de $F_i$ se trouve située sur la $i$-ème face ($n-1$-dimensionnelle) du simplex $E^i$.

6) Le membre droit de l'égalité (2) est nommé par G. Boole (en Algèbre de
la Logique) "constituant de l'univers du discours $E$ relatif au système $A_1, \ldots, A_n$.
A chaque constituant correspond évidemment de façon univoque un système
d'indices $i_1, \ldots, i_k$ (qui peut d'ailleurs être vide). Le théorème permet d'interpréter géométriquement cette correspondance: le système d'indices étant considéré comme simplex géométrique, la correspondance représente une fonction continue satisfaisant à la formule (2) ou, ce qui est équivalent, les simplexes étant disjoints, à l'inclusion

\[ f(A_{i_1} \cap \ldots \cap A_{i_k}) \subset \bigcup_{i \neq j} A_i \cap A_j \]

L'hypothèse que l'égalité (1) a lieu, c. à d. que le constituant égal au produit des complémentaires des ensembles $A_i$ est vide — est essentielle, puisqu'a ce constituant correspond le simplexe vide.

7) Nous avons supposé jusqu'à présent que les chiffres $0, \ldots, n$
désignent les sommets d'un simplexe $n$-dimensional non-singulier, c. à d. qu'ils ne sont pas situés dans un espace euclidien à $n-1$
dimensions. Si l'on ne fait pas cette restriction, c. à d. si $0 \ldots n$
est un simplexe arbitrairement singulier ou non, le théorème reste encore vrai,

\[ \text{lorsque l'égalité (2) est remplacée par l'inclusion (6)} \]

Pour démontrer ce qui précède, on répète presque textuellement la démonstration
du théorème (on peut aussi, en s'appuyant sur le théorème, appliquer une transformation simpliciale du simplexe ordinaire en
le simplexe singulier en question).

1) C'est une transformation "dual" à celle du système $\gamma$. Elle intervient dans
la démonstration du théorème de M. Hurewicz sur le "plongement" d'espaces arbi-

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On Continuous Curves Irreducible about Subsets.

By Leo Zippin (Princeton)

Some years ago, confining himself to the plane and using arguments valid there only, Gehman proved the following theorem:

If $C$ is a compact (plane) continuous curve and $H$ a closed subset, then in order that $C$ shall contain an acyclic continuous curve containing $H$ it is necessary and sufficient that $H$ have the following structure: 1) the components of $H$ are acyclic continuous curves or points, 2) not more than a finite number of these components exceed in diameter a preassigned positive number. We have had occasion to require analogous theorems in special applications where the curve $C$ was of rather arbitrary nature but the point set $H$, on the contrary, sharply delimited: for example, totally disconnected or, again, what we have called a Moore-Kline set. More recently W. H. Hurwitz has needed a complete extension of this theorem free of the restriction that $C$ be planar. We communicated to him that the result held and inasmuch as he has since used it, crediting it to us 1), it is proper that a proof now be given. We shall approach a more general problem, which we solve only in part, but whose partial solution includes as very special case the desired extension. This part solution appears to us to include the most likely applications of our "projected" theorem (see I, under section 1). None of these theorems is fairly fundamental in the field of continuous curves and seems to call for a definite answer.

Preliminary: We shall make the definition that a continuous curve $C$ is irreducible about a self-compact subset $H$ provided no
proper continuous subcurve of \( C \) contains \( H \). Now if a continuous curve, even the most general (i.e., one in whose definition completeness is allowed to replace the more usual compactness) contains a self-compact subset \( H \) it contains a compact continuous curve containing \( H \) (and differing from \( H \), in fact, by a subset of a countable sum of arcs). This is known, and follows readily by an argument which differs but slightly from one given by Whyburn and Ayres in another, related, connection \(^1\). Then if \( C \) is irreducible about \( H \) in the sense above, we must suppose it compact. If any point \( z \) of \( C - H \) fails to separate two points of \( H \), \( C - x \) contains a component whose closure is a continuous curve containing \( H \). Then this must coincide with \( C \) and \( x \) cannot be a cut-point of \( C \). But in this case there exists a neighborhood of \( x \) whose complement is a continuum (Kuratowski) which we may suppose to contain \( H \), since \( x \) is not in \( H \). By the theorem of Whyburn and Ayres, above, this must belong to a continuous curve of \( C \) not containing \( x \). This again is impossible, and we conclude that every point of \( C - H \) is a cutpoint of \( C \) and that \( C \) is irreducible about \( H \). We have further that every cyclic element of \( C \) (not a single point) must belong to \( H \). It follows that the components of \( H \) are continuous curves (or points) and that not more than a finite number of these exceed in diameter a preassigned positive number \(^2\).

