$M \cdot \overline{U}_p$. Then by the preceding paragraph, $M \cdot \overline{U}_p$ is a simple Peano continuum or is cyclicly connected. As it contains two end points, it cannot be cyclicly connected. Then $M \cdot \overline{U}_p = A_1$. Similarly $M \cdot (E_n - U_p) = A_2$. Thus M is the simple closed curve $J = A_1 + A_2$ and our proof is complete.

Corollary. *If every local cut point of a Peano continuum M in E_n is an ordinary point of M, then every point of M is of order c or every point is of order $\leqq 2$.*

The following simple example shows that neither the theorem nor the corollary is true unless we assume that the continuum M is a Peano continuum. In E_2 let M consist of the points of the curve $y = \sin(1/x)$ for $0 < x \leqq 1$ together with the points (x, y) for $-1 \leqq x \leqq 0$, $-1 \leqq y \leqq 1$.

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On joining finite subsets of a Peano space by arcs and simple closed curves [1].

By

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1. Introduction. Two of the most fundamental results in the theory of Peano spaces [2]) relate to the joining of two points by an arc or a simple closed curve. One of these is that every two points of a Peano space can be joined by an arc lying in the space. The other is that if two points are not separated by the omission of any single point, then they can be joined by a simple closed curve. It is the purpose of this paper to give generalizations of these results to sets of n points.

Our conditions are only sufficient. If the Peano space is itself a simple closed curve, then none of our conditions are true but all of the results hold. The determination of necessary and sufficient conditions for n-points seems to be a very difficult problem. Also it seems likely that if the problem were solved, the conditions would be so complex as to be of little interest.

2. Historical Note. The theorem that every two points may be joined by an arc was proved by Mazurkiewicz, Kaluzsay, Moore, Tietze and Vietoris [3]). The theorem for the joining

[1]) Presented to Akademie der Wissenschaften in Wien July 4, 1929, and American Mathematical Society August 30, 1929.

[2]) Following Rosenthal, *Math. Zeit.*, vol. 10 (1921), pp. 102—4, and Kuratowski, *Fund. Math.*, vol. 13 (1929), pp. 307—18, we shall call a metric space, which is the continuous image of a closed interval, a *Peano space*. Other terms commonly used are *continuous curve*, *im kleinen zusammenhängendes Kontinuum*, and *continu de Jordan*.

[3]) S. Mazurkiewicz, *O arytmetyzacji kontinuów*, C. R. de la Société de

of two points by a simple closed curve was proved by G. T. Whyburn[4]) for the case where the Peano space is a subset of the Euclidean plane. This result was announced at about the same time by W. L. Ayres[5]), but was never published. For regular curves it was proved by K. Menger[6]). The theorem for a general Peano space was proved by W. L. Ayres[7]). Conditions under which three and four points lie on an arc have been given by W. L. Ayres[8]) and E. W. Miller[9]). Conditions under which three points lie on a simple closed curve have been given by Ayres[10]). Finally conditions have been given under which a closed set is a subset of an arc in the space considered, by Moore and Kline[11]) where the space is the Euclidean plane, and by Miller[12]) where the

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Sciences de Varsovie, vol. 6 (1913), pp. 305—311; and *Sur les lignes de Jordan*, Fund. Math., vol. 1 (1920), pp. 169—209.

C. Kaluzsay, *A felületre vonatkozó Jordan-tétel megfordítása*, Mathematikai és Physikai Lapok, vol. 24 (1915), pp. 101—141.

R. L. Moore, *A theorem concerning continuous curves*, Bull. Amer. Math. Soc., vol. 23 (1917), pp. 233—236.

H. Tietze, *Ueber stetige Kurven, Jordansche Kurvenbögen und geschlossene Jordansche Kurven*, Math. Zeit., vol. 5 (1919), pp. 284—291.

L. Vietoris, *Bereiche zweiter Ordnung*, Monatshefte f. Math. u. Physik, vol. 32 (1922), pp. 258—280.

[4]) G. T. Whyburn, *Some properties of continuous curves*, Bull. Amer. Math. Soc., vol. 33 (1927), pp. 305—308.

[5]) *On the separation of points of a continuous curve by arcs and simple closed curves*, Bull. Amer. Math. Soc., vol. 33 (1927), p. 266.

[6]) *Zur allgemeinen Kurventheorie*, Fund. Math., vol. 10 (1927), pp. 95—115, Satz β.

[7]) W. L. Ayres, *Concerning continuous curves in metric space*, Amer. Jour. of Math., vol. 51 (1929), pp. 577—594.

