closed curve which I shall call J_2. There exists (in 4 dimensions) a set K which is homeomorphic with a plane, contains no point of H, but is such that $K+J_2$ is homeomorphic with a circle plus its interior in a plane. Let M denote $H+K$. Then obviously M is not a manifold. But it has the arc property, and contains a simple closed curve J_1 such that $M-J_1$ is connected and every simple closed curve in $M-J_1$ separates $M-J_1$.

Consider the following condition: „The set M contains k mutually exclusive simple closed curves $\alpha_1, \alpha_2, \ldots, \alpha_k$ whose sum does not separate M, but such that $M-\Sigma_1^k \alpha_i$ is separated by every simple closed curve which it contains“. The example given above shows that if this condition replaces the last condition of the theorem the conclusion no longer follows.

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Concerning the proposition that every closed, compact, and totally disconnected set of points is a subset of an arc.

By

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1. The theorem that every closed, compact and totally disconnected set of points in a space Σ is a subset of a simple continuous arc in Σ was stated by Riesz[1]) in 1906, and by Denjoy[2]) in 1910 and was generalized and proved by Moore and Kline[3]) in 1919 for the case where Σ is the plane. It has been well recognized among topologists that this theorem holds true in case Σ is a euclidean space of any number of dimensions[4]). Evidently it is not valid in case Σ is the space composed of the points of a continuous curve M [= a connected, locally connected, locally compact, metric and separable space] unless some restriction be placed on the continuous curve. For if M is the sum of three arcs ax, bx and cx, where $ax \cdot bx = bx \cdot cx = ax \cdot cx = x$, then obviously no arc in M contains the set $a+b+c$.

The problem of finding a simple and not too restrictive condition on a continuous curve M in order that this proposition be valid in M has been the source of considerable discussion among topologists in recent years. In this article I shall give a solution to this problem embodied in the condition that the continuous curve M

[1]) Comptes Rendus, vol. 141, pp. 650—655.

[2]) Ibid, vol. 151, pp. 138—140.

[3]) Ann. of Math., vol. 20, pp. 218—223.

[4]) So far as the author knows, however, no proof has been given, up to the present time, even for this case of the theorem.

should have no local separating point [1]). In fact we shall show that with this restriction on the space M the following stronger proposition is true:

Every closed, compact and totally disconnected set of points K is a subset of an arc pq in M, where p and q are points arbitrarily chosen in the set K.

Before proceeding to the proof of this result we first establish some preliminary theorems and lemmas which are needed. We conclude the paper with a discussion of the proposition announced in the title from the point of view of the cyclic-element theory and give a necessary and sufficient condition for the validity of this proposition in uni-coherent spaces M.

2. *Preliminary Theorems (Concerning Continuous Curves in General).* We consider a locally compact metric, separable, connected and locally connected space which we denote by M and which we call a continuous curve. Any connected open subset of M will be called a region; and if R is a region, $F(R)$ will denote the boundary of R.

(I). *If R is a region and K is any closed and compact subset of R, then there exists a region U such that $K \subset U \subset \overline{U} \subset R$* [2]).

For each positive number d, let $B(d)$ denote the set of all points of R at a distance d from $F(R)$. Let $f = 1/2\,\varrho[F(R), K]$. Then K is contained in a finite number $C_1, C_2, \ldots, C_n$ of the components of $R - B(f)$. For each integer i, $1 < i \leqslant n$, there exists an arc t_i in R joining a point of C_i to a point of C_1. Let $G = \sum_1^n C_j + \sum_2^n t_i$, let $h = 1/2\,\varrho[G, F(R)]$, and let U be the component of $R - B(h)$ containing G. Then U has the desired properties.

(II). *If R is any region, K is a closed and compact subset of R, and A and B are any two components of K, then there exist two mutually exclusive regions R_a and R_b containing A and B respectively and such that $R_a + R_b \supset K$ and $R \cdot F(R_a) = R \cdot F(R_b) = R - [R_a + R_b]$.*

[1]) A local separating point of a continuous curve M is a point which is a cut point of some region (= connected open subset) of M. An interesting example of a continuous curve having no local separating point is that of the universal plane 1-dimensional curve due to Sierpiński. See Comptes Rendus, vol. 162, p. 629.

[2]) This result is an immediate consequence also of a theorem of R. L. Wilder's; see Bull. Amer. Math. Soc., vol. 34 (1928), p. 652.

