On a certain neighborhood property [1]).

By

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In a recent paper [2]) we proved that if M is a metric continuum with end points p and q and every separation of M between p and q contains an ordinary point of M, then M is an arc with end points p and q. Shortly afterward [3]) it was pointed out that M need be supposed only a connected set and closure may be proved. In this note we propose to examine sets which have the slightly stronger property that the frontier of every proper neighborhood contains an ordinary point ot the set. We shall derive the properties of such sets and then apply them to obtain new characterizations of the arc and of the simple closed curve. As the property is purely intrinsic we may use the set itself as our space M. An open subset D of the space M such that M contains a point not belonging to $\overline{D}$ will be called a *proper neighborhood.* An ordinary point of M is a point of order 2 in the Menger-Urysohn sense. The set of all ordinary points of M is denoted by M^2. To avoid trivial cases we assume that M contains at least two points.

Below we give two examples of sets having this property:

a) Let M consist of the curve $y = \sin(1/x)$ for $0 < x \leqslant 1$ together with the points $(0, 1)$ and $(0, -1)$. This example shows that condition (b) of the second theorem is necessary.

[1]) Presented to the American Mathematical Society November 28, 1931.

[2]) W. L. Ayres, *On the regular points of a continuum*, Trans. Amer. Math. Soc., 33 (1931), p. 257.

[3]) K. Menger, *Remarks concerning the paper of W. L. Ayres On the regular points of a continuum*, Trans. Amer. Math. Soc., 33 (1931), p. 666.

b) Let M consist of the curve $y = \sin(1/x)$ for $0 < x \leqq 1$ and $-1 \leqq x < 0$ together with the interval from $(0, -1)$ to $(0, 1)$ and an arc joining the points $(1, \sin 1)$ and $(-1, -\sin 1)$ which has no other points in common with the curve.

Theorem. *If the frontier of every proper neighborhood of the metric space M contains an ordinary point, then* (1) *M is connected,* (2) $\overline{M^2} = M$, (3) *M^2 is connected,* (4) *M^2 is homeomorphic with a circle or a straight line.*

(1). Suppose $M = H + K$, where $\overline{H}K + H\overline{K} = 0$. Then K is an open subset of M for $M - K = H$ and $\overline{H}K = 0$. Also $F(K) = \overline{K}\cdot(M-K) = \overline{K}H = 0$. But this is impossible as $F(K)$ must contain an ordinary point.

(2). Let $p \in M$ and U_p be a proper neighborhood of M. Let $\overline{V}_p \subset U_p$. By hypothesis $F(V_p)$ contains a point of M^2.

(3). Suppose $M^2 = H + K$, where $\overline{H}K + H\overline{K} = 0$. As $K\overline{H} = 0$, if $p \in K$ then $d_p = \varrho(p, \overline{H}) > 0$. Let $D = \Sigma S(p, \frac{1}{2}d_p)$ for every $p \in K$. From the definition of D it follows that if $q \in D$ and $r \in H$ there exists a $p \in K$ such that $\varrho(r, q) > \frac{1}{2}\varrho(r, p)$. Now let $x \in F(D)$. From the preceding inequality we have $x \in \overline{K}$. As $H\overline{K} = 0$, x non-$\in H$ and x non-$\in K$ since $D \supset K$. Then $F(D)\cdot M^2 = 0$, which is contrary to hypothesis as $M - \overline{D} \supset H \neq 0$.

(4). Since the order of a point with respect to a subset of M is never greater than its order in M, the order of every point in M^2 (with respect to M^2) is $\leqq 2$. From (2) and the assumption that M is non-degenerate, we know that M^2 contains at least two points. As M^2 is also connected we have from the result of Frankl [1]) that M^2 is homeomorphic with a) a circle, b) a closed interval, c) a half-closed interval or d) an open interval. The second case is impossible for then M^2 is compact and by (2) $M^2 = M$. Then by the theorem of Menger-Urysohn [2]) M would be homeomorphic with a circle.

