W. Sierpiński.

les valeurs d'une fonction d'une variable réelle à valeurs distinctes qui satisfaient à la condition de Baire. (On dit qu'une fonction d'une variable réelle f(x) satisfait à la condition de Baire, si, quel que soit l'ensemble parfait M, f(x) est continue sur M, lorsqu'on néglige un ensemble de 1ère catégorie relativement à M). Nous donnerons une solution de ce problème, en admettant l'hypothèse du continu. Nous démontrerons ce

Théorème III. Si $2^n = \aleph_0$, la condition nécessaire et suffisante pour qu'un ensemble linéaire soit un ensemble de valeurs d'une fonction d'une variable réelle à valeurs distinctes qui satisfaire à la condition de Baire est qu'il contienne un sous-ensemble parfait.

La nécessité de notre condition résulte tout de suite du fait que si f(x) est une fonction satisfaisant à la condition de Baire, il existe un ensemble complémentaire à un ensemble de 1ère catégorie de Baire (donc contenant un sous-ensemble parfait) sur lequel f(x) est continue, et du fait qu'une fonction définie et continue sur un ensemble parfait et borné transforme cet ensemble en un ensemble parfait.

Admettons maintenant que $2^n = \aleph_0$, et soit B un ensemble linéaire contenant un sous-ensemble parfait M.

M. Luzin a démontré que si $\aleph_0 > \aleph_0$, il existe un ensemble N de puissance du continu qui est de 1ère catégorie sur tout ensemble parfait.

Comme on voit sans peine, il existe une fonction φ(x) d'une variable réelle à valeurs distinctes qui est une fonction de Baire, donc satisfaisant à la condition de Baire, et dont l'ensemble de valeurs est l'ensemble parfait P.

L'ensemble φ(N), donc aussi l'ensemble Q = (B - P) + φ(N) est évidemment de puissance du continu, et il existe une fonction ψ(x) qui établit une corresp. biunivoque entre les points de l'ensemble N et ceux de l'ensemble Q. Posons

$$f(x) = \psi(x) \quad \text{pour} \quad x \in N$$

et

$$f(x) = \phi(x) \quad \text{pour} \quad x \not\in N$$

L'ensemble N étant de 1ère catégorie sur tout ensemble parfait et la fonction φ(x) satisfaisant à la condition de Baire, il résulte tout de suite de la définition de la fonction f(x) qu'elle satisfaire à la condition de Baire. Or, f(x) est, comme on voit sans peine, une fonction à valeurs distinctes et l'ensemble de toutes les valeurs de f(x) (pour x réel) est B. Notre théorème III est ainsi démontré.

Il est à remarquer que si l'on savait démontrer notre théorème sans l'hypothèse du continu, il en résulterait tout de suite (sans l'aide de l'hypothèse du continu) que l'ensemble de toutes les fonctions d'une variable réelle qui satisfont à la condition de Baire a la puissance $2^\aleph_0$, ce qu'on ne sait pas démontrer sans utiliser l'hypothèse du continu.

Concerning S-regions in locally connected continua.

By

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1. A locally compact connected separable metric space which is locally connected will be called a locally connected continuum. Any connected open subset of such a space will be called a region in that space, and any region which has property $S^1$, i. e., which, for any given $\varepsilon > 0$, is the sum of a finite number of connected sets of diameter $< \varepsilon$, will be called an S-region.

In this paper a method of construction of S-regions will be given which yields, for any given point $p$ of a locally connected continuum M and for any $\varepsilon > 0$ such that the set $V_\varepsilon(p)$ of all points of M at a distance $< \varepsilon$ from p is compact, an S-region $E$ containing $p$ and lying in $V_\varepsilon(p)$. Furthermore, these regions $[E]$ are monotone increasing in the sense that from $\sigma < \varepsilon$ follows $E_\sigma \subset E_\varepsilon$.