A simple application of the Borel Theorem shows that there exists an open subset G of R containing A but not B whose boundary H is a subset of R and contains no point of K. Let q be a positive number which is less than $\varrho[H, K + F(R)]$. For each number d, $0 < d < q$, let $L(d)$ denote the set of all points of M at a distance d from the point set H. Then for each d, $L(d)$ is a cutting [1]) of R between A and B which contains no point of K. For each d, let $V(d)$ be the component of $R - L(d)$ containing A, let $R_b(d)$ be the component of $R - \overline{V(d)}$ containing B, and let $R_a(d)$ be the component of $R - \overline{R_b(d)}$ containing A. Finally, let $X(d) = R \cdot F[R_a(d)] = R \cdot F[R_b(d)]$. Then for each d, $X(d)$ is a cutting of R between A and B. Furthermore, the collection of cuttings $[X(d)]$ is non-separated [1]). For take any two d's, d_1 and d_2, and suppose $d_1 > d_2$. Then $\overline{V(d_1)} \supset X(d_1)$, and $\overline{V(d_1)}$ contains no point of $X(d_2)$. Hence $X(d_1) \subset R_a(d_2)$, and thus $X(d_2)$ does not separate $X(d_1)$. Since $X(d_1) \subset R_a(d_2)$, therefore $\overline{R_b(d_2)} = R_b(d_2) + X(d_2) \subset R_b(d_1)$, and thus $X(d_1)$ does not separate $X(d_2)$. Therefore $[X(d)]$ is non-separated. Accordingly, since this collection is uncountable, there exists [2]) a number d such that $R - X(d)$ is the sum of two connected point sets, and these must be the sets $R_a(d)$ and $R_b(d)$. The regions $R_a = R_a(d)$ and $R_b = R_b(d)$ satisfy all the conditions on the regions desired, for $X(d) \cdot K = 0$.

(III). *Under the same hypothesis as in (II), there exist regions U_a and U_b containing A and B respectively and such that $\overline{U}_a + \overline{U}_b \subset R$, $\overline{U}_a \cdot \overline{U}_b = 0$, and $U_a + U_b \supset K$.*

First apply (II), getting regions R_a and R_b. Set $K_a = K \cdot R_a$, $K_b = K \cdot R_b$, and then apply (I) to the sets R_a and R_b getting regions U_a and U_b containing K_a and K_b respectively and so that $\overline{U}_a \subset R_a$ and $\overline{U}_b \subset R_b$. Then the regions U_a and U_b will have the desired properties.

[1]) A set X is called a cutting of a region R between two connected sets A and B if A and B belong to different components of $R - X$, and X is called simply a cutting of R if $R - X$ is not connected. A collection G of cuttings of R is said to be non-separated if for each pair of elements X, Y of G, X lies in a single component of $R - Y$ and Y lies in a single component of $R - X$. See the abstract of my paper *Non-separated cuttings of continua*, Bull. Amer. Math. Soc., vol. 36 (1930), p. 216.

[2]) Loc. cit.

(IV). *Let R be any compact region and let N be a closed subset of $M-R$ such that $N\cdot\overline{R}$ is totally disconnected. Then there exists a compact region G containing R but containing no point of N and such that* (1) $\overline{G}\cdot N=N\cdot\overline{R}$, (2) $G+N\cdot\overline{R}$ *contains a continuous curve H which contains $N\cdot\overline{R}$ and is such that $H-N\cdot\overline{R}$ is connected and contains R, and* (3) *each point of $N\cdot\overline{R}$ is accessible from $H-N\cdot\overline{R}$ and hence also from G.*

Proof. Let K_1 denote the set of all points of $\overline{R}$ at a distance $\geqslant 1$ from the point set $\overline{R}\cdot N$; and for each integer $n>1$, let K_n denote the set of all points x of $\overline{R}$ such that $1/n\leqslant\varrho(x,\overline{R}\cdot N)\leqslant\leqslant 1/(n-1)$.

A simple application of the Borel Theorem proves the existence, for each positive integer n, of a finite number of compact continua $C_1^n, C_2^n,\dots, C_m^n$ each containing a point of R and whose sum C^n contains K_n in its interior (rel. M) but contains no point whose distance from K_n is greater than $1/4n$. For each i, $1<i\leqslant m$, let t_i be an arc in R joining a point of C_i^n to a point of C_1^n. Add all these arcs t_i to C^n and call the point set thus obtained D_n. Then D_n is a compact continuum which contains K_n but contains neither a point of N nor any point of $M-R$ whose distance from K_n is greater than $1/4n$. By a theorem due to Ayres and the author [1]), M contains, for each n, a compact continuous curve H_n containing D_n but which contains neither a point of N nor any point of $M-R$ whose distance from K_n is greater than $1/2n$.

Let $H_0=\sum_1^\infty H_n$, and let $H=H_0+\overline{R}\cdot N$. For each n, let G_n be a compact region containing H_n but containing no points or boundary points in N and containing no point whose distance from H_n is greater than $1/2n$. Let $G=\sum_1^\infty G_n$. Then the sets G and H have the desired properties.