[8]) *Concerning the arc-curves and basic sets of a continuous curve, Second paper*, Trans. Amer. Math. Soc., vol. 31 (1929), pp. 595—612, Theorem 12; and *Continuous curves which are cyclicly connected*, Bull. Acad. Sc. Polonaise, 1928, pp. 127—142.

[9]) E. W. Miller, *Concerning subsets of a continuous curve which lie on an arc of the continuous curve*, Dissertation, University of Michigan, 1930.

[10]) *Continuous curves which are cyclicly connected*, loc. cit.

[11]) R. L. Moore and J. R. Kline, *On the most general closed point-set through which it is possible to pass a simple continuous arc*, Annals of Math., vol. 20 (1919), pp. 218—223.

[12]) Loc. cit.

space is Euclidean n-space or an acyclic Peano space or the boundary of a plane domain.

3. Theorem. *Let K be a set consisting of n points $(n > 1)$ of the Peano space M. If no two points of K can be separated by the omission of $n-1$ points of M, then there exists a simple closed curve of M containing K.*

4. *The case $n = 2$.* This case follows as a special case of theorem 6 of my paper, *Concerning continuous curves in metric space* (loc. cit.). But we shall give here a simpler proof following along somewhat the same lines as it is this proof, slightly generalized, that we require for the general case of the theorem of § 3.

4. 1. Lemma. *If p is a frontier point of a domain D*[13]) *and for each $\varepsilon > 0$ there exists a $\delta > 0$ such that only a finite number of components of $D \cdot S(p, \varepsilon)$ contain points of $S(p, \delta)$, then p is accessible from D.*

For a proof of this lemma, see theorem 1 of my paper, *Concerning continuous curves in metric space.*

4. 2. Definition. If α is an arc of M, $p \in \alpha$ and the set $D \subset M - \alpha$, then p is said to be *chain-wise accessible* from D if there exists a countable set of arcs $x_i y_i$ $(i = 1, 2, \ldots$ ad. inf.$)$ such that (1) $x_i + y_i \subset \alpha$, $\langle x_i y_i \rangle$[14]) $\subset D$, (2) $(x_i y_i) \cdot (x_j y_j) = 0$ if $j \neq i+2$, and $(x_i y_i) \cdot (x_j y_j)$ is vacuous or the single point $y_i = x_j$ if $j = i+2$, (3) on the arc α we have the order $x_1 x_2 y_1 x_3 y_2 x_4 \ldots p$, (4) $\lim_{i \to \infty} \operatorname{diam} x_i y_i = 0$, $\lim x_i = \lim y_i = p$. We say that the set $\{x_i y_i\}$ is a *simple chain* of arcs of D from x_1 to p. Also a finite set $x_i y_i$ $(i = 1, 2, \ldots, n)$ of arcs is said to be a simple chain from x_1 to p if (1), (2) and (3) are satisfied and $y_n = p$. (See diagram in § 4.6).

4. 3. Lemma. *If α is an arc one of whose end points is p, D is a component or set of components of $M - \alpha$, and there exists an $\varepsilon > 0$ such that for every $0 < \delta < \varepsilon$ infinitely many of the components of $D \cdot S(p, \varepsilon)$ that have limit points in $M - S(p, \varepsilon)$ contain points of $S(p, \delta)$, then p is chain-wise accessible from D.*

[13]) It is assumed of course that D is a subset of the Peano space M. All of our sets are in M so it is not necessary to state this on each occasion.

[14]) If xy is an arc with end points x and y, then $xy\rangle$, $\langle xy$ and $\langle xy \rangle$ denote $xy - y$, $xy - x$ and $xy - x - y$ respectively.

There exists a set of numbers δ_i such that (a) $\varepsilon > \delta_i > 2\delta_{i+1} > 0$, (b) if $u + v \subset \alpha$, u non-ε $S(p, \delta_i)$, $v \in S(p, \delta_{i+1})$, then we have the order uvp on α. By hypothesis there exists an infinite set of distinct components $H_1, H_2, \ldots$ of $D \cdot S(p, \varepsilon)$ such that

$$(1) \qquad (M - S(p, \varepsilon)) \cdot \overline{H}_i \neq 0,$$

$$(2) \qquad S(p, \delta_i) \cdot H_i \supset p_i.$$

Let $H_{ji} = C(p_i, H_i \cdot S(p, \delta_j))$, i. e. the component of $H_i \cdot S(p, \delta_j)$ containing p_i $(j \leqq i)$. There exists an arc $x_1 y_1$ such that $x_1 \in \alpha \cdot \cdot (S(p, \delta_1) - S(p, \delta_2))$, $y_1 \in \alpha \cdot (S(p, \delta_7) - S(p, \delta_9))$ and the arc-segment $x_1 y_1$ belongs to one of the sets H_{1i} for $i \geqq 8$.

