les valeurs d'une fonction d'une variable réelle à valeurs distinctes qui satisfait à la condition de Baire. (On dit qu'une fonction d'une variable réelle $f(x)$ satisfait à la condition de Baire, si, quel que soit l'ensemble parfait P, $f(x)$ est continue sur P, lorsqu'on néglige un ensemble de 1re catégorie relativement à P)[1]). Nous donnerons ici une solution de ce problème, en admettant l'hypothèse du continu. Nous démontrerons ce

Théorème III. *Si* $2^{\aleph_0} = \aleph_1$, *la condition nécessaire et suffisante pour qu'un ensemble linéaire soit un ensemble de valeurs d'une fonction d'une variable réelle à valeurs distinctes qui satisfait à la condition de Baire est qu'il contienne un sous-ensemble parfait.*

La nécessité de notre condition résulte tout de suite du fait que si $f(x)$ est une fonction satisfaisant à la condition de Baire, il existe un ensemble complémentaire à un ensemble de 1re catégorie de Baire (donc contenant un sous-ensemble parfait) sur lequel $f(x)$ est continue, et du fait qu'une fonction définie et continue sur un ensemble parfait et borné transforme cet ensemble en un ensemble parfait.

Admettons maintenant que $2^{\aleph_0} = \aleph_1$ et soit E un ensemble linéaire contenant un sous-ensemble parfait P.

M. Lusin a démontré que si $2^{\aleph_0} = \aleph_1$, il existe un ensemble N de puissance du continu qui est de 1re catégorie sur tout ensemble parfait[2]).

Comme on voit sans peine, il existe une fonction $\varphi(x)$ d'une variable réelle à valeurs distinctes qui est une fonction de Baire, donc satisfait à la condition de Baire, et dont l'ensemble de valeurs est l'ensemble parfait P.

L'ensemble $\varphi(N)$, donc aussi l'ensemble $Q = (E - P) + \varphi(N)$ est évidemment de puissance du continu, et il existe une fonction $\psi(x)$ qui établit une correspondance biunivoque entre les points de l'ensemble N et ceux de l'ensemble Q. Posons

$$f(x) = \psi(x) \quad \text{pour} \quad x \in N$$

et

$$f(x) = \varphi(x) \quad \text{pour} \quad x \text{ non } \in N.$$

L'ensemble N étant de 1re catégorie sur tout ensemble parfait et la fonction $\varphi(x)$ satisfaisant à la condition de Baire, il résulte tout de suite de la définition de la fonction $f(x)$ qu'elle satisfait à la condition de Baire. Or, $f(x)$ est, comme on voit sans peine, une fonction à valeurs distinctes et l'ensemble de toutes les valeurs de $f(x)$ (pour x réels) est E. Notre théorème III est ainsi démontré.

Il est à remarquer que si l'on savait démontrer notre théorème sans admettre l'hypothèse du continu, il en résulterait tout de suite (sans l'aide de l'hypothèse du continu) que l'ensemble de toutes les fonctions d'une variable réelle qui satisfont à la condition de Baire a la puissance $2^{2^{\aleph_0}}$, ce qu'on ne sait pas démontrer sans utiliser l'hypothèse du continu[3]).

[1]) Voir *Fund. Math.*, t. V, p. 20.

[2]) *Comptes rendus* t. 158, p. 1256; cf. *Fund. Math.* t. II, p. 155.

[3]) Cf. *Fund. Math.*, t. IV, p. 368 (Problème 24).

Concerning *S*-regions in locally connected continua.

By

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1. A locally compact connected separable metric space which is locally connected will be called a *locally connected continuum*. Any connected open subset of such a space will be called a *region* in that space; and any region which has property S[1]), i. e., which, for any given $\varepsilon > 0$, is the sum of a finite number of connected sets of diameter $< \varepsilon$, will be called an *S-region*.

