Some properties of derivative functions.

By

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The first investigations of properties belonging to the remarkable class of derivative functions are to be found in the work of Cauchy, the great foundator of modern Differential Calculus. The study of them has been pursued by Duhamel, Dini and has attained to a very considerable development in our time especially by the work of Lebesgue and his successors. In the famous book of Lebesgue „*Sur l'intégration*“ we can find nearly all results of the known properties of these functions and we are able to recognise how many questions are hitherto unsolved.

The modest scope of my paper is to pursue in some points the considerations of the subject.

1. To begin with the precise definition we put the following:

(α) A function of one real variable $f(x)$, defined in the interval $[ab]$ {limits both included} is called „Duhamelian“ or briefly (D) in the interval, if there exists a function $F(x)$ such that:

$$f(x) = \frac{d}{dx} F(x) \quad \text{in} \quad [ab],$$

$f(x)$ being finite in any point of $[ab]$.

(β) We call $f(x)$ a quadratic Duhamelian or (D^2) if not only $f(x)$ but also $f^2(x)$ is (D) in $[ab]$.

(γ) The set of limited (D)'s or (D^2)'s we distinguish by (Dl) ev. (D^2l).

2. I come now to a brief account of the known properties concerning (D). Demonstrations are to be found in the quoted book of

Lebesgue and (partly) in Dini's „Fondamenti per una teoria delle funzioni“.

(1) Any continuous function in $[ab]$ is (D) in the same interval.

(2) Any (D) is of the first class in the classification of Baire.

(2^{bis}) Any (D) is almost pointwise discontinuous in respect of any perfect set of points in $[ab]$.

(3) The discontinuities of a (D) are (if at all) of the second kind.

(4) Any (Dl) in $[ab]$ is continuous in $[ab]$ in the sense considered by Darboux, i. e. it takes every intermediate value between the values on the extremes of any subintervall of $[ab]$.

(5) If (α) $f_n(x)$ are (D) in $[ab]$ $n=1,2\ldots$

(β) $\lim\limits_{n\to\infty} f_n(x) = f(x)$ uniformly in $[ab]$,

then: $f(x)$ is (D) in $[ab]$.

(6) Any (D) is totalisable in the sense of Denjoy and the necessary and sufficient condition for a $f(x)$ to be (D) in $[ab]$ is

$$\frac{d}{dx}\int\limits_a^x f\,dx = f(x) \quad \text{in } [ab],$$

where the integrale is taken in the widest sense of Denjoy (Cf. Denjoy, C. R. de Paris 1912).

(7) If $f(x)$ and $g(x)$ are (D) in $[ab]$ there are also $Af+Bg$, where A, B are arbitrary constants.

3. We come next to the consideration of the product of two (D) or (Dl).

The curious fact, that the product of two (D) even of two (Dl) may be not a (D), must be first exhibited.

It will be sufficient to prove that a quadrat of some (Dl) is not (D). But this is shown by the following example:

Take: $F(x) = x^2 \sin\frac{1}{x}$ in $[0,1]$ $(F(0)=0)$,

then $$f(x) = F'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & \text{if } x \neq 0 \\ = 0 & \text{if } x = 0, \end{cases}$$

but $$\varphi(x) = \begin{cases} 2x\sin\frac{1}{x} & \text{for } x \neq 0 \\ = 0 & \text{for } x = 0, \end{cases}$$

is continous in $[ab]$ and therefore (D).

Then:

$$\Phi(x) = \varphi(x) - f(x) = \cos\frac{1}{x} \quad x \neq 0$$
$$= 0 \quad x = 0,$$

is (D) in $[0,1]$.

