stration de M. Lindenbaum (qui paraîtra dans le vol. XX de ce journal) est cependant fort compliquée.

Les théorèmes I et II peuvent être sans peine généralisés. Au lieu des images continues on peut notamment prendre les images de Baire et, plus généralement, une famille quelconque de puissance du continu de transformations des ensembles à l'aide de fonctions mesurables d'une variable réelle. En effet, on voit sans peine que si $f(x)$ est une fonction mesurable d'une variable réelle, dont l'ensemble de valeurs est non dénombrable, il existe toujours un nombre réel $a$, tel que l'ensemble de tous les $x$ réels, pour lesquels $f(x)=a$, est de mesure nulle. La démonstration que nous avons donné pour le théorème I s'applique donc dans ce cas.

Or, le problème se pose: les théorèmes I et II restent-ils vrais pour les familles de puissance du continu (ou, seulement, pour les familles dénombrables) de transformations des ensembles à l'aide de fonctions quelconques d'une variable réelle? D'un théorème que nous avons trouvé récemment avec M^lle^ Braun[1]) résulte que ce n'est pas le cas pour le théorème II. En effet, comme nous avons démontré, si $2^{\aleph_0}=\aleph_1$, il existe une suite infinie de fonctions d'une variable réelle, $f_1(x)$, $f_2(x)$, $f_3(x), \ldots$, telle que, quel que soit l'ensemble linéaire non dénombrable $N$, il existe un indice $k$ (dépendant de $N$), tel que la fonction $f_k(x)$ transforme $N$ en l'ensemble de tous les nombres réels.

En s'appuyant sur ce résultat, on voit tout de suite que *la négation de l'hypothèse du continu équivaut à la proposition suivante*:

*$F$ étant une famille de puissance du continu d'ensembles linéaires de puissance du continu, et $\Phi$ une famille de puissance du continu de fonctions d'une variable réelle, il existe toujours un ensemble linéaire non dénombrable, $E$, tel que toute fonction de la famille $\Phi$ transforme $E$ en un ensemble distinct de tout ensemble de la famille $F$*[2]).

[1]) ce volume, p. 1.

[2]) Cf. ma communication au II Congrès de Mathématiciens Roumains à Turnu Severin, Mai 1932.

# On the functions of Besicovitch in the space of continuous functions.

By

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1. In their very interesting new proofs for the existence of continuous functions without derivatives, Banach and Mazurkiewicz showed that the class of continuous functions without finite one-sided derivative in any point is the complement of a set of the 1^st^ category of Baire in the space $\mathfrak{C}$ of continuous functions[1]). The same method, and with the same result, could be applied to the evaluation of the class of continuous functions without both-sided finite or infinite derivatives in any point which we shall call briefly functions of Weierstrass' type.

The problem was set by Banach and Steinhaus whether these results may be extended to the functions of Besicovitch's type i. e. continuous functions without one-sided derivatives (finite or infinite) in any point. We shall give here a negative answer to this problem, showing in the first part of this paper that the complement of the class of Besicovitch's functions is everywhere of the 2^nd^ category in the space $\mathfrak{C}$.

Banach has informed me in a letter that this theorem may be considerably strenghtened. First of all, as showed by this author,

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[1]) Mazurkiewicz, *Sur les fonctions non dérivables*, *Studia Mathematica*, t. III, (1931), pp. 92–94; Banach, *Über die Baire'sche Kategorie gewisser Funktionenmengen*, ibid., pp. 174—179; Steinhaus, *Anwendungen der Funktionalanalysis auf einige Fragen der reellen Funktionentheorie*, ibid., t. I (1929), pp. 51—81.

the class of Besicovitch's functions is the complement of an analytic set (or of a Souslin set or a set $(A)$)[2]) and therefore satisfies the property of Baire[3]). Consequently a sphere $S$ exists in the space $\mathfrak{C}$ where either this class or its complement is of the 1st category. But by the theorem which will be proved below, this complement is everywhere of the 2nd category and therefore the class of Besicovitch's functions must be of the 1st category in $S$. Now, being of the 1st category in one sphere, it is as we see easily of the same category in every other sphere and therefore in the whole space $\mathfrak{C}$.

This result shows that the class of Weierstrass' functions being the complement of a set of the 1st category is in a certain sense much larger than that of Besicovitch's functions which is of the 1st category itself. This explains perhaps the difficulties connected with finding the first example of a continuous function without either finite or infinite one-sided derivatives in any point[4]).