Definition: A self-compact point set \( H \) having the properties
\( a) \) its components are continuous curves or points, \( b) \) not more than a finite number of these exceed in diameter a preassigned positive number, shall be called a curve-set.

1) We are interested in the following proposition:
I: Every curve-set of a continuous curve \( C \) belongs to a continuous subcurve \( H \) of \( C \) which is irreducible about \( H \).

We do not intend here to settle this proposition completely. We propose to effect a considerable simplification of it and finally to prove it in case the curve-set is of dimension one (Menger-Urysohn). This case embraces curve-sets whose components are acyclic, regular, perfect, or rational and in each instance the irreducible \(^3\) continuous curve \( G \) is of the same character. It is these continuous curves principally which have been the objects of highly specialized study.

Definition: A curve-set which consists of a countable set of components, \( F_1, F_2, \ldots \), such that if \( (x_n) \) is a sequence of points of the set converging to a limit point \( x \) then either \( x \) belongs to \( F \) (which we shall call the principal curve of the set) or all but a finite number of the points of \( (x_n) \) belong to some one of the other continuous curves \( F_i \) we shall call a restricted curve-set.

2) We show first that \( I \) is implied by
II: Every restricted curve-set of a continuous curve \( C \) belongs to a continuous subcurve \( G \) of \( C \) which is irreducible about \( H \).

For, let \( H \) be an arbitrary curve-set of the continuous curve \( C \) and let \( F_1 \) be a countable set of components of \( H \) which is dense in \( H \) and includes all components not single points. Let \( S_n = S(H, 1/n) \) and \( E_n = C - S_n \). Let \( K_n \) be the set of points of \( H \) which cannot be \((1/n)\)-chained to a point of \( F_1 \), but which can be \((1/n - 1)\)-chained \(^3\). The set \( K_n \) is open and closed in \( H \). We can define a countable set of mutually exclusive continuous curves \( T_i \) such that \( K_n \subseteq \bigcup_{i=0}^{n} T_i \subseteq S_n \), where \( i_0 = 0 \). It is readily seen that \( H = F_1 + S T \) is a restricted curve-set with \( F_1 \) as principal curve. Then by II \( C \) contains a continuous curve \( C_1 \) irreducible about \( H \). \( C_1 \) contains \( H \).

Definition: A continuous \( G \) will be said to be perfect about a point set \( K \) provided every subcontinuum of \( G \) which contains \( K \) has every point of \( K \) for regular point (i.e., for a point at which it is locally connected).

It is clear that \( C_1 \) is perfect about \( F_1 \) and perfect about every point of \( C_1 \cdot E_1 \). We can continue inductively on the curves \( F_n \) and establish the existence of a monotonic decreasing sequence of continuous curves \( C_1 \) such that \( C_1 \) is perfect about \( F_1 \) and about \( F_i \), \( i < n \), as subcontinuum of \( C_1 \) and perfect about every point of \( C_1 \cdot E_1 \) and contains \( H \). The infinite product of these curves \( C_1 \)


\(^2\) We are assuming an acquaintance with the subject of Cyclic elements (Whyburn).

\(^3\) \( S_n \) is the set of points of \( C \) whose distance from some point of \( H \) is less than \( 1/n \).

\(^3\) We use this in its customary sense.