In this paper a method of construction of S-regions will be given which yields, for any given point p of a locally connected continuum M and for any $\varepsilon > 0$ such that the set $V_\varepsilon(p)$ of all points of M at a distance $< \varepsilon$ from p is compact, an S-region R containing p and lying in $V_\varepsilon(p)$. Furthermore, these regions $[R_\varepsilon]$ are monotone increasing in the sense that from $\sigma < \varepsilon$ follows $\overline{R}_\sigma \subset R_\varepsilon$. In view of the fact[2]) that any set having property S is locally connected and if a set S has property S, so also does any set S_0 such that $S \subset S_0 \subset \overline{S}$, it follows in particular that each of our sets $\overline{R}_\varepsilon$ is a compact locally connected continuum. Thus we are able to construct, in the neighborhood of any point p of M, an uncountable collection of compact locally connected subcontinua of M containing p and such that for any two of these, one is contained wholly in the interior (rel. M) of the other. This property is applied in § 4 to yield arbitrarily small S-regions containing any point p of a locally connected continuum M and whose exteriors are connected when p

[1]) See Sierpiński, Fund. Math., vol. 1 (1920), p. 44; and R. L. Moore, Fund. Math., vol. 3 (1922), p. 232.

[2]) See Moore, loc. cit.

is not a cut point of M and in any case have only a finite number of components. By an application of this same result it is shown in § 3 that if M has no *local separating point* (i. e., if no region in M has a cut point) then this property can be *localized* at any point of M, i. e., any point p of M is contained in an arbitrarily small compact locally connected subcontinuum of M which has no local separating point; also this property may be localized in a similar sense about any compact subcontinuum of M. In the concluding section, (§ 5), it will be shown that any compact region in a locally connected continuum whose boundary is totally disconnected is an S-region, and with the aid of this result it is proved that *every region* in such a continuum is homeomorphic with an S-region in some locally connected continuum.

2. Let M be a locally connected continuum and let p be any point of M. For any positive number ε such that $V_\varepsilon(p)$ is compact, let R_ε denote the set of all points x of M such that x can be joined to p by a simple chain [3]) of regions $L_1, L_2, \dots, L_n$, where L_1 contains p and L_n contains x, and where, for each i, $1 \leqslant i \leqslant n$, $\delta(L_i) < \varepsilon - \sum_1^i \delta(L_j)$.

(2.1). **Theorem.** *Every set R_ε has property S.*

P r o o f. Let δ be any positive number. Let $\alpha = \varepsilon - \delta/8$. Let E denote the set of all those points x of R_ε which can be joined to p by a simple chain of the type mentioned in the above definition and the sum of the diameters of whose links is less than α. Then $\overline{E}$ is a subset of R_ε. For obviously E is a subset of R_ε. And if y is any point of $\overline{E} - E$, y must belong to R_ε. For let G be a region containing y and of diameter $<(\varepsilon - \alpha)/2$. Then G contains a point x of E, and x can be joined to p by a chain of regions $L_1, L_2, \dots, L_k$, satisfying the condition in our definition of R_ε and furthermore such that the sum of the diameters of its links is $<\alpha$. Let L_n be the first link in this chain which has a point in common with G. Then clearly the chain $L_1, L_2, \dots, L_n$, G is a simple chain from p to y satisfying the condition in our definition, because $2\delta(G) < \varepsilon - \alpha < \varepsilon - \sum_1^n \delta(L_i)$, which gives $\delta(G) < \varepsilon - [\sum_1^n \delta(L_i) + \delta(G)]$. Therefore y belongs to R_ε and hence $\overline{E}$ is a subset of R_ε.

[3]) That is, a chain $L_1, L_2, \dots, L_n$, where for each i and j, $1 \leqslant i$, $j \leqslant n$, $L_i \cdot L_j \neq 0$ if and only if $|i-j| \leqslant 1$. See R. L. Moore, Trans. Amer. Math. Soc., vol. 17 (1916), Theorem 10.

