, and write $M_i = B_i + \Sigma' g_n^i$, where i in the last term is fixed and the summation extends over those sets for which a corresponding n exists. We ommit the simple proof that M_i is a continuous curve, and that the set of these curves satisfies again the first three conditions of our lemma. Their sum includes all points of P not in Σt_i or in $\Sigma t_n'$.

Among the sets T_i (which we previously set aside) there may be some which have no points of F on their boundary. We shall disregard these entirely, their sum contains at most a finite number of points of P. The remainder of these and the components t_n' we wish to modify and to add back to F. It is convenient to rename all of these components so as to have a single „handy" sequence: (t_n''). Then every t_n'' is either a component t_n' or one of the T_i with points of F on its boundary, and every such component is some t_n''. Let $P_n'' = t_n'' \cdot P$: this may be a finite or countably infinite point set. If it is infinite let $\overline{P}_n''$ denote, as customarily, its closure. This will differ from P_n'', of course, in a closed totally disconnected subset of F. But if P_n' is finite, let us make the convention that $\overline{P}_n''$ denotes $P_n'' + a_n$ where a_n is an arbitrary point of F on the boundary of t_n''. Then in any case, $\overline{P}_n''$ is self-compact and contains a point of F. We assert that $t_n'' + \overline{P}_n''$ contains a compact continuous curve g_n'' which contains $\overline{P}_n''$. To show this it is sufficient to know that $t_n'' + \overline{P}_n''$ is a generalised continuous curve: i. e. a complete metric separable connected and locally connected space. Now $t_n'' + \overline{P}_n''$ differs from $\overline{t}_n''$, the closure of t_n'', by a subset of the boundary of t_n'', and this subset is easily recognised to be an F_σ. It is immediate that $t_n'' + \overline{P}_n''$ is a G_δ-subset of C and therefore complete (Alexandroff). Certainly it is connected and locally connected at all points of t_n''. That it is also locally connected at all points of $\overline{P}''$ is readily deduced from the fact that this is a totally disconnected point set.

Now if g_n'' has been defined for all the components t_n'' it is almost apparent that $F + \Sigma g_n''$ is a compact continuous curve F'', and that $F'' \cdot M_i$ is contained in F. *Moreover*, all but a finite number of the points of P belong to $F'' + \Sigma M_i$. Now to complete the proof of our lemma it merely remains to show that we can find in F'' a finite set of continuous curves $M_{k+1}, \ldots, M_n$ such that these are mutually exclusive, of diameter less than e, intersect F in a continuous curve, and contain all but a finite number of the points of P which belong to F''. Hastily regarded, this is the same problem as our lemma. There is this important difference, of course, that if Q denotes the (closed) set of points of F which are limit points of $P \cdot F''$, *then Q is totally disconnected*. Now with each point q of Q we can associate a continuous curve of F'' which is of diameter less than a preassigned positive number d, which has q for inner point (relative to F''), and which intersects F in a continuous curve. Out of the totality of these „covering" sets we can extract a finite covering set, and assemble this into a finite system of mutually exclusive continuous curves. That there exists a d such that the resulting „covering" continuous curves of such a system are of diameter less than e is obvious, and our lemma is established.

We can now prove:

If H is a special curve-set of a one dimensional compact continuous curve C, then C contains a continuous curve G irreducible about H.

For, let $H = F + P$, as in the lemma. Let $M_1^1, M_2^1, \ldots, M_{n_1}^1$ be the set of continuous curves of that lemma, where e is taken as $1/2$, and the superscript is introduced to facilitate the statement of an inductive argument, Let P_1 be the finite set of points of P that do not belong to $M^1 = \Sigma_1^{n_1} M_i^1$. There exists in C a finite set of arcs L_1 which irreducibly connects the sets P_1 and M^1; i. e. if x denotes any point of P_1 there exists one and only one point x_1 of M^1 such that L_1 contains an arc xx_1, and L_1 contains only one such arc. For a given point x of P_1, the corresponding point x_1 of M^1 will be called its *projection* (on M^1: x_1 may, of course, be a point of F). Let Q_1 denote the set of projection points (on M^1) of P_1. Now $H_1 = F + P \cdot M^1 + Q_1$ is obviously a special curve-set of M^1. With M^1 replacing C let $M_1^2, \ldots, M_{n_2}^2$ be the set of continuous curves of our lemma for $e = 1/2^2$. Let P_2 be the finite set of points of $P \cdot M^1 + Q_1$ not contained in $M^2 = \Sigma M_i^2$. Let L_2 be a finite set of arcs irreducibly connecting the set P_2 to M^2. If x_1 is any point of P_2 there is associated with a unique point x_2 of M^2, its projection on M^2, and a unique arc $x_1 x_2$ of L_2. We may suppose L_n to have been defined for all values of n, M^n being defined by the lemma, given M^{n-1} and e taken as $1/2^n$, and we let $G = F + L_n$. We assert that G is the desired continuous curve.