\(^3\) The proof of this is quite simple, and we suppose it to exist somewhere.
contains a continuum $G$ irreducible about $H$ (Wilson) and $G$ must be perfect about every point of $G \rightarrow H$ and about every component $F_i$ of $H$. Now if $G$ could fail to be a continuous curve it would have to contain a continuum of irregular points (Maschke; Weyl). This continuum would have to be a subset of $H$, and therefore of some component $F_i$ of $H$. This is impossible, and we have proven that $\Pi$ implies $I$.

3) **Definition**: A restricted curve-set all but one of whose components are single points we shall call a **special curve-set**.

We prove that $\Pi$ is implied by

III: A special curve-set of a continuous curve $C$ belongs to a continuous subcurve $G$ irreducible about it.

For, let $H$ be a restricted curve-set and $F_i$ its principal curve, $F_1, F_2, \ldots$, its other components. We subject the curve $C$ (which we may suppose compact, as always) to an upper semicontinuous decomposition in which every component $F_i$ becomes a single point $f_i$ while all other points of $C$ (including the points of $F_i$) remain points. Then under this transformation $C$ becomes a continuous curve $C'$ and the set $H$ a special curve-set $H'$ of $C'$. By III $C'$ contains a continuous curve $G'$ irreducible about $H'$. Each point $f_i$ of $H'$ is an isolated point of $H'$ and in its neighborhood of $G'$ is a finite aecyclic continuous curve. Let $x_i$ be a finite set of points of $G'$ which irreducibly separate $f_i$ from every other point of $H'$ and which, further, $(1/n)$-separate $f_i$ (Urysohn). Let $Q_i$, be the component of $G' - X_i$, which contains $f_i$ and let $K_{i}'$ be a continuous curve of $G'$ which has $f_i$ for inner point, whose diameter does not exceed $1/n$, and such that $K_{i}' - G' = \bar{Q}_i$. We may suppose, proceeding inductively, that the curves $K_{i}'$ are mutually exclusive. Now $G' + \Sigma K_{i}'$ is a continuous curve containing $H'$ and irreducible about $F$ (the transform of $F$) plus $\Sigma K_{i}'$. If $G$ is the image of $G''$ under the decomposition of $C$, it is not difficult to verify that $G$ contains $H$, that it is a continuous curve, that it is perfect about $F_i$, and that it is irreducible about $F + \Sigma K_{i}'$ (these being the image sets of the $K_{i}'$ of $C'$). Nor is it difficult to see that if in $G$ (in $C$) we replace each continuous curve $K_{i}'$ by a subcurve $M_{i}$ irreducible about $F_i + X_i$ (the image in $C$ of the point set $X_i$ of $C$; this image is uniquely and continuously defined for every point $x'$ of $C$ which is not a point of $H'$) there results a continuous curve $G$ which is irreducible about $H$.

We now abandon the general proposition I, and prove the following lemma: If, in a one dimensional continuous curve $C$, $H$ is a special curve-set with $F$ as its principal curve and $P$ the countable set of its remaining point-components, then $C$ contains a finite set of continuous curves $M_1, M_2, \ldots, M_n$ such that

1) the diameter of $M_i$ is less than a preassigned positive number, say $\varepsilon$,

2) $M_i \cdot F$ is a continuous curve,

3) $M_i \cdot F_i$ is contained in $F_i$, $i \neq j$,

4) all but a finite number of the points of $P$ belong to $\Sigma M_i$.