For it is easily seen that H_0 is connected, that H is a continuum, and that $H=\overline{H}_0\cdot H$ must be a continuous curve. For if not, it fails to be locally connected at some one of its points p belonging to H_0, because $H-H_0=N\cdot\overline{R}$ is totally disconnected. There exists an n so that p belongs to H_n. If p belongs to $\overline{R}$, it is contained in the interior of H_n relative to M and in this case H is locally

[1]) See Bull. Amer. Math. Soc., vol. 34, (1928), p. 350.

connected at p. If p does not belong to $\overline{R}$, then $\varrho(p,K_n)\leqslant 1/2n$ and hence $\varrho(p,\overline{R}\cdot N)\geqslant 1/2n$. Hence p cannot belong to H_m, for any $m>6n$. Then since $\sum_1^{6n} H_i$ is locally connected at p, H must be locally connected at p. Thus H is a continuous curve. Since the boundary $H-H_0$ of H_0 relative to H is totally disconnected, it follows [1]) that every point of $H-H_0$ $(=\overline{R}\cdot N)$ is accessible from H and hence also from G.

(V). *If K is any closed, compact and totally disconnected subset of a region R, then for each $\varepsilon>0$ there exists a finite collection of compact regions $R_1, R_2,\dots, R_n$ covering K each of diameter $<\varepsilon$ and such that for each i, $j\leqslant n$, $\overline{R}_i\cdot\overline{R}_j=0$ and $\overline{R}_i\subset R$.*

Proof. It is well known that K is the sum of a finite number of mutually exclusive closed and compact points sets $K_1, K_2,\dots, K_m$ each of diameter $<\varepsilon/4$. With the aid of the Borel Theorem it follows from the local connectivity of R that there exists a finite collection of compact regions $U_1, U_2,\dots, U_k$ in R covering K and each containing at least one point of K, each of diameter $<\varepsilon/4$, and such that for each $i\leqslant k$ and each $j\leqslant m$ it is true that if U_i contains a point of K_j, then $\delta(U_i)<1/2\,\varrho(K_j, K-K_j)$. Now let $V_1, V_2,\dots, V_n$ denote the components of the point set $\sum_1^k U_i$. Then since $\delta(K_j)<\varepsilon/4$ for every $j\leqslant m$, it follows at once that $\delta(V_r)<\varepsilon$ for every $r\leqslant n$. Now for each $r\leqslant n$, let $K_r=K\cdot V_r$. Then K_r is closed and compact; and applying (I) we get a regions R_r such that $K_r\subset R_r\subset\overline{R}_r\subset\subset V_r\subset R$. The sets $R_1, R_2,\dots, R_n$ thus defined have the desired properties.

3. *Preliminary theorem (concerning continuous curves without local separating points).*

(VI). *If the space M has no local separating point and if H and K are mutually exclusive non-degenerate* [2]) *subsets of a region R, then R contains two mutually exclusive arcs each joining a point of H and a point of K.*

[1]) See the author's paper *A generalized notion of accessibility*, Fund. Math., vol. 14 (1929), p. 315.

[2]) A set of points is degenerate or non-degenerate according as it does or does not reduce to a single point.

For since M has no local separating point, R can have no cut point. Hence (VI) follows at once from a theorem of Ayres[1]).

Definition. By an *arc-region chain* C in a region R of a continuous curve M between two points p and q is meant the sum of a finite number, $n+1$, of mutually exclusive closed and compact regions, $\overline{R}_0, \overline{R}_1, \ldots, \overline{R}_n$ and of n mutually exclusive arcs $A_1, A_2, A_3, \ldots, A_n$ such that for each i, $0 < i \leqslant n$, A_i has one endpoint p_i in $\overline{R}_{i-1}$ and the other q_i in $\overline{R}_i$ but otherwise has no point in common with $\sum_0^n \overline{R}_j$, such that $p \subset R_0$, $q \subset R_n$ and R_0 and R_n may or may not be degenerate but R_i $(0 < i < n)$ is non-degenerate, and finally such that for each i, $0 < i \leqslant n$, the points q_i and p_{i+1} are accessible from R_i. The sets $[\overline{R}_i]$ and $[A_i]$ will be called the *region-links* and the *arc-links* respectively of C.

(VII). *If K is any closed, compact, and totally disconnected subset of a region R in a continuous curve having no local separating point, G is any collection of regions covering K which satisfy the conditions of* (V), *and p and q are any two points of K, then there exists an arc-region chain C in R from p to q such that each region-link of C is a subset of some region of G and such that the sum of all the region-links of C contains K in its interior.*

Lemma to (VII). If X is any arc region chain in R from p to q, S is a sub-region of R such that $\overline{S} \cdot X = 0$ and $(\overline{S} - S) \cdot K = 0$, T is any arc in R one endpoint of which belongs to $\overline{S}$ and the other to X but which otherwise, contains no points of $X + \overline{S}$, and U is any region whatever containing T, then there exists an arc-region chain X' from p to q which is a subset of $X + U + S$ such that every region-link of X' is a subset either of S or of some region-link of X and such that the sum of all the region-links of X' contains in its interior the point set $K \cdot S + K \cdot (V_0 + V_1 + \ldots + V_n)$, where $\overline{V}_0, \overline{V}_1, \ldots, \overline{V}_n$ are the region-links of X.