In view of the fact $^1$ that any set having property $S$ is locally connected and if a set $S$ has property $S$, so also does any set $S_0$ such that $S_0 \subset S$, it follows in particular that each of our sets $E_\varepsilon$ is a compact locally connected continuum. Thus we are able to construct, in the neighborhood of any point $p$ of M, a uncountable collection of compact locally connected subcontinua of M containing $p$ and such that for any two of these, one is contained wholly in the interior (rel. M) of the other. This property is applied in $\S$ 4 to yield arbitrarily small S-regions containing any point $p$ of a locally connected continuum M and whose exteriors are connected when $p$


$^*$ See Moore, loc. cit.
is not a cut point of $M$ and in any case have only a finite number of components. By an application of a similar result it is shown in § 3 that if $M$ has no local separating point (i.e., if no region in $M$ has a cut point) then this property can be localized at any point of $M$, i.e., any point $p$ of $M$ is contained in an arbitrarily small compact locally connected subcontinuum of $M$ which has no local separating point; also this property may be localized in a similar sense about any compact subcontinuum of $M$. In the concluding section, (§ 5), it will be shown that any compact region in a locally connected continuum whose boundary is totally disconnected is an $S$-region, and with the aid of this result it is proved that every region in such a continuum is homeomorphic with an $S$-region in some locally connected continuum.

2. Let $M$ be a locally connected continuum and let $p$ be any point of $M$. For any positive number $e$ such that $V_e(p)$ is compact, let $R_e$ denote the set of all points $x$ of $M$ such that $x$ can be joined to $p$ by a simple chain $x$ of regions $L_1, L_2, \ldots, L_n$, where $L_i$ contains $p$, $L_i$ contains $x$, and where, for each $i, 1 \leq i \leq n$, $\delta(L_i) < e - \frac{S}{2} \delta(L_i)$, which gives $\delta(G) < e - [\Sigma \delta(L_i) + \delta(G)]$. Therefore $y$ belongs to $R_e$ and hence $\overline{E}$ is a subset of $R_e$.

It follows by the Borel Theorem that $\overline{E}$ is contained in the sum of a finite number of regions $W_1, W_2, \ldots, W_k$, each of which is a subset of $R_e$ and of diameter $\theta / 4$. For each $i, 1 \leq i \leq k$, let $Q_i$ be the set of all points in $R_e$ which can be joined to some point of $W_i$ by a connected subset of $R_e$ of diameter $\theta / 4$. Then clearly each of the sets $Q_i$ is connected and of diameter $\delta$. It remains only to show that each point of $R_e$ is contained in at least one of the sets $Q_i$. To this end let $x$ be any point of $R_e$. Obviously we need only consider the case in which $x$ does not belong to $\overline{E}$.

There exists a simple chain of regions $L_1, L_2, \ldots, L_n$ satisfying the conditions in our definition of $R_e$. Let $L_1$ be the last link in this chain which has a point in common with $E$. Then since $L_1$ must contain at least one point of $R_e$ which does not belong to $E$, it follows by the definition of $E$ that $\Sigma \delta(L_i) \geq a$. Therefore $\delta(L_i) < e - \frac{S}{2} \delta(L_i)$, and also $\Sigma \delta(L_i) < e - \frac{S}{2} \delta(L_i)$, and since $\Sigma L_i$ is connected and contains $x$ and at least one point $y$ of $E$ and $y$ belongs to some set $W_i$, say $W_i$, it follows that $x$ belongs to $Q_i$. This completes the proof.

(2.1) Theorem. Every set $R_e$ has property $S$.

Proof. Let $\delta$ be any positive number. Let $\alpha = e - \delta / 8$. Let $\overline{E}$ denote the set of all those points $x$ of $R_e$ which can be joined to $p$ by a simple chain of the type mentioned in the above definition and the sum of the diameters of whose regions is less than $\alpha$. Then $\overline{E}$ is a subset of $R_e$. For obviously $E$ is a subset of $R_e$. And if $y$ is any point of $\overline{E}$, $y$ must belong to $R_e$. For let $G$ be a region containing $y$ and of diameter $< (e - \alpha) / 2$. Then $G$ contains a point $x$ of $E$, and $x$ can be joined to $p$ by a chain of regions $L_1, L_2, \ldots, L_n$, satisfying the condition in our definition of $R_e$, and furthermore such that the sum of the diameters of the links is $< \alpha$. Let $L_1$ be the first link in this chain which has a point in common with $G$. Then clearly the chain $L_1, L_1, \ldots, L_n$, $G$ is a simple chain from $p$ to $y$ satisfying the condition in our definition, because $2 \delta(G) < e - \alpha < e - \frac{S}{2} \delta(L_i)$.