After this take

$$\Phi^2(x) = \tfrac{1}{2}\left(1 + \cos^2\frac{2}{x}\right) \quad x \neq 0$$
$$= 0 \quad x = 0.$$

If $\Phi^2(x)$ were (D) then exists $A(x)$ such that

$$\frac{d}{dx}A(x) = \Phi^2(x) \quad \text{in } [0,1],$$

$\Phi(x)$ is (D), then if $B'(x) = \Phi(x)$ in $[ab]$ we have:

$$\frac{d}{dx}B\left(\frac{x}{2}\right) = \tfrac{1}{2}\left(\frac{d}{dz}B(z)\right)_{z=\frac{x}{2}}$$
$$= \tfrac{1}{2}\cos^2\frac{2}{x} \quad x \neq 0$$
$$= 0 \quad x = 0.$$

We shall have

$$\frac{d}{dx}A(x) = \tfrac{1}{2} + \frac{d}{dx}B\left(\frac{x}{2}\right) \quad x \neq 0$$
$$= 0 + \frac{d}{dx}B\left(\frac{x}{2}\right) \quad x = 0,$$

then:

$$\frac{d}{dx}\left[A(x) - B\left(\frac{x}{2}\right)\right] = \tfrac{1}{2}$$
$$= 0 \quad \text{shall be } (D)$$

but it is impossible for we have a discontinuity of the first kind (§ 2 (3)).

The non-Duhamelian character of $\varphi^2(x)$ is by this demonstrated![1]

From this we conclude that before some restrictions are not given we shall not expect generally a (D) in the product of (D) factors.

These restrictions are f. ex. includel in the following theorem.

[1]) The example by Lebesgue op. cit. p 95 is erroneous.

Prop. I. If 1. $f(x)$ and $g(x)$ are (Dl) in $[ab]$,
2. In any point of $[ab]$ either $f(x)$ or $g(x)$ is upper or lover semidiscontinuous (in the sense of Baire),

then the product $f(x)\,\varphi(x)$ is (D) in $[ab]$.

Dem: We take z in $[ab]$ and suppose $g(x)$ to be f. ex. upper semidiscontinuous in z.

Consider:

$$I_h = \frac{1}{h}\int_z^{z+h} [f(x)\,g(x) - f(z)\,g(z)]\,dx \quad \begin{cases} h \gtrless 0 \\ z+h \text{ in } [ab]. \end{cases}$$

May be:

$$I_h = I_h + K_k,$$

$$I_h = \frac{1}{h}\int_z^{z+h} [f(x) - f(z)]\,g(z)\,dx,$$

$$K_h = \frac{1}{h}\int_z^{z+h} f(x)\,[g(x) - g(z)]\,dx.$$

Then:

(α) $$\lim_{h=0} I_h = g(z) \lim_{h=0} \frac{1}{h}\int_z^{z+h} [f(x) - f(z)]\,dx$$

$$= 0 \quad \{f(x) \text{ being } (D).$$

(β) We take $\varepsilon > 0$ and $|h|$ sufficiently small to be

$$g(z) - g(x) < \varepsilon \quad \text{in } [z, z+h]$$

{on the bare of upper semi-disc.}.

(γ) Let be: $K_h = L_h + M_h$

$$L_h = \frac{1}{h}\int_z^{z+h} f(x)\,\{g(x) - g(z) + \varepsilon\}\,dx$$

$$M_h = \frac{1}{h}\,\varepsilon\int_z^{z+h} f(x)\,dx,$$

then:

(δ) $$\lim_{h=0} M_h = \varepsilon f(x),$$

(ε) $$g(x) - g(z) + \varepsilon \geqq 0,$$

also by the first theorem of medium

$$L_h = \frac{\theta_h}{h} \int_z^{z+h} [g(x) - g(z) + \varepsilon]\, dx \quad |\theta_h| < A$$

where $A > |f(x)|$ in $[a\,b]$.

$$(\vartheta) \qquad \lim_{h=0} \frac{1}{h} \int_z^{z+h} [g(x) - g(z) + \varepsilon]\, dx = \varepsilon.$$

We take $|h|$ sufficienty small to be

$$|I_h| < \varepsilon \qquad |M_h| < \varepsilon\, |f(x)| + \varepsilon,$$
$$|L_h| < A \,.\, 2\varepsilon,$$

then:

$$|I_h| < (3A + 2)\,\varepsilon.$$

We see:

$$\lim_{h=0} I_h = 0.$$

Also $f(x)\,g(x)$ is in every z in $[a\,b]$ thé differential-coefficient of its integral and $f(x)\,g(x)$ is (D) in $[a\,b]$.