Proof of the lemma. We may suppose without loss of generality that for some i, $0 < i < n$, the point $T \cdot X$ belongs to $\overline{V}_i$, for an obvious modification of the argument which is to follow takes care of the other simpler cases which might arise. We may suppose also that the end point of T which belongs to $\overline{V}_i$ is neither of the points q_i and p_{i+1} (i. e. neither of the endpoints of the two arc-links α_i and α_{i+1} of X ending in $\overline{V}_i$); for with the aid of (VI) it follows that there exist in U two mutually exclusive arcs from $\overline{S}$ to $\overline{V}_i$; and if U is chosen so that it does not contain both q_i and p_{i+1}, then one of these arcs, say α, fails to contain either q_i or p_{i+1}; and hence either α contains a subarc from $\overline{S}$ to $\overline{V}_i$ having only one point in $\overline{V}_i$ or it contains a subarc T' from a point a' of $\overline{S}$ to a point b' which is an interior point of some arc-link α_j of X and such that $T' \cdot X = b'$; and in the latter case, it will be seen that if in the argument below T is replaced by T', the proof is only simpler than in the case there treated. Finally let us suppose U so chosen that $\overline{U}$ contains no points of $q_i + p_{i+1} + (X - \overline{V}_i)$.

Now clearly the sets $U \cdot V_i$ and $U \cdot S$ are non-degenerate, and thus by (VI) there exist in U two mutually exclusive arcs $x_1 y_1$ and $x_2 y_2$ such that $x_1 + x_2 \subset U \cdot V_i$ and $y_1 + y_2 \subset U \cdot S$. Since q_i and p_{i+1} are accessible from V_i, it follows that there exist mutually exclusive arcs $h_1 q_i$ and $h_2 p_{i+1}$ which are subsets of $V_i + q_i - V_i \cdot \overline{U}$ and $V_i + p_{i+1} - V_i \cdot \overline{U}$ respectively. Now the set of points $K^* = K \cdot V_i + h_1 + h_2 + x_1 + x_2$ is a closed subset of V_i. Hence, by (I), there exists a region Q containing K^* and such that $\overline{Q} \subset V_i$. Now in the orders from the points h_1, h_2, x_1, x_2 to the other ends respectively on the arcs $h_1 q_i$, $h_2 p_{i+1}$, $x_1 y_1$ and $x_2 y_2$, let z_1, z_2, w_1, and w_2 respectively denote the last points belonging to $\overline{Q}$. Let N denote the set of points which is the sum of the four arcs $z_1 q_i$, $z_2 p_{i+1}$, $w_1 y_1$ and $w_2 y_2$ together with all points of the space which are not in V_i. Then N is closed and $N \cdot \overline{Q} = z_1 + z_2 + w_1 + w_2$. Hence by (IV) it follows that V_i contains a continuous curve H such that $H \cdot N = z_1 + z_2 + w_1 + w_2$ and $H - H \cdot N$ is connected and contains Q.

Now since H contains Q and Q can have no cut point, it follows that no point of H separates the point sets $z_1 + z_2$ and $w_1 + w_2$ in H. Hence by a theorem of Ayres[1]), there exists in H two mutually exclusive arcs joining these two sets. The two possible cases here are alike, so we shall suppose that mutually exclusive arcs $z_1 w_1$ and $z_2 w_2$ exist in H. Let K^0 denote the point set $K \cdot V_i + z_1 w_1 + z_2 w_2$. Then K^0 is a closed subset of H and $z_1 w_1$ and $z_2 w_2$ are components of K^0. Thus, by (II), there exist

[1]) See Amer. Journ. Math., vol. 51, (1929), p. 590.