We know that we can express $F$ as the sum of a finite set of continuous curves $A_1, A_2, A_3$ such that the product of any two of these is totally disconnected and such that we may associate with each of these a set $D_i$ open in $C$, containing $A_i$, and of diameter less than $\varepsilon$ (Vaneck). These sets $A_i$ have the first three properties of our lemma, but certainly not the fourth: no point of $P$ belongs to any of them. It is not difficult to show (we skip the slight argument) that there exists in $C$ a finite set of continuous curves $B_1, B_2, B_n$ such that: 1) $A_i \subset B_i \subset D_i$, 2) if $z$ is an inner point of $B_i$ relative to $F$, then it is an inner point of $B_i$ (in $C$), 3) $B_i \cdot B_j = = A_i \cdot A_j$, 4) the boundary of $B_i$ is totally disconnected. Now the boundary of $B = \Sigma B_i$ is totally disconnected. Therefore $C - B$ is the sum of a countable set of components (whose boundaries are totally disconnected) the set of whose diameters converges to zero, and such that each has at least one point of some $B_i$ on its boundary. We are interested, of course, only in those components which contain points of $P$. We observe that of the components of $C - B$ there can be an at most finite number which do not have this property that they belong to some $D_i$, and have as a boundary point at least one point of the corresponding $B_i$. Let us designate these exceptional ones by $T_1, T_n$, and for the moment disregard them.

The remaining components we shall separate conveniently into two sets, a set $(A_i)$ such that no one of these has any point of $F$ on its boundary, and a set $(B_i)$ such that each of these does have
some point of \( F \) on its boundary. We shall treat them separately, the latter set (with some modifications) we shall add back to \( F \) for a "fresh start". Let \( P_0 = P \cdot t_a \). It is clear that \( P_0 \) is a finite point set. Let for a given \( t_a \), let \( b_a \) be the least integer such that \( t_a + b_a \) belongs to \( B \) and has some point \( b_0 \) of \( B \) on its boundary. Clearly \( b_0 \) is arcwise accessible from \( t_a \) and \( t_a + b_a \) contains a continuous curve \( g_a \) which contains \( P_0 + b_0 \). Now \( g_a \) has no point in common with any \( B \) excepting \( B_0 \). We may suppose that with each \( t_a \) we have associated a \( g_a \) as above, and write \( M_1 = B_0 + \bigcup g_a \), where \( i \) is the last term is fixed and the summation extends over those sets for which a corresponding \( n \) exists. We omit the simple proof that \( M_1 \) is a continuous curve, and that the set of these curves satisfies again the first three conditions of our lemma. Their sum includes all points of \( P \) not in \( \Sigma t_i \) or in \( \Sigma t_j \).

Among the sets \( T_i \) (which we previously set aside) there may be some which have no points of \( F \) on their boundary. We shall disregard these entirely, their sum contains at most a finite number of points of \( P \). The remainder of these and the components \( t_i \) we wish to modify and to add back to \( F \). It is convenient to rename all of these components so as to have a single "handy" sequence: \( t_i \). Then every \( t_i \) is either a component \( t_i \) or one of the \( T_i \) with points of \( F \) on its boundary, and every such component is some \( t_i \). Let \( P_0 = t_i \cdot P \); this may be a finite or countably infinite point set. If it is infinite let \( P_0 \) denote, as customary, its closure. This will differ from \( P_0 \), of course, in a closed totally disconnected subset of \( F \). But if \( P_0 \) is finite, let us make the convention that \( P_0 \) denotes \( P_0 + a_0 \) where \( a_0 \) is an arbitrary point of \( F \) on the boundary of \( t_i \). Then in any case, \( P_0 \) is self-compact and contains a point of \( F \). We assert that \( t_i + P_0 \) contains a compact continuous curve \( g_0 \) which contains \( P_0 \). To show this it is sufficient to know that \( t_i + P_0 \) is a generalised continuous curve; i.e., a complete metric separable connected and locally connected space. Now \( t_i + P_0 \) differs from \( t_i \), the closure of \( t_i \), by a subset of the boundary of \( t_i \), and this subset is easily recognised to be an \( F \). It is immediate that \( t_i + P_0 \) is a \( G_a \)-subset of \( t_i \) and therefore complete (Alexandroff). Certainly it is connected and locally connected at all points of \( t_i \). It is also locally connected at all points of \( F \) readily deduced from the fact that this is a totally disconnected point set.