This may be proved as follows: From 1) and 2) it follows that H_{1i} $(i \geqq 8)$ contains a point $x_{2i} \in S'(p, \delta_2)$ and a point $x_{8i} \in S'(p, \delta_8)$, where $S'(p, K)$ denotes the set of all points q such that $\varrho(p, q) = k$. As M is locally connected and the points x_{2i} belong to different components of $M - \alpha$, every limit point of the set Σx_{2i} belongs to α, and hence to $\alpha \cdot S'(p, \delta_2)$. Let x_2 be one such limit point and let $x_{2i_1}, x_{2i_2}, x_{2i_3}, \ldots$ be a subsequence of Σx_{2i} such that $\lim_{n \to \infty} x_{2i_n} = x_2$. Let x_8 be a limit point of Σx_{8i_n} $(x_8 \in \alpha \cdot S'(p, \delta_8))$ and let $x_{8k_1}, x_{8k_2}, \ldots$ be a subsequence of Σx_{8i_n} such that $\lim x_{8k_n} = x_8$. Obviously we have $\lim x_{2k_n} = x_2$. Let η_2 and η_8 be positive numbers such that $S(x_2, \eta_2) \subset S(p, \delta_1) - S(p, \delta_2)$ and $S(x_8, \eta_8) \subset \subset S(p, \delta_7) - S(p, \delta_9)$. There exist positive numbers G_2 and G_8 such that any point of $S(x_s, G_s)$ may be joined to x_s by an arc $\subset S(x_s, \eta_s)$ for $s = 2, 8$. There exists a number n such that $x_{sk_n} \in S(x_s, G_s)$, $s = 2, 8$. Let α_s be an arc with end points x_{sk_n} and x_s such that $\alpha_s \subset S(x_s, \eta_s)$. There exists an arc $\beta_{k_n} \subset H_{1k_n}$ with end points x_{2k_n} and x_{8k_n}. In the order from x_{2k_n} to x_2, let x_1 be the first point of α on α_2, and in the order from x_1 to x_{2k_n} let z_2 be the first point of β_{k_n} on the subarc $x_1 x_{2k_n}$ of α_2. Similarly we may define a subarc $y_1 z_8$ of α_8 such that $(y_1 z_8) \cdot \alpha = y_1$ and $(y_1 z_8) \cdot \beta_{k_n} = z_8$. Then the subarc $x_1 z_2$ of α_2 plus the subarc $z_2 z_8$ of β_{k_n} plus the subarc $z_8 y_1$ of α_8 is the required arc $x_1 y_1$.

Similarly there exists an arc $x_2 y_2$ such that $x_2 \in \alpha \cdot (S(p, \delta_4) - S(p, \delta_6))$, $y_2 \in \alpha \cdot (S(p, \delta_{13}) - S(p, \delta_{15})$ and the set $\langle x_2 y_2 \rangle$ belongs to one of the sets H_{4i} for $i \geqq 13$ and $i \neq k_n$. Continue this process. In general there exists an arc $x_t y_t$ $(t > 1)$ such that (c) $x_t \in \alpha \cdot \cdot (S(p, \delta_{6t-8}) - S(p, \delta_{6t-6}))$, (d) $y_t \in \alpha \cdot (S(p, \delta_{6t+1}) - S(p, \delta_{6t+3}))$, and (e) the set $\langle x_t y_t \rangle$ belongs to one of the sets $H_{6t-8,i}$ for $i \geqq 6t + 1$ and (f) not equal to any value of i such that $H_i \supset x_r y_r$ for $r < t$[15]).

[15]) For example, when $r = 1$, $H_{k_n} \supset x_1 y_1$.

Now let us consider the conditions in the definition of chain-wise accessibility. Condition (1) follows from (c), (d) and (e). From (f), it follows that

$$(3) \qquad \langle \text{arc } x_i y_i \rangle \cdot \langle \text{arc } x_j y_j \rangle = 0.$$

From (c) and (d), we have

$$(4) \qquad (x_i + y_i) \cdot (x_j + y_j) = 0.$$

Then from (3) and (4), we have

$$(\text{arc } x_i y_i) \cdot (\text{arc } x_j y_j) = 0,$$

which satisfies condition (2) of the definition. Condition (3) is satisfied from (b), (c) and (d). From (a), since arc $x_i y_i \subset S(p, \delta_{6i-8})$, we have condition (4) satisfied. Then p is chain-wise accessible from $M - \alpha$.