[1]) Loc. cit.

mutually exclusive regions H_1 and H_2 (rel. H) containing $z_1 w_1$ and $z_2 w_2$ respectively and such that $H_1 + H_2 \supset K^0$. It follows by (I) that there exist regions L_1 and L_2 (rel. H) such that $H_1 \supset \overline{L}_1 \supset$ $\supset L_1 \supset H_1 \cdot K^0$ and $H_2 \supset \overline{L}_2 \supset L_2 \supset H_2 \cdot K^0$, and thus $\overline{L}_1 \cdot \overline{L}_2 = 0$. Now it is easily seen that there exist regions Q_1 and Q_2 (rel. R) containing L_1 and L_2 respectively and such that $\overline{Q}_1 \cdot \overline{Q}_2 = 0$ and $\overline{Q}_1 + \overline{Q}_2 \subset V_i$. On the arcs $p_i q_i z_1$ and $y_1 w_1$ in the orders $p_i z_1$ and y_1, w_1 let q_i' and p_{i+1}' respectively denote the first points belonging to $\overline{Q}_1$. Similarly on the arcs $q_{i+1} p_{i+1} z_2$ and $y_2 w_2$ let p'_{+3} and q'_{i+2} denote the first points of $\overline{Q}_2$. Now with the aid of (IV) it follows that there exist regions G_1 and G_2 (rel. R) containing Q_1 and Q_2 respectively and such that if F denotes the sum of the four arcs $p_i q_i'$, $y_1 p_{i+1}'$, $y_2 q_{i+2}'$, and $p_{i+3}' q_{i+1}$, then $\overline{G}_1 \cdot F = q' + p_{i+1}'$ and $\overline{G}_2 \cdot F =$ $= q_{i+2}' + p_{i+3}'$, such that $\overline{G}_1 \cdot \overline{G}_2 = 0$, $\overline{G}_1 + \overline{G}_2 \subset V_i$ and such that q_i' and p_{i+1}' are accessible from G_1 and q_{i+2}' and p_{i+3}' are accessible from G_2.

By virtue of (I), there exists a region D such that $K \cdot S + y_1 +$ $+ y_2 \subset D \subset \overline{D} \subset S$. Let q_{i+1}' and p_{i+2}' respectively denote the first points on the arcs $p_{i+1}' y_1$ and $q_{i+2}' y_2$ in these orders which belong to $\overline{D}$. It follows with the aid of (IV) that there exists a region E which is a subset of S and is such that $\overline{E} \cdot (p_{i+1}' q_{i+1}' + p_{i+2}' q_{i+2}') =$ $= q_{i+1}' + p_{i+2}'$ and the points q_{i+1}' and p_{i+2}' are accessible from E.

Now, for each integer j, $0 \leqslant j < i$, set $V_j = V'$, and for $0 < j < i$, set $p_j = p'$ and $q_j = q_j$. Set $p_i = p_i'$, $G_1 = V_i'$, $E = V_{i+1}'$, $G_2 = V_{i+2}'$, $q_{i+1} = q_{i+3}'$; and for each integer m, $i < m \leqslant n$, set $V_m = V_{m+2}'$ and for $i + 1 < m \leqslant n$, set $p_m = p_{m+2}'$ and $q_m = q_{m+2}'$. Finally, set

$$X' = \sum_0^{n+2} \overline{V'} + \sum_1^{n+2} p' q_j'.$$

Then clearly X' is an arc-region chain from p to q which lies in R and has all the desired properties for the lemma.

Proof of (VII). Let the regions of the collection G be denoted by $G_0, G_1, G_2, \ldots, G_m$, where G_0 contains p and G_1 contains q. By virtue of (I) there exist regions V_0 and V_1 containing the sets $G_0 \cdot K$ and $G_1 \cdot K$ respectively and such that $\overline{V}_0 \subset G_0$ and $\overline{V}_1 \subset G_1$. There exists an arc A_1 in R having one endpoint, p_1, in $\overline{V}_0$ and the other, q_1, in $\overline{V}_1$ but otherwise containing no point of $\overline{V}_1 + \overline{V}_2$. By (IV), there evist regions R_0 and R_1 which lie respectively in G_0 and G_1 and contain V_0 and V_1 respectively and are such that $\overline{R}_0 \cdot A_1 = p_1$, $\overline{R}_1 \cdot A_1 = q_1$ and the points p_1 and q_1 are accessible from R_0 and R_1 respectively. Clearly the point set $C = \overline{R}_0 + A_1 + \overline{R}_1$ is an arc-region chain in R from p to q every region-link of which is a subset of some one of the regions G_0 and G_1 and such that the sum of all the region-links of C contains in its interior the point set $K \cdot G_0 + K \cdot G_1$.

Suppose now that for any integer n, $0 < n < m$, we can define such a chain C in R every region-link of which is a subset of some one of the regions $G_0, G_1, \ldots, G_n$ and which contains in the interior of the sum of all its region-links the point set $K \cdot G_0 + K \cdot G_1 + \ldots + K \cdot G_n$. We shall proceed to show that under these conditions we can define such a chain C such that the sum of all the regions-links of C such that the sum of all the regions-links of C contains the set $K \cdot G_0 + K \cdot G_1 + \ldots + K \cdot G_{n+1}$ in its interior and hence by induction establish (VII).