It follows by the Borel Theorem that $\overline{E}$ is contained in the sum of a finite number of regions $W_1, W_2, \dots, W_k$ each of which is a subset of R_ε and is of diameter $< \delta/4$. For each i, $1 \leqslant i \leqslant k$, let Q_i be the set of all points in R_ε which can be joined to some point of W_i by a connected subset of R_ε of diameter $< \delta/4$. Then clearly each of the sets Q_i is connected and of diameter $< \delta$. It remains only to show that each point of R_ε is contained in at least one of the sets Q_i. To this end let x be any point of R_ε. Obviously we need only consider the case in which x does not belong to $\overline{E}$. There exists a simple chain of regions $L_1, L_2, \dots, L_n$ satisfying the conditions in our definition of R_ε. Let L_r be the last link in this chain which has a point in common with E. Then since L_r must contain at least one point of R_ε which does not belong to E, it follows by the definition of E that $\sum_1^r \delta(L_i) \geqslant \alpha$. Therefore $\delta(L_r) < \varepsilon - \sum_1^r \delta(L_i) \leqslant \delta/8$, and also $\sum_{r+1}^n \delta(L_i) < \delta/8$. Whence $\sum_r^n \delta(L_i) < \delta/4$. Thus $\delta(\sum_r^n L_i) \leqslant \sum_r^n \delta(L_i) < \delta/4$, and since $\sum_r^n L_i$ is connected and contains x and at least one point y of E and y belongs to some set W_i, say W_t, it follows that x belongs to Q_t. This completes the proof.

(2.2). **Theorem.** *If $\sigma < \varepsilon$, then $\overline{R}_\sigma$ is a subset of R_ε.*

For let y be any point of $\overline{R}_\sigma$ and let G be a region containing y and of diameter $<(\varepsilon - \sigma)/2$. Then G contains a point x of R and x can be joined to p by a chain of regions $L_1, L_2, \dots, L_k$, where L_1 contains p, L_k contains x, and for each i, $\delta(L_i) < \sigma - \sum_1^i \delta(L_j)$. Let L_n be the first link in this chain which contains a point of G. Then clearly the chain of regions $L_1, L_2, \dots, L_n$, G is a simple chain of regions from p to y satisfying the conditions in our definition of R_ε, because $2\delta(G) < \varepsilon - \sigma < \varepsilon - \sum_1^n \delta(L_i)$, which gives $\delta(G) < \varepsilon - [\sum_1^n \delta(L_i) + \delta(G)]$.

Therefore y belongs to R_ε and hence $\overline{R}_\sigma$ is contained in R_ε.

(2.3). **Corollary.** *If K is any compact subcontinuum of a locally connected continuum M, if ε is any positive number such that the set $V_\varepsilon(K)$ of all points of M whose distance from K is $<\varepsilon$ is compact, and if $R_\varepsilon(K)$ denotes the set of all points x of M which can be joined to some point of K by a simple chain of regions $L_1, L_2, \ldots, L_n$ where $L_1 \cdot K \neq 0$, $L_n \supset x$ and for each i, $1 \leqslant i \leqslant n$, $\delta(L_i) < \varepsilon - \sum_1^i \delta(L_j)$, then $R_\varepsilon(K)$ is an S-region. Furthermore, if $\sigma < \varepsilon$, then $\overline{R}_\sigma(K) \subset R_\varepsilon(K)$.*

Clearly only very slight modifications in the proofs given above suffice to establish this corollary.

3. We now make an application of the results in the preceeding section to obtain a useful property of locally connected continua which have no local separating points, i. e., no points p which are cut points of some region in M. Continua of this type are known to have many interesting and important properties. For example [4]) any such continuum lying in the plane contains, topologically, every plane one-dimensional set; also [5]) the Riesz-Denjoy theorem is valid in any such continuum, i. e., any closed compact and totally disconnected set is a subset of some simple continuous arc lying in the continuum.

(3.1). **Theorem.** *Each point p of a locally connected continuum M which has no local separating point is contained in an arbitrarily small region R such that $\overline{R}$ is a locally connected continuum having no local separating point.*

Proof. Let d be any positive number, which we shall suppose is small enough that the set of all points of M at a distance $<d$ from p is compact. We shall now consider the sets $[R_\varepsilon]$ for all $\varepsilon < d$ and shall show that, for at least one of these, the set $\overline{R}_\varepsilon$ has no local separating points. Indeed, we shall show that for at most a countable number of ε's can the set $\overline{R}_\varepsilon$ have a local separating point.

[4]) See Sierpiński, Comptes Rendus, vol. 162 (1916), p. 629.
[5]) See G. T. Whyburn, Fund. Math., vol. 17.