4. 4. Lemma. *Let α be an arc one of whose end points is p and S be a set of components of $M - \alpha$ such that* (a) *for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $D \in S$ and $\operatorname{diam}(D) > \varepsilon$ then $S(p, \delta) \cdot D = 0$,* (b) *there is a point $q \neq p$ of α such that if $x \in \alpha$ between p and q then there exists a $D \in S$ such that x lies between two limit points of D on α. Under these conditions, p is chain-wise accessible from S.*

On α we will define order as being from q to p. Let y_0 be a point of α between q and p. Let S_1 be the set of all components of S such that y_0 lies between two limit points of the component on α. By hypothesis (b), $S_1 \neq 0$. Let K_1 be the set of all limit points of the components of S_1, and let z_1 be the last point $\overline{K_1}$ on α. By hypothesis (a), $z_1 \neq p$. Let $S_{z_1} \in S$ such that z_1 lies between u_1 and v_1, the first and last limit points of S_{z_1} on α. The set S_1 contains a component S_{y_0} which has a limit point w_1 on α between u_1 and z_1 and such that $\varrho(w_1, z_1) < \frac{1}{2}$. And, as the accessible points are dense on the boundary of a domain, there exists an arc $x_1 y_1$ such that $x_1 + y_1 \subset \alpha$, $\langle \text{arc } x_1 y_1 \rangle \subset S_{y_0}$, x_1 precedes y_0, y_1 follows u_1, $\varrho(y_1, w_1) < \frac{1}{2}$.

Let S_2 be the set of all components of S such that y_1 lies between two limit points of the component on α. Let K_2 be the set of all limit points of components of S_2 on α, and let z_2 be the last point of $\overline{K_2}$. Since $S_{z_1} \in S_2$, z_1 precedes z_2. Let $S_{z_2} \in S$ such that z_2 lies between u_2 and v_2, its first and last limit points on α. There is a component $S_{y_1} \epsilon S_2$, which has a limit point w_2 on α such that both z_1 and u_2 precede w_2 and $\varrho(w_2, z_2) < 1/4$. Then there is an arc $x_2 y_2$ such that x_2 precedes y_1 on α, y_2 follows u_2 and z_1, $\varrho(y_2, w_2) < 1/4$, $<\text{arc}\, x_2 y_2> \subset S_{y_1}$. As S_{y_1} non-ϵ S_1, y_0 precedes x_2 or $y_0 = x_2$.

Continue this process. In general let S_n be the set of all components of S such that y_{n-1} lies between two limit points of the component on α. Let K_n be the set of all limit points of the components of S_n on α, and let z_n be the last point of $\overline{K}_n$. By hypothesis (a), $z_n \neq p$. And since $S_{z_{n-1}} \epsilon S_n$, z_{n-1} precedes z_n. Let $S_{z_n} \epsilon S$ such that z_n lies between u_n and v_n, the first and last limit points of S_{z_n} on α. There is a component $S_{y_{n-1}} \epsilon S_n$ which has a limit point w_n on α such that both z_{n-1} and u_n precede w_n and $\varrho(w_n, z_n) < 2^{-n}$. Then there is an arc $x_n y_n$ such that x_n precedes y_{n-1} on α, y_n follows both z_{n-1} and u_n, $\varrho(y_n, w_n) < 2^{-n}$, $<\text{arc}\, x_n y_n> \subset S_{y_{n-1}}$. As y_n follows z_{n-1}, $S_{y_{n-1}}$ non-ϵ S_{n-1}. Hence y_{n-2} precedes x_n or $y_{n-2} = x_n$.

We shall see that the conditions in the definition of chain-wise accessibility are satisfied. Condition (1) is evident. Conditions (2) and (3) follow from the order properties proved above. We shall prove now that $\lim y_n = p$. Since y_n precedes y_{n+1}, it is evident that $\lim y_n$ exists. Suppose $\lim y_n = b \neq p$. As $\varrho(y_n, z_n) \leqq \varrho(y_n, w_n) + \varrho(w_n, z_n) \leqq 2^{1-n}$, we have $\lim z_n = b$. And as z_n precedes z_{n+1}, we have z_n precedes b for every n. Let $S_b \epsilon S$ such that b lies between u_b and v_b, its first and last limit points on α. There exists an integer m such that y_m lies between u_b and b since $\lim y_n = b$. Then $S_b \epsilon S_{m+1}$ and $v_b \epsilon K_{m+1}$. Thus z_{m+1} follows v_b or $z_{m+1} = v_b$, and z_{m+1} follows b. But this is a contradiction as every z_n precedes b. Now as $\lim y_n = p$, we have $\lim \text{diam}\, (\text{arc}\, x_n y_n) = 0$ from hypothesis (a). From $\lim y_n = p$ and $\lim \text{diam}\, (\text{arc}\, x_n y_n) = 0$, we have $\lim x_n = p$. This completes the proof of condition (4). Then p is chain-wise accessible from S.