(2.2) Theorem. If $\alpha < e$, then $\overline{E}$ is a subset of $R_e$.

For let $y$ be any point of $\overline{E}$ and let $G$ be a region containing $y$ and of diameter $< (e - \alpha) / 2$. Then $G$ contains a point $x$ of $E$ and $x$ can be joined to $p$ by a chain of regions $L_1, L_2, \ldots, L_n$, where $L_0$ contains $p$, $L_0$ contains $x$, and for each $i, \delta(L_i) < e - \frac{S}{2} \delta(L_i)$, which gives $\delta(G) < e - \frac{S}{2} \delta(L_i)$, which gives $\delta(G) < e - \frac{S}{2} \delta(L_i)$.
Therefore \( y \) belongs to \( R_\varepsilon \) and hence \( \overline{R}_\varepsilon \) is contained in \( R_\varepsilon \).

(2.3) Corollary. If \( K \) is any compact subcontinuum of a locally connected continuum \( M \), if \( \varepsilon \) is any positive number such that the set \( V_\varepsilon(K) \) of all points of \( M \) whose distance from \( K \) is \( < \varepsilon \) is compact, and if \( R_\varepsilon(K) \) denotes the set of all points \( x \) of \( M \) which can be joined to some point of \( K \) by a simple chain of regions \( L_1, L_2, \ldots, L_n \) where \( L_i \cdot K \neq 0, \ L_n \supset x \) and for each \( i, 1 \leq i \leq n, \ d(L_i) < \varepsilon - \sum_{i=1}^{n} d(L_i) \), then \( R_\varepsilon(K) \) is an S-region. Furthermore, if \( \sigma < \varepsilon \), then \( \overline{R}_\varepsilon(K) \subset R_{\sigma}(K) \).

Clearly only very slight modifications in the proofs given above suffice to establish this corollary.

3. We now make an application of the results in the preceding section to obtain a useful property of locally connected continua which have no local separating points, i.e., no points \( p \) which are cut points of some region in \( M \). Continua of this type are known to have many interesting and important properties. For example, the Riesz-Deny theorem is valid in any such continuum, i.e., any closed compact and totally disconnected set is a subset of some simple continuum arc lying in the continuum.

(3.1). Theorem. Each point \( p \) of a locally connected continuum \( M \) which has no local separating point is contained in an arbitrarily small region \( R \) such that \( \overline{R} \) is a locally connected continuum having no local separating point.

Proof. Let \( d \) be any positive number, which we shall suppose is small enough that the set of all points of \( M \) at a distance \( < d \) from \( p \) is compact. We shall now consider the sets \( [R_p] \) for all \( \varepsilon < d \) and shall show that, for at least one of these, the set \( \overline{R}_\varepsilon \) has no local separating points. Indeed, we shall show that for at most a countable number of \( \varepsilon \)'s the set \( \overline{R}_\varepsilon \) has a local separating point.

Let us suppose, on the contrary, that there exists an uncountable set \( A \) of positive numbers \( < d \) such that for each number \( \varepsilon \) of \( A \), \( \overline{R}_\varepsilon \) has a local separating point \( x_\varepsilon \). Since clearly no region in \( M \) can have a local separating point, it follows that \( x_\varepsilon \) belongs to \( F(R_\varepsilon) \), for every \( \varepsilon \) in \( A \). Now for each \( \varepsilon \) in \( A \), there exists a region \( G_\varepsilon \subset M \) such that \( x_\varepsilon \) is a cut point of the component of \( G_\varepsilon \cdot \overline{R} \), which contains \( x_\varepsilon \). Since \( A \) is uncountable it follows that there exists a compact region \( G \) and an uncountable subset \( B \) of \( A \) such that for any \( \varepsilon \) in \( B \), \( x_\varepsilon \) in the component \( C_\varepsilon \) of \( G \cdot \overline{R} \) containing \( x_\varepsilon \), such that there exists a positive number \( \varepsilon \) so that every point of \( B \) belongs to the set \( Q \) of all points of \( G \) at a distance \( > \varepsilon \) from \( F(G) \). For each \( \varepsilon \) in \( B \), there exists at least two components of \( C_\varepsilon \) each of which contains points of \( Q \cdot R_{\sigma} \), where \( \sigma \) is some positive number \( < \varepsilon \). There exists two fixed positive numbers \( \sigma \) and \( f \) and an uncountable subset \( D \) of \( B \) such that for every \( \varepsilon \) in \( D \), \( \varepsilon - \sigma > f \) and such that at least two components of \( C_\varepsilon - x_\varepsilon \) contain points of \( Q \cdot R_\sigma \). Since \( D \) is uncountable it therefore contains a monotone decreasing sequence of numbers \( \varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \ldots \). For each \( i \), let \( x_\varepsilon = x_{\varepsilon_i} \).