We have then, by supposition, a chain $C = \sum_0^{n'} \overline{R}_i + \sum_1^{n'} A_i$ such that $\sum_0^{n'} R_i \supset \sum_0^{n} K \cdot G_i$ and such that for every $i \leqslant n'$ there exists a $j \leqslant n$ so that $R_i \subset G_j$. By virtue of the condition last stated, $\overline{G}_{n+1}$ can have no point in common with the set $\sum_0^{n'} \overline{R}_i$. It may, however, have points in common with $\sum_1^{n'} A_i$. Now if $\overline{G}_{n+1}$ has at most one point or has only boundary points in common with C (and therefore with $\sum_1^{n'} A_i$), then with the aid of (I) we can replace G_{n+1} by a subregion G_{n+1}^* of G_{n+1} which will contain $K \cdot G_{n+1}$ but will contain no points of C and will have no limit points in C and thus reduce this case to the case in which $C \cdot \overline{G}_{n+1} = 0$. In this case, since there exists in R an arc T joining a point a of $\overline{G}_{n+1}$ and a point b of C but which otherwise is free of points of $\overline{G}_{n+1} + C$, then by applying the lemma, using $G_{n+1} = S$, we obtain an arc-region chain having the desired properties.

Thus we may suppose that $C \cdot G_{n+1}$ contains at least two points. Let A_i and A_j respectively denote the first and the last of the arc-links $A_1, A_2, \ldots, A_{n'}$ which contains points of G_{n+1}. Let x_i and x_j denote points belonging to $A_i \cdot G_{n+1}$ and $A_j \cdot G_{n+1}$ respectively. Now by applications of (I) and (IV) we can obtain a region E

which lies together with its limit points in G_{n+1} and contains $K\cdot G_{n+1}+x_i+x_j$ and is such that the first points q_i' and p_j' on the arcs A_i and A_j respectively in the orders p_i, q_i and q_j, p_j which belong to $\overline{E}$ are accessible from E. Let X denote the arc-region chain $[\overline{R}_0+\ldots+\overline{R}_{i-1}+\overline{E}+\overline{R}_j+\overline{R}_{j+1}+\ldots+\overline{R}_{n'}]+[A_1+A_2+ +\ldots+A_{i-1}+\text{arc } p_i q_i' \text{ of } A_i+\text{arc } p_j' q_j \text{ of } A_j+A_{j+1}+A_{j+2}+\ldots+A_{n'}]$. Then X is an arc-region chain in R from p to q every region link of which lies in some region of the collection $[G_r]$, $(0\leqslant r\leqslant n+1)$, and such that the sum of all the region-links of X contains in its interior the point set

$$\sum_{t=1}^{n+1} K\cdot G_t-\sum_{t=i}^{j-1} K\cdot R_t.$$

Now on the arc A_i, in the order q_i, p_i. let z denote the first point belonging to $\overline{E}$ and let T denote the arc zq_i of A_i. There exists a region U containing T and such that

$$X\cdot U\subset\overline{E} \quad\text{and}\quad U\cdot C\subset A_i+\overline{R}_i-p_{i+1}.$$

By the lemma there exists an arc-region chain X^1 from p to q lying in $X+U+R_i$ every region-link of which is a subset either of R_i or of some region-link of X and such that the sum of all the region-links of X^1 contains in its interior the point set $K\cdot R_i+K$. (the sum of the region-links of X), which is identically the set $\sum_{t=1}^{n+1} K\cdot G_t-\sum_{t=i+1}^{j-1} K\cdot R_t$.

Now, using the fact that the point p_{i+1} is accessible from R_i, it is readily seen that R_i+p_{i+1} contains an arc yp_{i+1} having only the point y in common with X^1. Then $yp_{i+1}+A_{i+1}$ is an arc joining the point q_{i+1} of $\overline{R}_{i+1}$ and the point y of X^1. Let w denote the first point on this arc in the order q_{i+1}, y belonging to X^1, and let T^1 denote the subarc of this arc from q_{i+1} to w. Let U^1 be a region in R containing T^1 and such that $U^1\cdot X^1\subset\overline{R}_i+\overline{E}$ and $U^1\cdot C\subset\overline{R}_i+\overline{R}_{+1}+A_{i+1}$. Then, applying the lemma, we obtain an arc-region chain X^2 from p to q lying in $X^1+U^1+R_{i+1}$, every region-link of which is a subset of either R_{i+1} or some region link of X^1 and such that the sum of all the region-links of X^2 contains in its interior the point set $K\cdot R_{i+1}+K$. (the sum of the region links of X^1), which is identically the set of points

$$\sum_{t=1}^{n+1} K\cdot G_t-\sum_{t=i+2}^{j-1} K\cdot R_t.$$

Now, continuing this process, we see that after $j-i$ steps we obtain an arc-region chain $X^{j-i-1}=C^*$ from p to q in R every region-link of which is a subset of some region of $[G_r]$, $(0<r<n+1)$, and such that the sum of all the region-links of C^* contains in its interior the point set

$$\sum_{1}^{n+1} K\cdot G_t-\sum_{j-1}^{j-1} K\cdot R_t=\sum_{1}^{n+1} K\cdot G_t.$$

Therefore, by the principle of induction, it follows that there exists an arc-region chain C from p to q in R satisfying all the conditions in (VII).