In case c) suppose M^2 is homeomorphic with the half-open interval $0 \leqq x < 1$ and let φ represent the homeomorphism which

[1]) F. Frankl, *Ueber die zusammenhängenden Mengen von höchstens zweiter Ordnung*, Fund. Math., 11 (1928), pp. 96—104.

[2]) K. Menger, Math. Ann., 95 (1925); and P. Urysohn, Amster. Verh., 13 (1927).

carries this interval into M^2. Let $p = \varphi(0)$. We shall define a point y of M^2 as preceding a point z of M^2 if $\varphi^{-1}(y) < \varphi^{-1}(z)$. If $[p_i]$ is a sequence of points of M^2 such that p_i precedes p_{i+1} and $\lim_{i\to\infty} \varphi^{-1}(p_i) = 1$, we shall say that $[p_i]$ is an *increasing sequence*. Since the image of a compact set is compact, by (2) every point of $M - M^2$ is a limit point of an increasing sequence. If p is not a limit point of an increasing sequence, there exists a point q of M^2 and a neighborhood U_p of p such that every point of M^2 in U_p precedes q Then there exists a neighborhood V_p such that $\overline{V}_p \subset U_p$ and $F(V_p) \cdot M^2$ consists of just one point. Since $p \in M^2$, for V_p sufficiently small, $F(V_p)$ contains at least one other point y of M. As $y \in M - M^2$, it is a limit point of an increasing sequence and U_p contains a point following q. Hence p is a limit point of an increasing sequence.

Now let $z \neq p$ be any point of M^2 and let η be any positive number $< \varrho(p, z)$. There exists a neighborhood $U_p \subset S(p, \eta)$ such that $F(U_p) = x_1 + x_2$. Since z non-$\in U_p$, $F(U_p)$ contains a point of M^2 preceding z. And as p is a limit point of an increasing sequence of M^2, the other point of $F(U_p)$, say x_2, is a point of M^2 following z. Now as the points of M^2 following x_2 form a connected set containing points of U_p (since p is a limit point of an increasing sequence) but no point of $F(U_p)$, it follows that U_p contains every point of M^2 following x_2.

As M^2 is not homeomorphic with a circle, $M - M^2 \neq 0$ and let $t \in M - M^2$. Then t is a limit point of an increasing sequence of M^2. But this is impossible since for each $\eta > 0$ there exists a point x_2 of M^2 such that $S(p, \eta)$ contains all points of M^2 following x_2. This contradiction shows that case c) is impossible.

Theorem. *In order that a metric space M be an arc it is necessary and sufficient that* (a) *the frontier of every proper neighborhood of M contain an ordinary point,* (b) *M be locally connected at two non-ordinary points.*

The conditions are necessary. If D is a proper neighborhood of an arc, then $F(D)$ contains at least one interior point of the arc, an ordinary point. The two end points of the arc are non-ordinary points at which the arc is locally connected.

The conditions are sufficient. From the preceding theorem we know that M^2 is homeomorphic with a straight-line or a circle.

The second case is impossible since $M - M^2 \neq 0$ by (b). Now let x and y be the two points whose existence is given by (b). Let us define increasing and decreasing sequences of M^2 as in the proof of the preceding theorem. Then either there is an increasing or a decreasing sequence of M^2 which has x as a limit point. Let us suppose an increasing sequence $[p_i]$, and suppose there exists an increasing sequence $[q_i]$ of M^2 which does not have x as a limit point. There exists a neighborhood U_x and subsequences of $[p_i]$ and $[q_i]$ (which we denote by the same symbols) such that $p_i \in U_p$, q_i non-$\in U_p$, p_i precedes q_i and q_i precedes p_{i+1} in M^2. As M is locally connected at x, there is a $V_x \subset U_x$ such that if $u \in V_x$ there is a connected subset K of M such that $u + x \subset K \subset V_x$. There is an integer n such that $p_n \in V_x$ and let K denote the connected subset of M such that $p_n + x \subset K \subset V_x$. As q_n non-$\in U_x$, K does not consist entirely of points of M^2 following q_{n-1}. Let H be the set of points of K which are points of M^2 between q_{n-1} and q_n and let $I = K - H$. We have $\overline{H} \cdot I = 0$ since the set of points of M^2 between q_{n-1} and q_n is the image of an arc and thus compact. Further since these points belong to M^2 it follows that no point of M^2 between q_{n-1} and q_n is a limit point of points of M lying elsewhere, i. e. $H \cdot \overline{I} = 0$. As