Now if \( g_0 \) has been defined for all the components \( t_i \) it is almost apparent that \( F + \Sigma g_0 \) is a compact continuous curve \( F' \), and that \( F' + M_1 \) is contained in \( F \). Moreover, all but a finite number of the points of \( P \) belong to \( F' + \Sigma M_1 \). Now to complete the proof of our lemma it merely remains to show that we can find in \( F' \) a finite set of continuous curves \( M_2, \ldots, M_k \) such that these are mutually exclusive, of diameter less than \( e \), intersect \( F \) in a continuous curve, and contain all but a finite number of the points of \( P \) which belong to \( F' \). Hastily regarded, this is the same problem as our lemma. There is this important difference, of course, that if \( Q \) denotes the (closed) set of points of \( F \) which are limit points of \( F' + F'' \), then \( Q \) is totally disconnected. Now with each point \( g \) of \( Q \) we can associate a continuous curve of \( F'' \) which is of diameter less than a preassigned positive number \( d \), which has \( g \) for inner point (relative to \( F'' \)), and which intersects \( F \) in a continuous curve. Out of the totality of these covering sets we can extract a finite covering set, and assemble this into a finite system of mutually exclusive continuous curves. That there exists a \( d \) such that the resulting covering continuous curves of such a system are of diameter less than \( e \) is obvious, and our lemma is established.

We can now prove:

If \( H \) is a special curve-set of a one dimensional compact continuous curve \( C \), then \( C \) contains a continuous curve \( G \) irreducible about \( H \).

For, let \( H = F + P \), as in the lemma. Let \( M_1^1, M_2^1, \ldots, M_1^2 \) be the set of continuous curves of that lemma, where \( e \) is taken as \( 1/2 \), and the superscript is introduced to facilitate the statement of an inductive argument. Let \( P_0 \) be the finite set of points of \( P \) that do not belong to \( M_1 = \Sigma M_1^1 \). There exists in \( C \) a finite set of arcs \( L_1 \) which irreducibly connects the sets \( P_0 \) and \( M_1^1 \); i.e., if \( a \) denotes any point of \( P_0 \), there exists one and only one point \( x_0 \) of \( M_i \) such that \( L_1 \) contains a point \( x_0 \), and \( L_1 \) contains only one such arc. For a given point \( a \) of \( P_0 \), the corresponding point \( x_1 \) of \( M_1^1 \) will be called its projection (on \( M_1 : x_1 \) may, of course, be a point of \( P \)). Let \( Q_1 \) denote the set of projection points (on \( M_i^1 \)) of \( P_0 \).

Now \( H = F + P \cdot M_1^1 + Q_1 \) is obviously a special curve-set of \( M_1^1 \). With \( M_1^1 \) replacing \( C \) let \( M_1^2, M_2^2, \ldots, M_1^3 \) be the set of continuous curves of our lemma for \( P = 1/2^2 \). Let \( P_1 \) be the finite set of points of \( P \cdot M_2^2 + Q_1 \) not contained in \( M_1^3 = \Sigma M_1^3 \). Let \( L_2 \) be a finite
set of arcs irreducibly connecting the set $P_{3}$ to $M^{a}$. If $x_{1}$ is any point of $P_{3}$ there is associated with a unique point $x_{4}$ of $M^{a}$, its projection on $M^{a}$, and a unique arc $x_{5}x_{6}$ of $L_{a}$. We may suppose $L_{a}$ to be defined for all values of $a$, $M^{a}$ being defined by the lemma, given $M^{a-2}$ and $z$ taken as $1/2^{a}$, and we let $G := F \setminus L_{a}$. We assert that $G$ is the desired continuous curve.