Corr: If $f(x)$ is (Dl) and $g(x)$ continuous in $[ab]$ then $f(x)\,g(x)$ is (Dl) in the same interval.

The next step in these questions shall be made after some considerations of (D^2) to which I now pass over.

4. Consider the class of (D^2). It is obvious that:

(α) If $f(x)$ and $g(x)$ are (D^2) in $[ab]$ there are also Af and $f(x) + A$ where A is an arbitrary constant.

(β) Every continuous function is (D^2).

(γ) If $f(x)$ is (D^2) in $[ab]$ then it is summable (L) with its quadrat in $[ab]$.

Dem: $f^2(x)$ shall be totalisable, but $f^2 \geqq 0$ then summable in $[ab]$ — also $f(x)$ must be summable every (D) being mesurable.

Definition. 1. Take $f(x)$ defined in the neighbourhood of $x = z$. Denote by $K_\delta(\varepsilon, z)$ the set of points in this neighbourhood where:

$$|f(x) - f(z)| > \varepsilon$$

being

$$z < x \leqq z + \delta \quad \text{if} \quad \delta > 0$$

r

$$z > x \geqq z + \delta \quad \text{if} \quad \delta < 0.$$

2. Then if:

$$\lim_{\delta=0} \frac{m' K_\delta(\varepsilon, z)}{|\delta|} = 0 \qquad \left\{ \begin{array}{l} m' K^{e} \text{ is the } (L) \\ \text{mesure of } K! \end{array} \right.$$

for any $\varepsilon > 0$

we say: $f(x)$ is asymptotically continuous in z.

$f(x)$ asympt. cont. in every z in $[ab]$ is called asympt. cont. in $[ab]$.

It is obvious that the function which is asymptotically continuous in $[ab]$ is also mesurable in the interval considered.

We can now prove the next fundamental theorem.

Prop. I. The necessary and sufficient condition for a limited function, to be $(D^2 l)$ in $[ab]$ is its asymptotical-continuity in $[ab]$.

Dem: 1. Sufficiency.

Let be: $f(x)$ asympt. cont. in $[ab]$.

Consider:

$$I_h = \frac{1}{h} \int_z^{z+h} \{f(x) - f(z\} \, dz,$$

$$I_h = \frac{1}{h} \int_{K_h(\varepsilon, z)} \{f(x) - f(z)\} \, dx + \int_{L_h(\varepsilon, z)} \{f(x) - f(z)\} \, dx$$

$$= A_h + B_h \qquad \left\{ \begin{array}{l} L_h = [z, z+h] - K_h \\ z \text{ arbitrary in } [ab] \end{array} \right.$$

but: in L_h we have. $|f(x) - f(z)| \leqq \varepsilon$ therefore:

$$|B_h| \leqq \frac{m' L_h}{|h|} \varepsilon \leqq \varepsilon,$$

$$|A_h| \leqq \frac{m' K_h}{|h|} \cdot 2A \qquad A > |f(x)| \quad \text{in } [ab],$$

then:

$$|I_h| \leqq \frac{m' K_h}{|h|} 2A + \varepsilon,$$

$$\lim_{h=0} I_h = 0$$

i. e.: $f(x)$ is (D) in $[ab]$.