Let us suppose, on the contrary, that there exists an uncountable set A of positive numbers $<d$ such that for each number ε of A, $\overline{R}_\varepsilon$ has a local separating point x_ε. Since clearly no region in M can have a local separating point, it follows that x_ε belongs to $F(R_\varepsilon)$, for every ε in A. Now for each ε in A, there exists a region G_ε in M such that x_ε is a cut point of the component of $G_\varepsilon \cdot \overline{R}_\varepsilon$ which contains x_ε. Since A is uncountable it follows that there exists a compact region G and an uncountable subset B of A such that for any ε in B, x_ε cuts the component C_ε of $G \cdot \overline{R}_\varepsilon$ containing x_ε and such that there exists a positive number e so that every point of B belongs to the set Q of all points of G at a distance $>e$ from $F(G)$. For each ε in B, there exists at least two components of $C_\varepsilon - x_\varepsilon$ each of which contains points of $Q \cdot R_{\sigma_\varepsilon}$, where σ_ε is some positive number $<\varepsilon$. There exists two fixed positive numbers σ and f and an uncountable subset D of B such that for every ε in D, $\varepsilon - \sigma > f$ and such that at least two components of $C_\varepsilon - x_\varepsilon$ contain points of $Q \cdot R_\sigma$. Since D is uncountable it therefore contains a monotone decreasing sequence of numbers $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \ldots$ For each i, let $x_i = x_{\varepsilon_i}$.

Since there are at least two components of $C_{\varepsilon_1} - x_1$ containing points of $Q \cdot R_\sigma$, at least one of these, say K_1, contains a point p_1 of $Q \cdot R_\sigma$ and is such that an infinite subset S_1 of $\sum_2^\infty x_i$ exists no point of which belongs to K_1. Let x_{n_2} be the point of least subscript in S_1. Likewise there exists a component K_2 of $C_{\varepsilon_{n_2}} - x_{n_2}$ which contains a point p_2 of $Q \cdot R_\sigma$ and is such that an infinite subset S_2 of $S_1 - x_{n_2}$ exists no point of which belongs to K_2. Let x_{n_3} be the point of least subscript in S_2. Again there exists a component K_3 of $C_{\varepsilon_{n_3}} - x_{n_3}$ containing a point p_3 of $Q \cdot R_\sigma$ and such that an infinite subset S_3 of $S_2 - x_{n_3}$ exists no point of which belongs to K_3, and so on. Let us continue this process indefinitely. The sequence $p_1, p_2, \ldots$ has at least one limit point p. which must belong to $\overline{Q} \cdot \overline{R}_\sigma$. Since $\overline{R}_\sigma$ is locally connected and $\varrho[p, F(G)] \geqslant e$, it follows that some two points of the sequence $[p_i]$, say p_j and p_k, where $j < k$, lie together in a connected subset H of $G \cdot \overline{R}_\sigma$. But then since, by (2.2), $\overline{R}_\sigma \subset \overline{R}_{\varepsilon_{n_j}} - x_{n_j}$, we have $H \subset G \cdot \overline{R}_{\varepsilon_{n_j}} - x_{n_j}$. Therefore $H + K_k + x_{n_k}$ is a connected subset of $G \cdot \overline{R}_{\varepsilon_{n_j}} - x_{n_j}$, because $K_k + x_{n_k} \subset \overline{R}_{\varepsilon_{n_k}} \subset R_{\varepsilon_{n_j}}$; hence this set is a subset of K_j, since it contains the point p_j of K_j. But this is contrary to the fact that x_{n_k} is a point of S_j and no point of S_j belongs to K_j. This contradiction proves our theorem.

Corollary. *If p is a point of a locally connected continuum M which has no local separating points and d is a number such that $V_d(p)$ is compact, then the set E of all positive numbers $\varepsilon < d$ such that $\overline{R}_\varepsilon$ has a local separating point is countable.*

(3.2). **Theorem.** *If K is any compact subcontinuum of a locally connected continuum M having no local separating point and if we use the same notation as in (2.3), then at most a countable number of the sets $\overline{R}_\varepsilon(K)$ can have local separating points. Thus K can be imbedded in a compact locally connected subcontinuum of M which has no local separating point and whose diameter surpasses that of K by an arbitrarily small amount.*

This theorem is proved from (2.3) by the same argument as just given to prove (3.1) from (2.1) and (2.2).