2. We shall extend this theorem in the second part of this paper and prove that every continuous function with the exception of a class of the 1st category in the space $\mathfrak{C}$, has a right-sided derivative $+\infty$ in a non-denumerable set of points. This assertion reduces to the proof that the set of such functions has the property of Baire, and then the complete proof follows by the same argument of Banach and by the theorem of § 4.

The first part of this paper is quite elementary. The second although shorter is really much less elementary because substantially based on the general properties of analytic sets and on a new method developed by Tarski and Kuratowski[5]) for the evaluation of classes of sets in abstract spaces.

[2]) For the definition of these sets see for instance Hausdorff, *Mengenlehre* 2. Aufl. 1927.

[3]) Nikodym O., *Sur une propriété de l'opération A*, *Fund. Math.* t. VII, (1925), pp. 149—154; Szpilrajn E., *O mierzalności i warunku Baire'a* (in Polish) *C. R. du Ier Congrès de Mathématiciens des Pays Slaves* (1930) pp. 297—303.

[4]) The first example of such functions was given by Besicovitch in 1922 (in Russian). See Besicovitch, *Discussion der stetigen Funktionen in Zusammenhang mit der Frage über ihre Differenzierbarkeit*, *Bull. de l'Ac. des Sc. de Russie*, 1925, p. 527. The reasoning of Besicovitch was simplified by E. D. Pepper (*On continuous functions without a derivative*, *Fund. Math.*, t. XII, (1928), pp. 244—253).



## I.

3. **Lemma.** *Let $x(t)$ be continuous in an interval $(a, b)$ and let*

$$(1)\qquad |x(t)-mt+n| < \frac{b-a}{8}\varepsilon \quad (m>0,\ \varepsilon>0),$$

*for every $a \leqslant t \leqslant b$. Then there exists in the interval $\left(a, \frac{a+b}{2}\right)$ a non-denumerable set of points $c$ with the property that*

$$(2)\qquad x(t)-x(c) \geqslant (m-\varepsilon)(t-c)$$

*for every $c \leqslant t \leqslant b$*[6]).

Proof. Suppose that the set of points $c$ $\left(a \leqslant c \leqslant \frac{a+b}{2}\right)$ satisfying (1) is at most denumerable and let $\{c_i\}$ $(i=1,2,\ldots)$ be the sequence of these points. Let $\eta$ be an arbitrary positive number. Denote by $t_0$ the upper bound of the set $T$ of points $t$ $(a \leqslant t \leqslant b)$ verifying the inequality

$$(3)\qquad x(t)-x(a) \leqslant (m-\varepsilon)(t-a)+\sum_{n}^{(t)}\frac{\eta}{2^n},$$

where the summation $\sum\limits_{n}^{(t)}$ is extended to all such values of $n$ for which $c_n < t$. We assert first that

$$(4)\qquad t_0 \geqslant \frac{a+b}{2}.$$

Indeed if $t$ is an arbitrary point of $\left(a, \frac{a+b}{2}\right)$ verifying (3) but

[5]) Kuratowski et Tarski, *Les opérations logiques et les ensembles projectifs*, *Fund. Math.*, t. XVII (1931), pp. 240—248; Kuratowski, *Evaluation de la classe borélienne ou projective d'un ensemble de points à l'aide des symbols logiques*, *ibid.*, pp. 249—272.

[6]) For our purposes (see § 4) it suffices only that there exist two different points $c$ with this property.

not belonging to the sequence $\{c_i\}$ then there exists by assumption a point $u > t$ such that

$$x(u) - x(t) < (m - \varepsilon) \cdot (u - t),$$

and we obtain from (3)

$$x(u) - x(a) < (m - \varepsilon)\ (u - a) + \sum_n^{(u)} \frac{\eta}{2^n},$$

that is to say $u$ belongs to $T_0$. On the other hand, if a point $c_k$ satisfies (3) i. e. if

$$x(c_k) - x(a) \leqslant (m - \varepsilon)\ (c_k - a) + \sum_n^{(c_k)} \frac{\eta}{2^n}, \tag{5}$$

then there exists a point $u > c_k$ such that

$$x(u) - x(c_k) < \frac{\eta}{2^k}$$

and we infer from (3)

$$x(u) - x(a) \leqslant (m - \varepsilon)\ (c_k - a) + \sum_n^{(c_k)} \frac{\eta}{2^n} + \frac{\eta}{2^k}$$
$$< (m - \varepsilon)\ (u - a) + \sum_n^{(u)} \frac{\eta}{2^n}.$$

It means however that $u$ belongs also to $T_0$ and therefore no point of $\left(a, \frac{a+b}{2}\right)$ could be the upper bound of the set $T_0$. This proves the inequality (4).