4. 5. **Lemma.** *If p and q are interior points of an arc α, and S is a set of components of $M - \alpha$ such that if x is any point of the subarc pq of α then there is a component of S which has x between two of its limit points, then there is a finite simple chain of arcs of S from a point of α preceding p to a point of α following q.*

The proof of lemma 4.5 follows closely that of 4.4. We take $y_0 = p$. From this point we follow the previous proof exactly and we

may prove that for some n, q precedes y_n on α in exactly the same way as we proved in 4.4 that $\lim y_n = p$. Then $x_1 y_1, x_2 y_2, \ldots, x_n y_n$ is the desired simple chain.

4. 6. On the basis of the preceding lemmas, we shall now prove the case $n = 2$ of our theorem. Let $K = p + q$ and let α be an arc of M with end points p and q. Since no point separates p and q, for each point x of $<\alpha>$ there is a component D_x of $M - \alpha$ such that x lies between two limit points of D_x on α.

Suppose p is a limit point of some component D of the set $\{D_x\}$. If p is accessible from D, let β_1 be an arc with end points p and some point r of D_x such that $\beta_1 - p \subset D$. By definition every component D_x has at least two limit points on α. Hence there is an arc β_2 with end points r and $x_1 \epsilon \alpha - p$ such that $\beta_2 - x_1 \subset D$. The set $\beta_1 + \beta_2$ contains an arc $x_1 y_1$ where $y_1 = p$. If p is not accessible from D, there exists, by 4. 1, a number $\varepsilon > 0$ such that infinitely many components of $D \cdot S(p, \varepsilon)$ contain points of $S(p, \delta)$ for every $0 < \delta < \varepsilon$. Then p is chain-wise accessible by 4. 3, and there exists a simple chain $\{x_i y_i\}$ of arcs of D from some point x_1 of α to p.

Now consider the case where no component of $\{D_x\}$ has p as a limit point. Then $\{D_x\}$ contains infinitely many distinct components. If for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $S(p, \delta) \cdot D_x = 0$ for every D_x such that $\text{diam}\, D_x > 2\varepsilon$, then p is chain-wise accessible by 4. 4. If there is an $\varepsilon > 0$ for which this is not true, then the hypothesis of 4. 3 is evidently satisfied and p is again chain-wise accessible. Let $\{x_i y_i\}$ be the simple chain from some point x_1 of α to p.

In precisely the same way we may determine a chain $\{u_i v_i\}$, from some point u_1 of α to q. The set $\{u_i v_i\}$ contains either a single arc or an infinite number.

We denote order on α as being from p to q. If u_1 precedes x_1 on α, let n be the largest integer such that u_n precedes x_1. There exists a largest n unless $x_1 = q$, in which case $\{x_i y_i\}$ is a simple chain from p to q. Let m be the largest integer such that x_m follows u_n. Then we have the order $p \ldots y_m\, x_{m+1}\, u_n\, x_m\, u_{n+1}\, v_n \ldots q$ on α. If $<x_i y_i> \cdot <u_j v_j> = 0$. for any $i \geqq m$ and $j \geqq n$, then $\{x_i y_i\} + \{u_j v_j\}$ for $i \geqq m$, $j \geqq n$ is a simple chain from p to q. If $<x_i y_i> \cdot <u_j v_j> \neq 0$ for every $i \geqq m$ and $j \geqq n$, let n^* be the largest integer j such that $<u_j v_j> \cdot \sum^{\infty} <x_i y_i> \neq 0$ and let m^* be the largest integer i such that $\langle u_{n^*} v_{n^*} \rangle \cdot \langle x_i y_i \rangle \neq 0$. Then the set $u_{n^*} v_{n^*} + x_{m^*} y_{m^*}$ contains an arc $y_{m^*} v_{n^*}$ with end points y_{m^*} and v_{n^*} such that $\langle y_{m^*} v_{n^*} \rangle \cdot \alpha = 0$. Then $\{x_i y_i\}$ for $i > m^*$ plus $\{u_j v_j\}$ for $j > n^*$ plus the arc $y_{m^*} v_{n^*}$ is a simple chain from p to q.