Since there are at least two components of \( C_{\varepsilon_i} \cdot x_{\varepsilon_i} \) containing points of \( Q \cdot R_\sigma \), at least one of these, say \( K_\varepsilon \), contains a point \( p_\varepsilon \) of \( Q \cdot R_\sigma \) and is such that an infinite subset \( S_1 \) of \( \sum_{i=1}^{\infty} x_{\varepsilon_i} \) exists no point of which belongs to \( K_\varepsilon \). Let \( x_{\sigma_\varepsilon} \) be the point of least subscript in \( S_1 \). Likewise there exists a component \( K_\varepsilon \subset C_{\varepsilon_\sigma} - x_{\sigma_\varepsilon} \) which contains a point \( p_{\sigma\varepsilon} \) of \( Q \cdot R_{\sigma} \) and is such that an infinite subset \( S_2 \subset S_1 - x_{\varepsilon_\sigma} \) exists no point of which belongs to \( K_\varepsilon \). Let \( x_{\sigma_\varepsilon} \) be the point of least subscript in \( S_2 \). Again there exists a component \( K_\varepsilon \subset C_{\varepsilon_\sigma} - x_{\varepsilon_\sigma} \) containing a point \( p_{\sigma\varepsilon} \) of \( Q \cdot R_{\sigma} \) and such that an infinite subset \( S_3 \subset S_2 - x_{\varepsilon_\sigma} \) exists no point of which belongs to \( K_\varepsilon \), and so on. Let us continue this process indefinitely. The sequence \( p_{\varepsilon_1}, p_{\varepsilon_2}, \ldots \) has at least one limit point \( p \), which must belong to \( Q \cdot R_\sigma \). Since \( \overline{R}_\sigma \) is locally connected and \( F(G) \supseteq \varepsilon \), it follows that some two points of the sequence \( \{p\} \), say \( p_j \) and \( p_k \), where \( j < k \), lie together in a connected subset \( H \subset G \cdot R_\sigma \). Therefore \( H \supset K_\varepsilon + x_{\varepsilon_\sigma} \) is a connected subset of \( G \cdot R_{\sigma_\varepsilon} - x_{\varepsilon_\sigma} \),
because $K_x + x_a \subseteq R_{x_a} \subseteq R_{x_j}$, hence this set is a subset of $K_j$, since it contains the point $p_j$ of $K_j$. But this is contrary to the fact that $x_a$ is a point of $S_j$ and no point of $S_j$ belongs to $K_j$. This contradiction proves our theorem.

**Corollary.** If $p$ is a point of a locally connected continuum $M$ which has no local separating point and $d$ is a number such that $V_d(p)$ is compact, then the set $E$ of all positive numbers $\epsilon < d$ such that $R_{\epsilon}$ has a local separating point is countable.

(3.2). **Theorem.** If $K$ is any compact subcontinuum of a locally connected continuum $M$ having no local separating point and if we use the same notation as in (2.3), then at most a countable number of the sets $R_{\epsilon}(K)$ can have local separating points. Thus $K$ can be imbedded in a compact locally connected subcontinuum of $M$ which has no local separating point and whose diameter is equal to $K$ by an arbitrarily small amount.

This theorem is proved from (2.3) by the same argument as just given to prove (3.1) from (2.1) and (2.2).

4. **Theorem.** Each non-cut point of a locally connected continuum $M$ is contained in an arbitrarily small $S$-region whose exterior is connected.