4. **Theorem.** *Each non-cut point of a locally connected continuum M is contained in an arbitrarily small S-region whose exterior is connected.*

Proof. Let p be any non-cut point of M and let d be any positive number which is small enough that $V_d(p)$ is compact. There exists [6]) a region $R \subset V_d(p)$ contaning p whose exterior H is connected; and there exists a positive number e such that for all positive numbers $\varepsilon < e$, the set R_ε is a subset of R. For any such ε, let H_ε be the component of $M - R_\varepsilon$ containing H, let Q_ε be the component of $M - H_\varepsilon$ containing R_ε, and let $X_\varepsilon = F(Q_\varepsilon)$. Then for each ε, Q_ε has property S, as will now be demonstrated.

To this end let δ be any positive number. Now $\overline{Q}_\varepsilon$ is locally connected, since $\overline{Q}_\varepsilon = \overline{R}_\varepsilon +$ a certain group of components of $M - \overline{R}_\varepsilon$, and $\overline{R}_\varepsilon$ is locally connected. Therefore, as $\overline{Q}_\varepsilon$ is a subset of $V_d(p)$ and thus is compact, it therefore has property S. Hence, $\overline{Q}_\varepsilon = Q_1 + Q_2 + \ldots + Q_n$, where, for each i, Q_i is a locally connected continuum of diameter $< \delta/4$. Let $Q_{n_1}, Q_{n_2}, \ldots, Q_{n_j}$ be those sets Q_i which contain points of X_ε and let T be the sum of all those sets Q_i which contain no point of X_ε. Then T is a subset of Q_ε, and since it is a compact locally connected continuum and therefore has property S, it follows that the set $T + (\overline{R}_\varepsilon - X_\varepsilon)$ has property S, because $\overline{R}_\varepsilon - X_\varepsilon$ has property S. Thus we can write $T + (\overline{R}_\varepsilon - X_\varepsilon) = W_1 + W_2 + \ldots + W_k$, where each set W_i is connected and of diameter $< \delta/4$. For each i, $1 \leqslant i \leqslant k$, let V_i be the set of all those points of Q_ε which can be joined to some point of W_i by a connected subset of Q_ε of diameter $< \delta/4$. Then clearly each set V_i is a connected subset of Q_ε of diameter $< \delta$. It remains only to show that every point of Q_ε belongs to at least one set V_i. Let x be any point of Q_ε which, we may suppose, does not belong to $T + (\overline{R}_\varepsilon - X_\varepsilon)$. Then x must belong to some set Q_{n_i}, say to Q_{n_r}. Let C denote the component of $Q_{n_r} - \overline{R}_\varepsilon \cdot Q_{n_r}$ which contains x. Then $\overline{R}_\varepsilon$ contains at least one limit point y of C, and y cannot belong to X_ε because in this case C would belong to H_ε as $X_\varepsilon \subset H_\varepsilon$. Thus y belongs to $\overline{R}_\varepsilon - X_\varepsilon$ and hence belongs to some set V_i, say to V_s. And since $C + y$ is a connected subset of Q_ε of diameter $< \delta/4$, it follows by definition of V_s that x belongs to V_s. Hence it is proved that Q_ε has property S.

Now the collection of sets $[X_\varepsilon]$ is non-separated [7]). For since, if $\sigma < \varepsilon$, $X_\sigma \subset \overline{R}_\sigma \subset R_\varepsilon$, X_ε cannot separate X_σ; and since from $\overline{R}_\sigma \subset R_\varepsilon$ follows $H_\varepsilon \cdot X_\sigma \subset H_\varepsilon \cdot \overline{R}_\sigma = 0$ and $X_\varepsilon \subset H_\varepsilon$, X_σ cannot separate X_ε. Therefore a theorem [8]) on non-separated cuttings applies to give us some set X_ε which cuts M into two connected regions, one of which is Q_ε, say into U_p and U_q, where $U_p = Q_\varepsilon$, and is such that $X_\varepsilon = F(U_p) = F(U_q)$. Thus Q_ε, or U_p, is an S-region of diameter $< d$ whose exterior U_q is connected, and the theorem is proved.