Thus it follows now from (1), (3) and (4)

$$\frac{b-a}{4}\varepsilon > |[x(t_0) - mt_0 + n] - [x(a) - ma + n]|$$
$$> m(t_0 - a) - |x(t_0) - x(a)|$$
$$> \varepsilon(t_0 - a) - \eta$$
$$> \frac{b-a}{2}\varepsilon - \eta.$$

This is however contradictory as the positive number $\eta$ may be arbitrarily small.



4. **Theorem.** *The class of functions continuous in the interval* (0, 1) *with the right sided derivatives existing and* $+\infty$ *in a set of the power of the continuum, is everywhere of the* 2<sup>nd</sup> *category in the space* $\mathfrak{C}$ *of continuous functions.*

Proof. Let $K$ be an arbitrary open sphere in the space $\mathfrak{C}$ of functions continuous in (0, 1) and $\{A_n\}$ a sequence of nowhere dense sets in $\mathfrak{C}$. We shall show that there exists then a function belonging to $K - \sum_n A_n$ whose right-sided derivative exists and becomes positively infinite in a perfect set of points. This will prove our theorem.

We shall define for this purpose by induction a dyadic system of subintervals $\{I_{n_1, n_2, \ldots, n_j} = (a_{n_1, n_2, \ldots, n_j}, b_{n_1, n_2, \ldots, n_j})\}$ in the interval (0, 1), a sequence of continuous functions $x_j(t)$ $(0 \leqslant t \leqslant 1)$ and a sequence of open spheres $K_j$ in the space $\mathfrak{C}$ satisfying the following conditions:

A) $K_0 = K, \quad K_j \subset K_{j-1}, \quad \overline{K}_j \times \overline{A}_j = 0$ [7] $\quad (j \geqslant 1)$;

B) $x_j(t)$ is the center of $K_j$ $\quad (j = 0, 1, \ldots)$;

C) $I_{n_1, n_2, \ldots, n_{j-1}, n_j} \subset I_{n_1, n_2, \ldots, n_{j-1}}; \quad I_{n_1, n_2, \ldots, n_{j-1}, 0} \times I_{n_1, n_2, \ldots, n_{j-1}, 1} = 0;$
$|I_{n_1, n_2, \ldots, n_j}| \leqslant \frac{1}{j} \quad (j \geqslant 1;\ n_j = 0, 1)$ [8].

D) $x_j(t)$ is linear with the coefficient $j$ $[x_j(t) = jt + \text{const.}]$ in each interval $I_{n_1, n_2, \ldots, n_j}$ of the $j$ th order;

E) if $t$ belongs to $I_{n_1, n_2, \ldots, n_j}$ and $u$ to the interval $(b_{n_1, n_2, \ldots, n_i}, b_{n_1, n_2, \ldots, n_{i+1}})$ $(1 < i < j)$ then

$$\frac{x_j(u) - x_j(t)}{u - t} > i - 2.$$

Suppose that the functions $x_j(t)$, spheres $K_j$ and sub-intervals $I_{n_1, n_2, \ldots, n_j}$ have been determined already for $j = 1, 2, \ldots, r$ and obey the above conditions. We shall define a continuous function $x_{r+1}$, a sphere $K_{r+1}$ and a system of intervals of the $r+1$ th order.

First of all the set $A_r$ being nowhere dense there exist in every neighborhood of $x_r(t)$ continuous functions $g(t)$ belonging to $K_r - \overline{A}_r$. Since, by the condition (D) (for $j = r$), $x_r(t)$ is linear and has the

[7]) $\overline{A}$ denotes as usually the closure of $A$.
[8]) $|I|$ denotes the length of the interval $I$.

coefficient $r$ in each interval $I_{n_1,n_2,\dots,n_r}$ of the $r$th order, we can choose $g(t)$ sufficiently close to $x_r(t)$ so that according to our lemma (§ 3) there exist in each interval $I_{n_1,n_2,\dots,n_r}$ two points, say $b_{n_1,n_2,\dots,n_r,0}$, $b_{n_1,n_2,\dots,n_r,1}$, such that

$$(1)\qquad \frac{g(t)-g(b_{n_1,n_2,\dots,n_r,n_{r+1}})}{t-b_{n_1,n_2,\dots,n_r,n_{r+1}}}>r-1 \quad \text{for} \quad b_{n_1,n_2,\dots,n_r,n_{r+1}}<t<b_{n_1,n_2,\dots,n_r},$$