Take now:

$$P_h = \frac{1}{h} \int_z^{z+h} [f^2(x) - f^2(z)] \, dx$$

We have:

$$P_h = \frac{1}{h}\int_z^{z+h} (f(x) - f(z)) (f(x) + f(z))\, dx,$$

$$|P_h| \leqq \frac{1}{|h|}\left|\int_z^{z+h} |f(x) - f(z)| - |f(x) + f(x)|\, dx\right.$$

$$\leqq \frac{2A}{|h|}\left|\int_z^{z+h} |f(x) - f(z)|\, dx.\right.$$

From this point onward the demonstration proceeds in analogous manner to the indicated above:

We split:

$$\frac{1}{h}\int |f(x) - f(z)|\, dx = \frac{1}{|h|}\int_{K_h} + \frac{1}{|h|}\int_{L_h} = Q_h + R_h,$$

$$|Q_h| \leqq \frac{m' K_h}{|h|} \cdot 2A \qquad |R_h| \leqq \varepsilon.$$

Then:

$$|P_h| \leqq 4A \cdot \frac{m' K_h}{|h|} + 2A\varepsilon,$$

$$\lim_{h=0} P_h = 0 \quad \text{therefore } f(x) \text{ is } (D^2).$$

II. Necessity.

Let be $f(x)$ $(D^2 l)$ in $[ab]$ and z arbitrary in $[ab]$.

Then $f(x) - f(z)$ is also $(D^2 l)$ and we shall have:

$$\lim_{h=0} I_h = \lim_{h=0} \frac{1}{h}\int_z^{z+h} \{f(x) - f(z)\}^2\, dx = 0.$$

Let be

$$A_h = \frac{1}{|h|}\int_{K_h(\varepsilon, z)}; \qquad B_g = \frac{1}{|h|}\int_{L_h(\varepsilon, z)} \qquad \text{as above.}$$

We have:

$$I_h = A_h + B_h,$$

$$A_h \geqq \varepsilon^2 \frac{m' K_h}{|h|} \qquad B_h < \varepsilon^2 = \lim B_h = \lim I_h = 0.$$

then:

$$\lim_{h=0} \frac{m' K_h(\varepsilon, z)}{|h|} = 0.$$

The following theorem that I regard as a key to the whole importance of (D^2l) is due to my devoted friend St. Banach:

Prop. II. If: (1) $f(x)$ is (Dl) in $[ab]$,
(2) $f(x)$ is (D^2l) in $[ab]$,
then: $f(x)\,g(x)$ is (D^2l) in $[ab]$.

Dem: Take z in $[ab]$ and put

$$I_h = \frac{1}{h}\int\limits_z^{z+h} [f(x)\,g(x) - f(z)\,g(z)]\,dx,$$

$$I_h = \frac{1}{h}\int\limits_z^{z+h} [g(x) - g(z)]\,f(x)\,dx,$$

$$L_g = \frac{1}{h}\int\limits_z^{z+h} g(z)\,\{f(x) - f(z)\}\,dx,$$

$$I_h = I_h + L_h,$$

then:

$$L_h = g(z)\cdot\frac{1}{h}\int\limits_z^{z+h} [f(x) - f(z)]\,dx,$$

$$\lim_{h=0} L_h = 0 \qquad \{\text{because } f(x) \text{ is } (D)\}$$

$$|I_h| \leqq \frac{A}{h}\int\limits_z^{z+h} |g(x) - g(z)|\,dx \qquad A \geqq |f(x)| \text{ in } [ab],$$

$$I_h \leqq \frac{A}{h}\sqrt{h\int\limits_z^{z+h} (g(x) - g(z))^2\,dx} \quad \Big\{ \text{ by the famous Schwarzian inequality}$$

$$I_h \leqq A\sqrt{\frac{1}{h}\int\limits_z^{z+h} (g(x) - g(z))^2\,dx},$$

$$\lim_{h=0} I_h = 0.$$

Then: $\lim\limits_{h=0} I_h = 0$ i. e. $f(x)\,g(x)$ is (D) in $[ab]$.

What follows is only an application of the Banach-theorem.

Prop. III. If $f(x)$ is (D^2l) in $[ab]$ then $f^n(x)$ where n is any positive integer is also (D^2l).

Dem: By the Prop. II if $f(x)$ and (D^2l) and $f(x)^k$ (Dl), then also $f(x)^{k+1}$ is (Dl).