Corollary. *Any point of a locally connected continuum is contained in an arbitrarily small S-region whose exterior is the sum of a finite number of connected sets.*

[6]) See H. M. Gehman, Proc. Ntl. Acad. Sci., vol. 14 (1928), pp. 431—433; and W. L. Ayres, Monat. für Math. und Phys., vol. 36 (1929), pp. 139—140; also G. T. Whyburn, Amer. Jour. Math., vol. 53 (1931), p. 429.

[7]) That is, for any two ε's, say ε_1 and ε_2, X_{ε_1} lies wholly in a single component of $M - X_{\varepsilon_2}$; See the author's paper on non-separated cuttings in Trans. Amer. Math. Soc., vol. 33 (1931), pp. 444—454.

[8]) Loc. Cit., § 4, result (β), p. 449.

For let p be any point of such a continuum M and let d be any positive number small enough that $V_d(p)$ is compact. Let $C_1, C_2, \ldots, C_n$ be the components of $M-p$ which are of diameter $> d/4$. For each i, $1 \leqslant i \leqslant n$, p is a non-cut point of the locally connected continuum $C_i + p$. Hence, by the theorem, there exists an S-region R_i in $C_i + p$ of diameter $< d/4$, which contains p and whose exterior $(C_i + p) - \overline{R}_i$ is connected. Now $M - \sum_1^n C_i$ is a compact locally connected continuum and hence has property S. Therefore the set $[M - \sum_1^n C_i] + \sum_1^n R_i = G$ has property S. Hence G is an S-region in M containing p, of diameter $< d$, and whose exterior is the sum of the connected sets $[(C_i + p) - \overline{R}_i]$.

5. In this section we consider regions in a general locally connected continuum. The proof for the theorems given here are independent of the results in the preceeding sections.

(5.1). **Theorem.** *Any compact region R in a locally connected continuum whose boundary is totally disconnected is an S-region.*

For if ε is any given positive number, it follows with the aid of the Borel Theorem together with the fact that $\overline{R}$ is locally connected that $F(R)$ is contained in the sum W of a finite number of connected regions $W_1, W_2, \ldots, W_n$ in $\overline{R}$ each of diameter $< \varepsilon$ and such that the boundary of no one of these sets contains a point of $F(R)$, i. e., $\overline{R} \cdot F(W_i) \subset R$ for every i. Likewise $R - W \cdot R$ is contained in the sum of a finite number of connected regions $V_1, V_2, \ldots, V_k$ each of which is a subset of R and is of diameter $< \varepsilon$. Now since, for each i, $1 \leqslant i \leqslant n$, W_i is connected and $F(W_i) \cdot F(R) = 0$, it follows that there are only a finite number of components of $R \cdot W_i$. Thus $R \cdot W$ is the sum of a finite number of connected sets of diameter $< \varepsilon$, namely, the components of $R \cdot W_i$ for $1 \leqslant i \leqslant n$, and these sets together with the sets $V_1, V_2, \ldots, V_k$ contain R. Therefore R is the sum of a finite number of connected sets of diameter $< \varepsilon$ and hence has property S.

Now let Q be any region in a locally connected continuum, which we shall suppose is compact, because we are interested only in interval properties of Q and clearly any region in a locally connected continuum is homeomorphic with a region in some compact locally

connected continuum. Let W^* be the space whose points are: (a) the points of Q and (b) the components of $F(Q)$, and whose neighborhoods are (1) all open subsets of Q and (2) all open subsets of $\overline{Q}$ whose boundaries contain no points of $F(Q)$. It is seen at once that W^* is a compact locally connected continuum and that the set $Q^* = W^* -$ [all points of class (b)] is a region in W^* whose boundary is totally disconnected. Therefore, by (5.1), Q^* has property S. But since Q and Q^* have the same points and since any Q-neighborhood of a point p of Q is also a W^* neighborhood of p and any W^*-neighborhood of p contains a Q-neighborhood of p, it follows that Q and Q^* are homeomorphic. Thus we have proved that:

(5.2). *Any region in a locally connected continuum is homeomorphic with an S-region in some locally connected continuum.*

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