$$(n_{r+1}=0,\,1).$$

Then we slightly modify $g(t)$ by replacing it by a function $h(t)$ linear and with coefficient $j+1$ in a couple of distinct subintervals $\delta'_{n_1,n_2,\dots,n_r}$, $\delta''_{n_1,n_2,\dots,n_r}$ of $I_{n_1,n_2,\dots,n_r}$ whose right ends are the points $b_{n_1,n_2,\dots,n_r,0}$, $b_{n_1,n_2,\dots,n_r,1}$ respectively. We can choose these intervals sufficiently small less than $\frac{1}{j+1}$ and further $h(t)$ sufficiently close to $g(t)$ so that

a) $h(t)$ belong to $K_r-\bar{A}_r$,

and

$$\text{b)}\quad \frac{h(t)-h(b_{n_1,n_2,\dots,n_r,n_{r+1}})}{t-b_{n_1,n_2,\dots,n_r,n_{r+1}}}>r-1 \quad \text{for} \quad b_{n_1,n_2,\dots,n_r,n_{r+1}}<t<b_{n_1,n_2,\dots,n_r}$$

$$(n_{r+1}=0,\,1).$$

The function $h(t)$ chosen in this manner will be defined as $x_{r+1}(t)$, the corresponding intervals $\delta'_{n_1,n_2,\dots,n_r}$, $\delta''_{n_1,n_2,\dots,n_r}$ as the intervals of the $r+1$th order $I_{n_1,n_2,\dots,n_r,0}$, $I_{n_1,n_2,\dots,n_r,1}$ respectively. Finally, we can choose easily an open sphere $K_{r+1}$ with the center $x_{r+1}(t)$ and the radius less than $\frac{1}{r+1}$ so that

$$K_{r+1}\subset K_r,\quad \bar{K}_{r+1}\times\bar{A}_{r+1}=0.$$

We see at once that then the conditions (A—E) remain verified for $j=r+1$.

Now by the conditions (A, B) the sequence $x_j(t)$ converges uniformly to a continuous function $x(t)\in K-\sum_n A_n$. We shall show that $x'_+(t)=\infty$ in a perfect set of points.

Indeed, set for every sequence $n_1, n_2,\dots, n_j,\dots$ $(n_j=0,\,1)$

$$(2)\qquad t_{n_1,n_2,\dots,n_j}=I_{n_1}\times I_{n_1,n_2}\times\dots\times I_{n_1,n_2,\dots,n_j}\times\dots$$



We have then from (1)

$$\frac{x_j(u)-x_j(t_{n_1,n_2,\dots,n_k,\dots})}{u-t_{n_1,n_2,\dots,n_k,\dots}}>i-2$$

for every sequence $\{n_k\}$ $(n_k=0,\,1)$, $j>i>1$ and $b_{n_1,n_2,\dots,n_i}>u>$ $>t_{n_1,n_2,\dots,n_k,\dots}$. It follows by passing to the limit

$$\frac{x(u)-x(t_{n_1,n_2,\dots,n_k,\dots})}{u-t_{n_1,n_2,\dots,n_k,\dots}}\geqslant i-2$$

for $i=1,2,\dots$ and $b_{n_1,n_2,\dots,n_i}>u>t_{n_1,n_2,\dots,n_k,\dots}$. Thus

$$x'_+(t_{n_1,n_2,\dots,n_k,\dots})=+\infty$$

in every point (2). The set of these points being obviously perfect our proof is completed.

## II.

5. We shall now prove

**Theorem.** *The class of continuous functions which have the right-sided derivatives positively infinite in a set of the power* $\aleph$, *is an analytic set in the space* $\mathfrak{C}$ *and therefore has the* B a i r e *property.*

This will be an immediate consequence of the following.

**Lemma.** $\mathfrak{T}$ *denoting the interval* $(0, 1)$ *the set* $Q$ *of points* $(x, t)$ *in the product space* $\mathfrak{T}\times\mathfrak{C}$ *such that*

$$x'_+(t)=+\infty\ ^{9}),$$

*is* $F_{\sigma\delta}$.

P r o o f. Denote for each couple of natural numbers $m, n$ by $Q_{mn}$ the set of points $(x, t)$ such that

$$0<h\leqslant\frac{1}{m} \quad \text{implies} \quad \frac{x(t+h)-x(t)}{h}\geqslant n.$$

Then

$$Q=\prod_n\sum_m Q_{mn},$$

[9]) $x'_+(t)$ denotes the right-sided derivative of $x(t)$.

and since $P_{mn}$ are closed sets, $Q$ is a set $F_{\sigma\delta}$ and our lemma is proved.