Corollary. *With hypothesis as in* (VII) *except that p and q are any two accessible boundary points of R, there exists an arc-region chain C in $R+p+q$ from p to q satisfying the conditions on the chain C in* (VII).

4. Theorem. *If K is any closed, compact and totally disconnected subset of a continuous curve M having no local separating point and p and q are any two points of K, then there exists in M an arc pq which contains K.*

Proof. Let G^1 be a finite collection of regions $G_0, G_1, G_2, \ldots, G_n$, covering K each of diameter <1 which satisfies all the conditions of (V). By (VII) there exists an arc-region chain $C_1=\sum_0^{m_1}\overline{R}_i+\sum_1^{m_1}A_i$ from p to q such that $\sum_0^{m_1}R_0\supset K$ and every region-link of C_1 is contained in some region of G^1. Now for each i, $0\leqslant i\leqslant m_1$, let G_i^2 be a collection of regions $G_{i0}, G_{i1}, \ldots, G_{in_2}$ covering $K\cdot R_i$ each of which is a subset of R_i and is of diameter $<1/2$, which satisfies all the conditions of (V). By (VII), and corollary, there exists an arc-region chain C_2^i from q_i to p_{i+1} (where q_i is the end point of A_i lying in $\overline{R}_i$ and $q_i=p$ if $i=0$, and p_{i+1} is the end point of A_{i+1} in $\overline{R}_i$ and $p_{i+1}=q$ if $i=n$) which contains $K\cdot R_i$ in the interior of the sum of all its region-links, is a subset of $R_i + q_i + p_{i+1}$, and each of its region-links is a subset of some region of G_i^2. Clearly the point set $C_2 = \sum_{i=1}^{m_1} C_2^i$ is an arc-region chain from p to q every region-link of which is a subset of some region-link of C_1. Continuing this process indefinitely, we obtain a sequence $C_1, C_2, C_3, \ldots$ of arc-region chains in M from p to q such that, for each i, $C_i \subset C_{i-1}$ and each region-link of C_i is of diameter $< 1/i$ and is a subset of some region-link of C_{i-1}.

Let $C = pq = \prod_1^\infty C_i$. Then C is an arc in M from p to q which contains K. For clearly C is a compact continuum containing K. And if x is any point of C distinct from p and from q, then either x is an interior point of some arc-link of C_n for some n, in which case clearly $C - x = \prod_n^\infty P_m + \prod_n^\infty Q_m$ (where P_m and Q_m are the components of $C_m - x$ containing p and q respectively) is not connected, or else there exists a sequence of regions $R_{n_1}, R_{n_1 n_2}, R_{n_1 n_2 n_3}, \ldots$ such that for each i, $\overline{R}_{n_1, \ldots, n_i}$ is a region-link of the chain C_i and such that $x = \prod_1^\infty \overline{R}_{n_1 n_2 \ldots n_i}$. But in the latter case it is easily seen that $C - x = [\text{Lim } P_n - x] + [\text{Lim } Q_n - x]$, where P_n and Q_n respectively denote the components of $C_n - R_{n_1 n_2 \ldots n_n}$ containing p and q and that $C - x$ is therefore not connected. Hence it follows [1]) that C is an arc pq from to q which lies in M and contains K, and our theorem is proved.

5. Conclusion. Let A denote the property of a continuous curve M to have no local separating point; let B denote the property of being such that if K is any closed, compact and totally disconnected subset of M and p and q are points of K then M contains an arc pq which contains K; and let C denote the property that each closed, compact and totally disconnected subset of M be contained in an arc in M. Then with the aid of our theorem established above in § 4 it is seen that property A implies property B but not conversely, and clearly property B implies property C but not conversely. Thus we have the relation $A \rightarrow B \rightarrow C$.

[1]) See R. L. Moore, Trans. Amer. Math. Soc., vol. 21 (1920), p. 340.