But $f(x)$ in D^2l — then by the principle of induction every $f^n(x)$ is (Dl), then also (D^2l). The theorem is therefore proved.

Prop. IV. If (1) $f(x)$ is (D^2l) in $[ab]$,

(2) $F(y)$ is continuous in the interval of variability of $f(x)$,

than: $\Phi(x) = F\{f(x)\}$ is also (D^2l) in $[ab]$.

Dem: Develop $F(y)$ in a sequence of uniformly convergent polynomials as follows:

$$P_1(y)\, P_2(y) \ldots P_n(y) \ldots$$

$$\lim_{n=\infty} P_n(y) = F(y).$$

If we put

$$\pi_n(x) = P_n\{f(x)\},$$

then by prop. II. $\pi_n(x)$ is (D^2l) and the sequence

$$\pi_2(x) \ldots \pi_n(x) \ldots$$

converges uniformly to $\Phi(x)$.

By the theorem in § 1 we have $\Phi(x)$ is (Dl).

Take: $G(y) = F^2(y)$, then $G(y)$ is also continuous and by what preceeds we have:

$$\Psi(x) = G\{f(x)\} = \Phi^2(x) \text{ is } (Dl) \text{ in } [ab].$$

These theorems prove the identity of every class $(D^2l)\,(D^3l)\ldots$ $(D^nl)\ldots$ with (D^2l) and als (CD^2l) i. e. continuous transformation of (D^2l).

5. A further theorem concerning (D^2) can be stated as follows:

Prop.: If $f(x)$ and $g(x)$ are (D^2) in $[ab]$ then $f(x)\,g(x)$ is (D) in the same interval.

Dem. Put:

$$I_h = \frac{1}{h}\int_z^{z+h} \{f(x)\,g(x) - f(z)\,g(z)\}\,dx,$$

$$I_h = \frac{1}{h}\int_z^{z+h} \{f(x) - f(z)\}\,g(x)\,dx + \frac{1}{h}\int_z^{z+h} f(z)\,\{g(x) - g(z)\}\,dx$$

$$= A_h + f(z)\,B_h.$$

But:

$$(\alpha) \qquad \lim_{h=0} B_h = 0,$$

$$(\beta) \qquad |A_h| \leq \sqrt{\lim \frac{1}{h}\int_z^{z+h} [f(x)-f(z)]^2\,dx \,.\, \lim \frac{1}{h}\int_z^{z+h} g^2\,dx} = 0.$$

This implies: $\lim I_h = 0$ i. e. $f(x)\,g(x)$ is (D) in $[ab]$.

6. Consider $f(x)$ and $g(x)$ both (D) in $[ab]$. If $f(x)-g(x)$ vanishes in $[ab]$, with possible exception of a set of points of mesure zero, we can affirm the identity of two functions in this interval. The truth of this theorem is evident if we remember that taking

$$F'(x) = f(x), \qquad G'(x) = g(x) \quad \text{in } [ab]$$

we have

$$f(x) - g(x) = (F - G)'$$

and by a theorem of De la Vallée Poussin [cf. his Cours d'Analyse V. I, 3. ed.] if a derivative of a function is always finite and vanishes „à mesure nulle près" then it vanishes identically — Then we have $f(x) \equiv g(x)$.

From this we conclude: If in a function (D) we change the definition in a set of mesure zero the changed function is not more (D).

Remains a question. Suppose we have a mesurable function $f(x)$ in $[ab]$, which is a (non-D) Can we change this function in a set of zero mesure to obtain a (D)?

The answer must be in some caser negative. Because if the changed function is $g(x)$ than:

$$F(x) = \int_a^x f\,dz \equiv \int_a^x g\,dz$$

and

$$F'(x) = g(x) \text{ in } [ab].$$

Then if the indefinite integral of our $f(x)$ is a function which has no derivative in some points of the interval the problem must remain impossible.

Further development of the subject I leave to the next paper.