If $u_1 = x_1$ or x_1 precedes u_1 on α, by **4**. 5 there exists a finite simple chain covering the subarc $x_1 u_1$ of α. With this finite chain and the two chains from x_1 to p and u_1 to q, we may determine a simple chain from p to q as above.

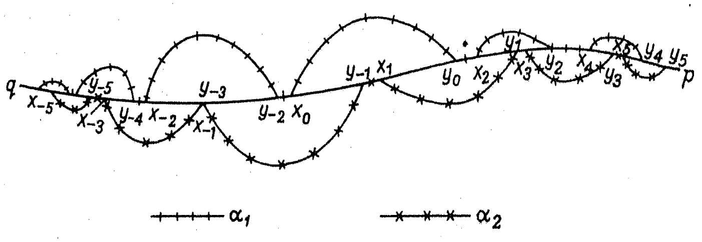

Fig. 1.

In any case we will denote this simple chain by $\{x_i y_i\}$, where i may run through all integers (positive and negative), be bounded above or below, or both above and below. We assume $\{x_i y_i\}$ satisfies conditions (1), (2) and (3) of **4**. 2, and condition (4) is replaced by the following: „(4′) If s is the largest value of i, then $y_s = p$. If i is unbounded above, then $\lim_{i \to +\infty} \operatorname{diam} x_i y_i = 0$, $\lim_{i \to +\infty} x_i = p$. If t is the smallest value of i, then $x_t = q$. If i is unbounded below, then $\lim_{i \to -\infty} \operatorname{diam} x_i y_i = 0$, $\lim_{i \to -\infty} x_i = q$". In case of i bounded we may assume that $s \geqq 0$ and $t \leqq 0$ without loss of generality. Now we may select two arcs from p to q as follows:

$$\alpha_1 = p + q + \sum_i \text{arc}\, x_{2i} y_{2i} + \sum_i \text{subarc}\, y_{2i} x_{2i-2} \text{ of } \alpha,$$

$$\alpha_2 = p + q + \sum_i \text{arc}\, x_{2i+1} y_{2i+1} + \sum_i \text{subarc}\, y_{2i+1} x_{2i-1} \text{ of } \alpha.$$

In order to have the above formulas hold in all cases we define $y_{s+1} = p$, $x_{t-1} = q$. Then $\alpha_1 \cdot \alpha_2 = p + q$, and $\alpha_1 + \alpha_2$ is a simple

closed curve containing $K = p + q$. This proves the case $n = 2$ of our theorem.

5. *The general case.* Since the theorem has already been established in § 4 for $n = 2$, we may prove the general case by mathematical induction. Suppose the theorem true for $n = k - 1$. Now let $K = p_1 + p_2 + \ldots + p_k$ and suppose no two points of K may be separated by the omission of $(k-1)$ points of M. Then, as the theorem is true for $n = k - 1$, there exists a simple closed curve J of M such that $J \supset p_1 + p_2 + \ldots + p_{k-1}$. On the curve J we may assume the cyclic order $p_1 p_2 \ldots p_{k-1} p_1$. Let I_i denote that arc of J with end points p_i and p_{i+1} (subscripts reduced modulo $k-1$) which contains no other point of $\sum_1^{k-1} p_j$. In case $p_k \varepsilon J$, our proof is complete. If p_k non-εJ, we shall proceed to our proof through several lemmas:

5. 1. **Lemma.** *Suppose* (a) *H is the set consisting of a simple closed curve and an arc α which has just one end point in common with it,* (b) *p denotes the other end point of α,* (c) *D is a component or set of components of $M - H$,* (d) *there exists an $\epsilon > 0$ such that for every $0 < \delta < \epsilon$ infinitely many components of $S(p, \epsilon) \cdot D$ that have limit points in $M - S(p, \epsilon)$ contain points of $S(p, \delta)$. Then p is chain-wise accessible from D.*

5. 2. **Lemma.** *Suppose* (a), (b), (c), *of* **5**. 1 *are true, and* (d) *for each $\epsilon > 0$ there exists a $\delta > 0$ such that if $D_1 \varepsilon D$ and diam $(D_1) > \epsilon$ then $S(p, \delta) \cdot D_1 = 0$,* (e) *there is a point $q \neq p$ of α such that if $x \varepsilon \alpha$ between p and q then there is a $D_1 \varepsilon D$ such that x lies between two limit points of D_1 on α. Then p is chain-wise accessible from D.*

5. 3. **Lemma.** *Suppose* (a), (b), (c) *of* **5**. 1 *are true, and* (d) *q and r are interior points of α,* (e) *if x is any point of the subarc or of α then there is a component of D which has x between two of its limit points on α. Then there is a finite simple chain of arcs of D from a point of α preceding q to a point of α following r.*

The three preceding lemmas are slight generalizations of **4**. 3, **4**. 4 and **4**. 5 and may be proved in the same way.