We shall give the proof in some detail since this will make clearer, if that is necessary, the exposition which has gone before. First, then, $G$ is closed. For if $y$ is a point of $G$ -- $G$, it cannot be a point of $F$. Therefore there is an integer $n$ such that it is not a point of $M^{n}$: we have only to take $n$ so large that the distance of $y$ to $F$ is greater than $1/2^{n}$. Then $y$ must be a limit point of $\sum_{i} L_{i}$. But this is closed and belongs to $G$. Second, $G$ is connected. For, let $x$ be any point of $F$. There exists a first integer $n$ such that $x$ does not belong to $M^{n}$: then it belongs to $P_{a}$ (defined by analogy with $P_{1}$ and $P_{2}$). Then in $L_{a}$ there is a unique arc $x_{a}$, where $x_{a}$ is the projection point of $x$ on $M^{a}$. If $x_{a}$ is a point of $F$, we have shown that $x$ may be arc-associated to $F$ in $G$. If not, there is a first integer $n$, such that $x_{n}$ is not a point of $M^{n}$. In $L_{n}$ there is a unique arc $x_{n}x_{n+1}$, where $x_{n}$ is the projection of $x_{a}$ on $M^{n}$. If at any time we arrive at a point of $F$, we have an arc joining $x$ to $F$ in $G$. If not we can continue indefinitely to discover a sequence of arcs (whose sum is connected clearly) such that two non-consecutive arcs are mutually exclusive and consecutive arcs have an endpoint only in common. The diameters of these arcs converge to zero at least as rapidly as the sequence $(1/2^{n})$: this requires a word of justification. Since $x_{n}$, above, belongs to $M^{a}$ it belongs to one of the continuous curves $M^{a}_{f}$ and it cannot belong to more than one of these unless it is a point of $F$. The second possibility no longer interests us. Therefore we may say that the arc $x_{n}x_{n+1}$ belongs to $M^{a}_{f}$ and its diameter is less than $1/2^{n}$. Then it is clear that the closure of the sum of these arcs is an arc joining $x$ to a single point of $F$. Now any point of $L_{a}$ is arc-associated in $L_{a}$ to a point of $P_{a}$, and this ultimately in $\sum_{i} L_{i}$ to some point of $P$. We see then that $G$ is arcwise connected.

Now if we suppose the points of $F$ to be enumerated, $p_{1}, p_{2}, \ldots, p_{n}, \ldots$, we have shown that in $G$ there is associated with each $p_{a}$ a unique arc $p_{a}f_{a}$, where $f_{a}$ is a point of $F$. And it is clear from our definition of the sets $L_{a}$ that every point of $G$ not in $F$ belongs to at least one such arc. Therefore $G := F \setminus \Sigma p_{a}f_{a}$. We can see by the argument above, that as $n$ becomes indefinitely large, the diameters of the arcs $p_{a}f_{a}$ converge to zero. It is an immediate consequence of this that $G$ is a continuous curve. There remains to see that $G$ is irreducible about $F \setminus F$. But it is irreducibly connected about $F \setminus F$. Hence it is irreducible. For let $x$ be any point of $G \setminus F$. $x$ must belong to some arc $p_{a}f_{a}$. If $x$ does not separate $p_{a}$ from $F$ in $G$, then there is an arc $p_{a}f_{a}$ which is different, at least in some points, from the arc $p_{a}f_{a}$. But we have seen that with each point $p_{a}$ there is associated one and only one arc connecting it, in $G$, to a point of $F$. Our assertion is proved.

Now it will be appreciated that the implications which we established in the first part of this paper are valid, even when we restrict ourselves to some special class of curve-sets. For it is essentially merely a method quite independent of any class of curve-sets for carrying out the desired construction (of the irreducible curve $G$) upon general curve-sets when one already has a method for doing this upon the special curve-sets. Such a method we have given, above, when the curve-sets are one-dimensional. We may then state the

**Theorem:** If $H$ is a one-dimensional curve-set of a continuous curve $C$ (even the most general) then $H$ belongs to a curve subcurve $G$ of $C$ which is irreducible about $H$.

We may remark finally, that this theorem can be slightly extended, on the basis of this paper, by designating as the kernel of a curve-set $H$ the set of points $K = (x)$ such that $x$ belongs to a component $X$ of $H$ and also to $H \setminus X$. It is sufficient to require that this kernel be one-dimensional.

We have left open the following problem: is every curve set $H$ of a continuous curve $C$ contained in a curve subcurve $G$ irreducible about it?