$$K = H + I \quad \text{and} \quad \overline{H} \cdot I + H \cdot \overline{I} = 0,$$

the set K is not connected contrary to hypothesis. Thus the assumption that an increasing sequence of M^2 not having x as a limit point exists leads to a contradiction.

In the same way we may show that every decreasing sequence has y as a limit point. Now consider any sequence of points of M^2. Either there is a subsequence having a point of M^2 as a limit point or an increasing subsequence or a decreasing subsequence. Thus every sequence of M^2 contains a subsequence having a limit point. From property (2) of the preceding theorem we have

$$M = M^2 + x + y$$

and M is compact. Thus we may extend the homeomorphism between M^2 and an open interval to M and the closed interval, which proves our result.

Corollary. *If the frontier of every proper neighborhood of M contains an ordinary point, then M is not locally connected at more than two non-ordinary points.*

Theorem. *In order that a metric space M be a simple closed curve it is necessary and sufficient that the frontier of every proper neighborhood of M contain at least two ordinary points.*

The condition is obviously necessary. We may show it to be sufficient as follows: From (4) it follows that M^2 is homeomorphic with a circle or a line. In the first case we have $M=M^2$ and our theorem is proved. We shall prove the second case impossible. Let p be any point of M^2. As pointed out in the preceding proof no point of M^2 is a limit point of an increasing or decreasing sequence of M^2. Then, if x is any point of M^2 preceding p, $d_x=\varrho(x, H)>0$, where H consists of p together with all points of M^2 following p. Let

$$D=\sum S(x, \tfrac{1}{2}d_x)$$

for all points x preceding p. The set D is a proper neighborhood of M and no point x of M^2 preceding p belongs to $F(D)$ since each such point belongs to $S(x, \frac{1}{2}d_x)$. Suppose some point y of M^2 following p belongs to $F(D)$. Let $[z_i]$ be a sequence of points of D with y as a limit point. Let x_i be a point so that $z_i \epsilon S(x_i, \frac{1}{2}d_{x_i})$. We have

$$\varrho(x_i, y)\leqq\varrho(x_i, z_i)+\varrho(z_i, y)<\tfrac{1}{2}\varrho(x_i, y)+\varrho(z_i, y),$$

or

$$\tfrac{1}{2}\varrho(x_i, y)<\varrho(z_i, y).$$

Hence

$$\lim_{i\to\infty}\varrho(x_i, y)=0,$$

which is possible only when $y=p$. Hence D is a proper neighborhood of M whose frontier contains at most one ordinary point.

Corollary. *There exists no metric space M in which the frontier of every proper neighborhood contains at least three ordinary points.*

Theorem. *In order that a Peano continuum M in Euclidean space E_n should be a simple Peano continuum* [1]) *it is necessary and sufficient that M contain one local cut point* [2]) *and each local cut point be an ordinary point of M.*

The condition is quite obviously necessary. We shall see that it is sufficient. If M contains no simple closed curve, every point of M is a cut point or an end point [3]). As every cut point is a local cut point, M consists entirely of points of orders 1 and 2. Then M is an arc, a ray or an open curve. If M contains a simple closed curve J^*, let K denote the maximal cyclic set of M containing J^*. If $M\neq K$, each component Q of $M-K$ has one limit point q in K and each such point q is a cut point of M. There are three distinct arcs of M having q as their common end point, one in $Q+q$ and two on a simple closed curve of K containing q. Then q cannot exist, as it would be a local cut point of order $\geqq 3$. Hence $M=K$, and M is cyclicly connected.