5. 4. **Lemma.** *Suppose there exists at least one arc α of M whose end points are p_k and a point of I_i $(i = 1, 2, \ldots, k-1)$ such* *that $\alpha \cdot J$ is this single end point belonging to I_i. Then either there is a point $q_i \neq p_k$ such that every such arc α contains q_i or there exist two such arcs α_1 and α_2 such that $\alpha_1 \cdot \alpha_2 = p_k$.*

Suppose the lemma false. Let β be one of the arc α with end points p_k and r. Since $r \neq q_i$, there exists an arc γ (of the type α) with end points p_k and s and not containing r. In the order from s to p_k let t be the first point of β on γ. As we have assumed the lemma false, we have $t \neq p_k$. Let D_γ be the component of $M-(J+\beta)$ containing the subset $<ts>$ of γ. Let D be the set of all components D_γ for all arcs γ of type α not containing r. If p_k is a limit point of some component D_1 of D, it follows from our assumption that the lemma is false that p_k is not accessible from D_1. From 4.1 then, the hypothesis of 5.1 is satisfied and p_k is chain-wise accessible from D_1. If p_k is a limit point of no component of D but $p_k \varepsilon \overline{D}$, then the hypothesis of 5.1 is again satisfied and p_k is chain-wise accessible from D. Then there is a simple chain of arcs $\{x_j y_j\}$ of D such that (1) $x_1 \varepsilon I_i - r$, $y_1 + \sum_{j>1}(x_j + y_j) \subset \subset <\beta>$, $<x_j y_j> \subset D$, and (2), (3) and (4) of 4.2 are satisfied with p_k replacing p.

If p_k non-$\varepsilon \overline{D}$, let u be the last point of $\overline{D}$ on β in the order r to p_k. Let $z \varepsilon$ subset $up_k>$ of arc β. As $z \neq q_i$, there is an arc η of type α so that z non-$\varepsilon\,\eta$. The arc η contains a segment which has no point in common with β and such that the end points of the segment are on β and z lies between these two points on β. Let S_η be the component of $M-(J+\beta)$ containing this segment, and let S be the set of all such sets S_η for all points z. With the use of 5.1, 5.2 and 5.3, we may obtain from S a simple chain $\{x_j y_j\}$ $(j = 2, 3, \ldots)$ from a point x_2, which precedes u on β, to p_k just as in 4.6. Evidently this may be done so that x_3 lies on the subarc up_k of β, for if this condition were not satisfied it could be obtained by omitting a finite number of the arcs $\{x_j y_j\}$ and re-lettering them. As $u \varepsilon \overline{D}$ there is an arc $x_1 y_1$ such that $x_1 \varepsilon I_i - r$, $y_1 \varepsilon$ subset $<x_2 u$ of β, $<x_1 y_1> \subset D$. As $D \cdot S = 0$, $(x_1 y_1) \cdot (x_j y_j) = 0$ except $(x_1 y_1) \cdot (x_2 y_2)$ may be $y_1 = x_2 = u$. Then when p_k non-$\varepsilon \overline{D}$ we have a set of arcs $\{x_j y_j\}$ $(j = 1, 2, 3, \ldots)$ exactly as in the case where $p_k \varepsilon \overline{D}$.

Let $r = y_0$. Then

$$\alpha_1 = p_k + \sum_n x_{2n} y_{2n} + \sum_n \text{subarc } y_{2n-2} x_{2n} \text{ of } \beta,$$

$$\alpha_2 = p_k + \sum_n x_{2n-1} y_{2n-1} + \sum_n \text{subarc } y_{2n-1} x_{2n+1} \text{ of } \beta$$

are two arcs of type α such that $\alpha_1 \cdot \alpha_2 = p_k$, which is contrary to our assumption that the lemma is false.