**Proof.** Let $p$ be any non-cut point of $M$ and let $d$ be any positive number which is small enough that $V_d(p)$ is compact. There exists a region $R \subseteq V_d(p)$ containing $p$ whose exterior $H$ is connected; and there exists a positive number $\epsilon$ such that for all positive numbers $\epsilon < \xi$, the set $R_\epsilon$ is a subset of $R$. For any such $\epsilon$, let $H_\epsilon$ be the component of $M - R_\epsilon$ containing $H$, let $Q_\xi$ be the component of $M - H_\epsilon$ containing $H_\epsilon$, and let $X_\epsilon = F(Q_\xi)$.

To this end let $\delta$ be any positive number. Now $Q_\xi$ is locally connected, since $Q_\xi = R_\epsilon + \xi$ a certain group of components of $M - R_\epsilon$, and $R_\epsilon$ is locally connected. Therefore, as $Q_\xi$ is a subset of $V_d(p)$ and thus is compact, it therefore has property $S$. Hence, $Q_\xi = Q_\xi + \xi Q_\xi + \ldots + Q_\xi$, where, for each $i$, $Q_\xi$ is a locally connected continuum of diameter $< \delta/4$. Let $Q_{\xi_1}, Q_{\xi_2}, \ldots, Q_{\xi_k}$ be those sets $Q_\xi$ which contain points of $X_\epsilon$, and let $T$ be the sum of all those sets $Q_\xi$ which contain no point of $X_\epsilon$. Then $T$ is a subset of $Q_{\xi_k}$, and since it is a compact locally connected continuum and therefore has property $S$, it follows that the set $T + (R_{\xi_k} - X_\epsilon)$ has property $S$, because $R_{\xi_k} - X_\epsilon$ has property $S$. Thus we can write $T + (R_{\xi_k} - X_\epsilon) = W_1 + W_2 + \ldots + W_n$, where each set $W_i$ is connected and of diameter $< \delta/4$. For each $i, 1 \leq i \leq k$, let $V_i$ be the set of all those points of $Q_\xi$ which can be joined to some point of $W_i$ by a connected subset of $Q_\xi$ of diameter $< \delta/4$. Then clearly each set $V_i$ is a connected subset of $Q_\xi$, of diameter $< \delta$. It remains only to show that every point of $Q_\xi$ belongs to at least one set $V_i$. Let $x$ be any point of $Q_\xi$ which, we may suppose, does not belong to $T + (R_{\xi_k} - X_\epsilon)$. Then $x$ must belong to some set $Q_{\xi_i}$, say to $Q_{\xi_i}$. Let $C$ denote the component of $Q_{\xi_i} - R_{\xi_k} - Q_{\xi_i}$ which contains $x$.

Then $R_{\xi_k}$ contains at least one limit point $y$ of $C$, and $y$ cannot belong to $X_\epsilon$, because in this case $C$ would belong to $H_\epsilon$ as $X_\epsilon \subseteq H_\epsilon$. Thus $y$ belongs to $R_{\xi_k} - X_\epsilon$ and hence belongs to some set $V_i$, say to $V_i$. And since $C + y$ is a connected subset of $Q_\xi$ of diameter $< \delta/4$, it follows by definition of $V_i$ that $x$ belongs to $V_i$. Hence it is proved that $Q_\xi$ has property $S$.

Now the collection of sets $[X_\epsilon]$ is non-separated.

Now, the collection of sets $[X_\epsilon]$ is non-separated.

For since, if $\sigma < \epsilon$, $X_\epsilon \subseteq R_\epsilon \subseteq R_\epsilon$, $X_\epsilon$ cannot separate $X_\sigma$; and since from $R_\epsilon \subseteq R_\epsilon$, $R_\epsilon - X_\epsilon$ contains $H_\epsilon$, and hence $X_\epsilon \subseteq H_\epsilon$, $X_\epsilon - X_\epsilon$ cannot separate $X_\epsilon$. Therefore a theorem of non-separated cuttings applies to give us the connected continua, one of which is $Q_\xi$, say into $U_\xi$ and $U_\sigma$, where $U_\xi = Q_\xi$, and is such that $X_\epsilon = F(U_\xi) = F(U_\sigma)$. Thus, $Q_\xi$, or $Q_\xi$, is an $S$-region of diameter $< \delta$ whose exterior $U_\xi$ is connected, and the theorem is proved.