Now every maximal cyclic curve of a continuous curve may have property A and yet M not have property C. For let M be a the sum of three sets P, Q, and R in the plane each of which is a simple closed curve plus its interior and such that $P \cdot Q = P \cdot R = R \cdot Q =$ one point x; and let K be the sum of three points p, q and r belonging to $P - x$, $Q - x$, and $R - x$ respectively. Then no arc in M can contain K. However, we may state the following

Theorem. (*a*). *If each maximal cyclic curve of a continuous curve M has property B, then M has property C if and only if the cyclic elements of M form a simple cyclic chain.*

The condition is necessary. For if the cyclic elements of M do not form a simple cyclic chain, then if E_a and E_b are nodes [1]) of M and X is the simple cyclic chain in M from E_a to E_b, there exists a component C of $M - X$ which, then, has just one limit point x in X. Then if a and b are points of E_a and E_b respectively which are non-cut points of M in case these sets or either of them is non-degenerate and c is a point of C, it is readily seen that no arc in M contains the set $a + b + c$, contrary to property C.

The condition is also sufficient. For suppose the cyclic elements of M from a simple cyclic chain X in M between two cyclic elements E_a and E_b. Let a and b be points of E_a and E_b which are non-cut points of M in case E_a and E_b or either is non-degenerate, let N be the set of all those points which separate a and b in M, and let K be any closed, compact and totally disconnected subset of M. Let $C_1, C_2, C_3, \ldots$ be the non-degenerate cyclic elements of M, and for each i, let a_i and b_i be the two points of $C_i \cdot (N + a + b)$. It follows from our hypothesis, that for each i, C_i contains an arc t_i from a_i to b_i which contains $K \cdot C_i + a_i + b_i$, since this set is closed compact and totally disconnected; and it is immediately seen that the point set $N + a + b + \sum_{i=1,2,\ldots} t_i$ is an arc in M which contains K.

Now in case the continuous curve M is uni-coherent [2]), it fol-

[1]) A *node* of a continuous curve M is either an end point of M or a maximal cyclic curve of M containing just one cut point of M. For definitions and theorems concerning the cyclic elements of a continuous curve the reader is referred to the author's paper in the Amer. Jour. Math., vol. 50 (1928). pp. 167—194.

[2]) That is, if the common part of every two continua H and K whose sum is M is a continuum: See C. Kuratowski, Fund. Math., vol. 12, p. 24.

lows at once that no cyclic element of M can have a local separating point, and thus every cyclic element of M has property A and hence also property B. Therefore from Theorem (α) we get

Theorem (β). *In order that the uni-coherent continuous curve M should have property C it is necessary and sufficient that the cyclic elements of M should form a simple cyclic chain.*

Remark. While the condition in our principal theorem above is not a necessary one, we see that the following condition stated in terms of local separating points is a necessary — though not a sufficient one: *in order that a continuous curve have property C* (or, of course, B) *it is necessary that M have no local separating point p which cuts M locally into more than two components* (i. e. such that a region R exists containing p and such that $R-p$ has more than two components). For if such a point p exists in M, then if X, Y, and Z are distinct components of $R-p$ and $[x_i]$, $[y_i]$, and $[z_i]$ are sequences of points in X, Y, and Z respectively each converging to p, it is easily seen that no arc in M contains the point set $p+\sum_{i=1}^{\infty}(x_i+y_i+z_i)$.

Über die symmetrisch allgemeinen Lösungen im Klassenkalkul.

Von

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Einleitung.

Vorliegende Arbeit ist ein kurzer Auszug eines Teiles meiner auf norwegisch geschriebenen Abhandlung „Undersökelser indenfor logikkens algebra“ vom Jahre 1913. Diese Abhandlung ist bisher nicht veröffentlicht worden, sondern wird im Archiv der Universität Oslo aufbewahrt. Eine vollständige Wiedergabe jener Arbeit wird in den Schriften der Akademie der Wissenschaften zu Oslo bald erscheinen.

Die Aufgabe, welche im folgenden behandelt wird, ist die, Gleichungen oder Subsumtionen im Klassenkalkul in bezug auf eine oder mehrere Unbekannten symmetrisch aufzulösen; man findet solche Aufgaben schon in Schröders „Algebra der Logik“. Abgesehen davon, daß ich $A\rightarrow B$ schreibe um auszudrücken, daß eine Aussage B aus der Aussage A folgt, sind die hier benutzten Bezeichnungen immer die Schröderschen. Weiter bemerke ich, daß ich die Addition der Klassen (Bildung ihrer Vereinigung) und ihre Multiplikation (Durchschnittsbildung) hier ohne Skrupel auf ganz beliebige Mengen von Klassen anwende, wie es ja nach den Axiomen der Mengenlehre möglich sein soll. Auf Grundlagenfragen soll hier nicht eingegangen werden.

§ 1.

In dem einfachsten Teile des Logikkalkuls (Gebiete- oder Klassenkalkul, auch identischer Kalkul genannt) hat man besonders die Aufgabe zu studieren, Gleichungen oder Subsumtionen in bezug auf eine oder mehrere Unbekannten aufzulösen. Ist die Zahl der Unbe-