5. 5. If there exists no arc α with end points p_k and a point of I_i such that $\alpha \cdot J$ is the end point belonging to I_i, let $q_i = 0$. If there is an arc α, then by 5.4 there is a point q_i such that every such arc α contains q_i or there are two such arcs α_1 and α_2 such that $\alpha_1 \cdot \alpha_2 = p_k$. If the latter is the case for any i, then $\alpha_1 + \alpha_2 +$ that arc of J from $\alpha_1 \cdot J$ to $\alpha_2 \cdot J$ which contains Σp_j is a simple closed curve containing K and our proof is complete. But this must be true for some i for if the point q_i is defined for every i (vacuous or otherwise), we shall show that Σq_i separates two points of K, which is contrary to the hypothesis of the theorem.

Suppose q_i is defined for every $i = 1, 2, \ldots, k-1$. If $p_n \varepsilon \Sigma q_i$ $(n < k)$, then $p_n = q_n$ or $p_n = q_{n-1}$. In the former case p_{n+1} non-$\varepsilon\, \Sigma q_i$, and in the latter case p_{n-1} non-$\varepsilon\, \Sigma q_i$. For suppose $p_n = q_n$, then every arc of type α for I_n contains p_n; and if $p_{n+1} \varepsilon \Sigma q_i$ there exists an arc with end points p_k and p_{n+1} having only p_{n+1} in common with J. But this is an arc of type α for I_n and must contain p_n, which is impossible. Then there is a point p_t of $K - p_k$ which does not belong to Σq_i. The set Σq_i separates p_t and p_k in M, for if it does not then there exists an arc ϑ with end points p_k and p_t so that $\vartheta \cdot \Sigma q_i = 0$. In the order from p_k to p_t, let v be the first point of J on ϑ. The point v exists for $p_t \varepsilon \vartheta \cdot J$ and $\vartheta \cdot J$ is closed. Suppose $v \varepsilon I_s$. Then the subarc $v p_k$ of ϑ is an arc α for I_s and thus $q_s \varepsilon$ subarc vp_k of ϑ. But this is impossible as $\vartheta \cdot \Sigma q_i = 0$. This completes the proof of our theorem.

6. Theorem. *Let K be a set consisting of n points $(n > 1)$ of the Peano space M. If no two points of K can be separated by the omission of $(n-2)$ points of M, then there exists an arc of M containing K.*

Let $p \varepsilon K$ and let $H = K - p$. Since no two points of H can be separated by the omission of $(n-2)$ points of M, by the result of § 3 there exists a simple closed curve J of M containing H. If $p \varepsilon J$, then there is evidently an arc of J containing K. If p non-εJ, let $q \varepsilon H$ and let α be an arc of M with end points p and q. Denote by r the first point of J on α in the order from p to q. In one direction on J from r, let s be the first point of H. Then the subarc pr of α plus that arc rs of J containing H is an arc of M containing K.

7. Examples. We shall give two examples to show that the numbers $(n-1)$ and $(n-2)$ in the two preceding theorems cannot be reduced. Let K consist of n distinct points p_i on a line ab. Let q_j denote $n-1$ distinct points on a circle with center on ab and lying in a plane perpendicular to the line ab. Let

$$M = \sum_{j=1}^{n-1} \sum_{i=1}^{n} p_i q_j,$$

where $p_i q_j$ denotes the straight-line interval from p_i to q_j. It is easy to see that no two points of K may be separated by $n-2$ points of M. If there were a simple closed curve of M containing K then it would consist of n arcs between the points p_i, which are distinct except for the points p_i. But every arc of M with end points p_m and p_n contains a point q_j. Then no such simple closed curve exists for there are but $n-1$ of the points q_j. We see that the number $n-1$ of the theorem of § 3 cannot be replaced by $n-2$.

Now if we take

$$N = \sum_{j=1}^{n-2} \sum_{i=1}^{n} p_i q_j$$

we have a Peano space containing a set K of n points such that no two points of K are separated by the omission of any $n-3$ points of M and there is no arc of M containing K. Thus the number $n-2$ of § 6 is as small as possible.

8. The Plane Case. It may be noted that in the preceding example M is not homeomorphic with any subcontinuum of the plane for $n > 3$; and similarly N cannot be mapped in the plane for $n > 4$. This leaves open the question as to whether the numbers $n-1$ and $n-2$ are as small as possible if we assume in

addition that the Peano space M is a subset of the Euclidean plane. In connection with this it would be interesting to determine whether the following is a true theorem:

Let the Peano space M be a subset of the plane and let K be any finite subset of M. Suppose no two points of K may be separated by the omission of any two points of M. Under these conditions there is a simple closed curve of M containing J.

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