By hypothesis M contains one local cut point p As M is cyclicly connected, there is a simple closed curve J of M containing p. Let e be chosen so that diam $(J)>e>0$. By hypothesis there is a neighborhood U_p such that diam $(U_p)<e$, $F(U_p)\cdot M=p_1+p_2$. Since $e<$ diam (J), it follows that $p_1+p_2\subset J$ and that one A_1 of the two arcs of J from p_1 to p_2 belongs to $\overline{U}_p$, the other A_2 to E_n-U_p. As p_i is an isolated point of $F(U_p)\cdot M$, it is a local cut point of M [4]). Then p_i is a point of order 2 of M and thus a point of order 1 in both $M\cdot\overline{U}_p$ and $M\cdot(E_n-U_p)$. As the condition of our theorem is a local property and we have seen that p_1 and p_2 (being points of order 1) are not local cut points of $M\cdot\overline{U}_p$, we see that every local cut point of $M\cdot\overline{U}_p$ is an ordinary point of

[1]) A set which is either an arc, a ray, a simple closed curve or an open curve is called a *simple Peano continuum*. A simple Peano continuum is characterized by being a closed set which is homeomorphic with a subcontinuum of a circle or a line.

[2]) A point p of a continuum M is said to be a ***local cut point*** of M if there exists a neighborhood U_p such that p is a cut point of the component of $M\cdot\overline{U}_p$ which contains p.

[3]) G. T. Whyburn, Trans. Amer. Math. Soc., vol. 29 (1927), p. 392; and W. L. Ayres, Annals of Math., vol. 28 (1927), p. 399.

[4]) G. T. Whyburn, Math. Ann., vol, 102 (1929), Th. 31, p. 331.

$M \cdot \overline{U}_p$. Then by the preceding paragraph, $M \cdot \overline{U}_p$ is a simple Peano continuum or is cyclicly connected. As it contains two end points, it cannot be cyclicly connected. Then $M \cdot \overline{U}_p = A_1$. Similarly $M \cdot (E_n - U_p) = A_2$. Thus M is the simple closed curve $J = A_1 + A_2$ and our proof is complete.

Corollary. *If every local cut point of a Peano continuum M in E_n is an ordinary point of M, then every point of M is of order c or every point is of order $\leqq 2$.*

The following simple example shows that neither the theorem nor the corollary is true unless we assume that the continuum M is a Peano continuum. In E_2 let M consist of the points of the curve $y = \sin(1/x)$ for $0 < x \leqq 1$ together with the points (x, y) for $-1 \leqq x \leqq 0$, $-1 \leqq y \leqq 1$.

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On joining finite subsets of a Peano space by arcs and simple closed curves [1]).

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1. Introduction. Two of the most fundamental results in the theory of Peano spaces [2]) relate to the joining of two points by an arc or a simple closed curve. One of these is that every two points of a Peano space can be joined by an arc lying in the space. The other is that if two points are not separated by the omission of any single point, then they can be joined by a simple closed curve. It is the purpose of this paper to give generalizations of these results to sets of n points.

Our conditions are only sufficient. If the Peano space is itself a simple closed curve, then none of our conditions are true but all of the results hold. The determination of necessary and sufficient conditions for n-points seems to be a very difficult problem. Also it seems likely that if the problem were solved, the conditions would be so complex as to be of little interest.

2. Historical Note. The theorem that every two points may be joined by an arc was proved by Mazurkiewicz, Kaluzsay, Moore, Tietze and Vietoris [3]). The theorem for the joining

[1]) Presented to Akademie der Wissenschaften in Wien July 4, 1929, and American Mathematical Society August 30, 1929.

[2]) Following Rosenthal, *Math. Zeit.*, vol. 10 (1921), pp. 102—4, and Kuratowski, *Fund. Math.*, vol. 13 (1929), pp. 307—18, we shall call a metric space, which is the continuous image of a closed interval, a *Peano space*. Other terms commonly used are *continuous curve*, *im kleinen zusammenhängendes Kontinuum*, and *continu de Jordan*.

[3]) S. Mazurkiewicz, *O arytmetyzacji kontinuów*, C. R. de la Société de