Now let $R$ be the set of functions $x$ in $\mathfrak{C}$ for which the sets $\underset{t}{E}[(x, t) \epsilon P]$ ($P$ denotes the same set in the product space $\mathfrak{C} \times \mathfrak{T}$ as above) are of the power $\aleph$. There follows from a theorem of Mazurkiewicz and Sierpiński as generalized recently by Kuratowski that this set is analytic [10]. This proves the theorem stated at the beginning of this section as $R$ is exactly the class of continuous functions with right-sided derivatives $+\infty$ in a set of the power $\aleph$. Furthermore, by the reasoning indicated in the introduction, il follows now from the theorem of § 4 that *the class $R$ is the complement of a set of the 1st category* Q. E. D.

The theorem of Mazurkiewicz and Sierpiński which we have used, was stated by Kuratowski in the following general form for abstract spaces:

*If $Q$ is an analytic set in the product space $\mathfrak{X} \times \mathfrak{U}$ of two compact metric spaces $\mathfrak{X}$ and $\mathfrak{U}$, then the set $X$ of points $x$ in $\mathfrak{X}$ such that the corresponding sets $\underset{u}{E}[(x, u) \epsilon Q]$ are of the power $\aleph$ is also analytic.*

A very slight modification of the reasoning of Kuratowski shows that the assumption of the compactness of the space $\mathfrak{X}$ is superfluous in the above theorem; it suffices to suppose only that this space is complete [11]. Indeed, $Q$ being by assumption an analytic set, it is the projection of a set $G_\delta$, $W$, in the product space $\mathfrak{X} \times \mathfrak{U} \times \mathfrak{T}$ where $\mathfrak{T}$ denotes the interval $(0, 1)$. If for some $x \epsilon \mathfrak{X}$ the set $\underset{u}{E}[(x, u) \epsilon Q]$ is of the power $\aleph$, then it contains a perfect subset $P$ which is the projection on $\mathfrak{U}$ of a closed set $F$ in the space $\mathfrak{U} \times \mathfrak{T}$. The spaces $\mathfrak{U}$ and $\mathfrak{U} \times \mathfrak{T}$ being by assumption compact both classes $\mathfrak{P}$ of perfect sets $P \subset \mathfrak{U}$ and $\mathfrak{F}$ of closed sets $F \subset \mathfrak{U} \times \mathfrak{T}$ may be considered as complete metric spaces [12].

Therefore using the logical notations of Kuratowski we have

$$x \epsilon X \equiv \sum_P \sum_F \prod_{u,t} \{[(u, t) \epsilon F] \rightarrow [(x, u, t) \epsilon W] \cdot (u \epsilon P)\}$$

[10]) Mazurkiewicz et Sierpiński, *Sur un problème concernant les fonctions continues*, *Fund. Math.*, t. VI (1924), pp. 161—169; Kuratowski, l. c., pp. 261—262.

[11]) It is important in our case as the space $\mathfrak{C}$ of continuous functions is not compact.

[12]) As for the class $\mathfrak{F}$ see Hausdorff, l c. pp. 145—150. As for $\mathfrak{P}$ Banach proved that generally the class of perfect sets in a compact space (in our case in $\mathfrak{U}$) is a $G_\delta$ in the space of all closed subsets of that space (see Kuratowski, l. c., 260). This means however (see Hausdorff, l. c., pp. 214—216) that it may be itself „métrisable" as a complete space.



where $x \epsilon \mathfrak{X}$, $u \epsilon \mathfrak{U}$, $t \epsilon \mathfrak{T}$, $P \epsilon \mathfrak{P}$, $F \epsilon \mathfrak{F}$. It follows

$$X \equiv PPCPC(CF + G_\delta \cdot F)$$

what shows that $X$ is an analytic set.

The reader will find the complete explanations and details in the papers of Tarski and Kuratowski quoted in § 2, especially Kuratowski, pp. 261—2. The only difference between the above reasoning and that of Kuratowski is that we consider the closed sets $F$ in the product space $\mathfrak{U} \times \mathfrak{T}$ which is compact with $\mathfrak{U}$ and $\mathfrak{T}$, whereas Kuratowski deals in this respect in the whole space $\mathfrak{X} \times \mathfrak{T} \times \mathfrak{U}$ and subsequently assumes that $\mathfrak{X}$ should be also compact.

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