We shall give the proof in some detail since this will make clearer, if that is necessary, the exposition which has gone before. *First*, then, G is closed. *For* if y is a point of $\overline{G} - G$, it cannot be a point of F. Therefore there is an integer n such that it is not a point of M^n: we have only to take n so large that the distance of y to F is greater than $1/2^n$. Then y must be a limit point of $\Sigma_1^n L_i$. But this is closed and belongs to G. *Second*, G is connected. For, let x be any point of P. There exists a first integer n such that x does not belong to M^n: then it belongs to P_n (defined by analogy with P_1 and P_2). Then in L_n there is a unique arc xx_n where x_n is the projection point of x on M^n. If x_n is a point of F, we have shown that x may be arc-joined to F in G. If not, there is a first integer n_1 such that x_n is not a point of M^{n_1}. In L_{n_1} there is a unique arc $x_n x_{n_1}$, where x_{n_1} is the projection of x_n on M^{n_1}. If at any time we arrive at a point of F, we have an arc joining x to F in G. If not we can continue indefinitely to discover a sequence of arcs (whose sum is connected clearly) such that two nonconsecutive arcs are mutually exclusive and consecutive arcs have an endpoint only in common. The diameters of these arcs converge to zero at least as rapidly as the sequence $(1/2^n)$: this requires a word of justification. Since x_n, above, belongs to M^n it belongs to one of the continuous curves M_i^n, and it cannot belong to more than one of these unless it is a point of F. The second possibility no longer interests us. Therefore we may say that the arc $x_n x_{n_1}$ belongs to M_i^n, and its diameter is less than $1/2^n$. Then it is clear that the closure of the sum of these arcs is an arc joining x to a single point of F. Now any point of L_k is arcjoined in L_k to a point of P_k, and this ultimately in $\Sigma_1^k L_i$ to some point of P. We see then that G is arcwise connected.

Now if we suppose the points of P to be enumerated, $p_1, p_2, \ldots, p_n, \ldots$, we have shown that in G there is associated with each p_n a unique arc $p_n f_n$, where f_n is a point of F. And it is clear from our definition of the sets L_n that every point of G not in F belongs to at least one such arc. Therefore $G = F + \Sigma p_n f_n$. We can see by the argument above, that as n becomes indefinitely large, the diameters of the arcs $p_n f_n$ converge to zero. It is an immediate consequence of this that G is a continuous curve. There remains to see that G is irreducible about $F + P$. But it is irreducibly connected about $F - P = H$. For let z be any point of $G - H$. z must belong to some arc $p_n f_n$. If z does not separate p_n from F in G, then there is in $G - z$ an arc $p_n f$ which is different, at least in some points, from the arc $p_n f_n$. But we have seen that with each point p_n there is associated one and only one arc connecting it, in G, to a point of F. Our assertion is proved.

Now it will be appreciated that the implications which we established in the first part of this paper are valid, even when we restrict ourselves to some special class of curve-sets. For it is essentially merely a method quite independent of any „class of curve-sets" for carrying out the desired construction (of the irreducible curve G) upon general curve-sets when one already has a method for doing this upon the special curve-sets. Such a method we have given, above, when the curve-sets are one-dimensional. We may then state the

Theorem: if H is a one-dimensional curve-set of a continuous curve C (even the most general) then H belongs to à continuous subcurve G of C which is irreducible about H.

We may remark, finally, that this theorem can be slightly extended, on the basis of this paper, by designating as the *kernel* of a curve-set H the set of points $K = (x)$ such that x belongs to a component X of H and also to $\overline{H - X}$. It is sufficient to require that this kernel be one-dimensional.

We have left open the following problem: is every curve set H of a continuous curve C contained in a continuous curve G irreducible about it?