**Corollary.** Any point of a locally connected continuum is contained in an arbitrarily small $S$-region whose exterior is the sum of a finite number of connected sets.

1) That is, for any two $\epsilon$, say $\epsilon$ and $\eta$, $X_\epsilon$ lies wholly in a single component of $M - X_\eta$; see the author's paper on non-separated cuttings in Trans. Amer. Math. Soc., vol. 33 (1931), pp. 444-454.

For let \( p \) be any point of such a continuum \( M \) and let \( d \) be any positive number small enough that \( V_d(p) \) is compact. Let \( C_1, C_2, \ldots, C_n \) be the components of \( M \) with \( p \) which are of diameter \( > d/4 \). For each \( i, 1 \leq i \leq n \), \( p \) is a non-cut point of the locally connected continuum \( C_i + p \). Hence, by the theorem, there exists an \( S \)-region \( R_i \) in \( C_i + p \) of diameter \( < d/4 \), which contains \( p \) and whose exterior \( (C_i + p) - R_i \) is connected. Now \( M = \bigcup_{i=1}^n C_i \) is a compact locally connected continuum and hence has property \( S \). Therefore the set \( \bigcup_{i=1}^n R_i \) has property \( S \). Hence \( G \) is an \( S \)-region in \( M \) containing \( p \) of diameter \( < d \), and whose exterior is the sum of the connected sets \( ((C_i + p) - R_i) \).

5. In this section we consider regions in a general locally connected continuum. The proof for the theorems given here are independent of the results in the proceeding sections.

(5.1). Theorem. Any compact region \( R \) in a locally connected continuum whose boundary is totally disconnected is an \( S \)-region.

For if \( \varepsilon \) is any given positive number, it follows with the aid of the Borel Theorem together with the fact that \( \overline{R} \) is locally connected that \( F(R) \) is contained in the sum \( W \) of a finite number of connected regions \( W_1, W_2, \ldots, W_n \) in \( \overline{R} \) each of diameter \( < \varepsilon \) and such that the boundary of no one of these sets contains a point of \( F(R) \), i.e., \( \overline{R} \cdot F(W_i) \subset R \) for every \( i \). Likewise \( R \cdot W \) is contained in the sum of a finite number of connected regions \( V_1, V_2, \ldots, V_k \) each of which is a subset of \( R \) and of diameter \( < \varepsilon \). Now since, for each \( i, 1 \leq i \leq n \), \( W_i \) is connected and \( F(W_i) \cdot F(R) = 0 \), it follows that there are only a finite number of components of \( R \cdot W_i \). Thus \( R \cdot W \) is the sum of a finite number of connected sets of diameter \( < \varepsilon \), namely, the components of \( R \cdot W_i \) for \( 1 \leq i \leq n \), and these sets together with the sets \( V_1, V_2, \ldots, V_k \) contain \( R \). Therefore \( R \) is the sum of a finite number of connected sets of diameter \( < \varepsilon \) and hence has property \( S \).

Now let \( Q \) be any region in a locally connected continuum, which we shall suppose is compact, because we are interested only in interval properties of \( Q \) and clearly any region in a locally connected continuum is homeomorphic with a region in some compact locally connected continuum. Let \( W^* \) be the space whose points are:

(a) the points of \( Q \) and (b) the components of \( F(Q) \), and whose neighborhoods are (1) all open subsets of \( Q \) and (2) all open subsets of \( Q \) whose boundaries contain no points of \( F(Q) \). It is seen at once that \( W^* \) is a compact locally connected continuum and that the set \( Q^* = W^* \cdot \{ \text{all points of class (b)} \} \) is a region in \( W^* \) whose boundary is totally disconnected. Therefore, by (5.1), \( Q^* \) has property \( S \). But since \( Q \) and \( Q^* \) have the same points and since any \( Q \)-neighborhood of a point \( p \) of \( Q \) is also a \( W^* \)-neighborhood of \( p \) and any \( W^* \)-neighborhood of \( p \) contains a \( Q \)-neighborhood of \( p \), it follows that \( Q \) and \( Q^* \) are homeomorphic. Thus we have proved that:

(5.2). Any region in a locally connected continuum is homeomorphic with an \( S \)-region in some locally connected continuum.

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