End-sets of continua irreducible between two points.

By

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I. Introduction.

A *continuum irreducible between two points* is a continuum which contains these two points but which contains no proper subcontinuum which contains them [1]). Throughout this paper *M will represent a continuum irreducible between two given points a and b. If x and y are any two points of M then K_{xy} will represent any subcontinuum of M containing $x+y$; and xy will represent any subcontinuum of M irreducible between x and y. Letters such as M, N,... will be used to represent point sets and letters such as m, n,... will be used to represent points.*

Although much work has been done on irreducible continua [2]) still because of the importance of this type of continua a much more detailed study should be made. In this paper a study is made from the standpoint of end-sets and related sets. Conditions necessary and sufficient not only that an irreducible continuum be indecomposable but also that various subcontinua of an irreducible continua be indecomposable are obtained. In the application of the theory of irreducible continua to various problems among other things it is often of importance to know between what points there

[1]) Such irreducible continua were first defined by L. Zoretti, Ann. de l'Ecole Normale, XXVI. For a generalization see W. A. Wilson, *On the oscillation of a continuum at a point*, Trans. Amer. Math. Soc., Vol. 27 (1925), p. 429—440; see also H. M. Gehman, *Concerning irreducible connected sets and irreducible continua*, Proc. National Academy of Sciences, Vol. 12 (1926), p. 544—547.

[2]) For bibliography see C. Kuratowski, Fund. Math. III., p. 200 and Fund. Math. X., p. 274.

exist one and only one irreducible continuum in a given point set. It is also of importance to have defined subsets of irreducible continua which will enable various types of irreducible continua to be characterized and thus used to describe the properties of other point sets. In this paper some consideration has been given to these problems.

C. Kuratowski defines and obtains properties of a set $I(a, M)$ of an irreducible continuum [1]). Let $(a) = I(a, M)$ and $(b) = I(b, M)$. In section II of this paper among other things it is shown that the following conditions are each necessary and sufficient in order that M be indecomposable:

(1) $$(a) \times (b) \neq 0$$

(2) $$(a) + (b) = M$$

(3) $$(a) \times ((b)' - (b)) \neq 0 \text{ [2])}$$

(4) $$((a)' - (a)) + (a) \times (b) + ((b)' - (b)) = M.$$

If $cd = M$ then $c+d$ will be called a non-remainder set of M. Yoneyama [3]) proved that if $c+d$ is a non-remainder set of M then either $a+c$ or $a+d$ is a non-remainder set of M. In section III among other things it is proven that in order that M be decomposable it is necessary and sufficient that $(a) \times (p+q)$ and $(b) \times (p+q)$ each contain one and only one point for each non-remainder set $p+q$ of M.

C. Kuratowski [4]) has proven the important theorem that if K is any subcontinuum of M containing a then a necessary and sufficient condition that K be contained in (a) is that $(M-K)' = M$. In section IV among other things it is proven that if (b) contains p and M contains a decomposable pb then (b) contains pb.

In section V basic-wise connected subsets of M are defined. It is proven that if M is bounded then a necessary and sufficient condition that M is indecomposable is that $(a)' - (a)$ be a maximal basic-wise connected subset of M everywhere dense in M. After obtaining other characterizations of indecomposable continua by means of these sets it is proven that if M is a bounded decompo-

[1]) Fund. Math. X., p. 230. See also section II of this paper.

[2]) If W is a point set W' will be used to denote the set composed of W and the limit points of W.

[3]) Tôhoku Math. Jour., Sendai 1917, p. 48, theorem 3.

[4]) Fund. Math. X., p. 233.

sable continuum then in order that M be the sum of two proper indecomposable continua it is necessary and sufficient that M contain a basic-wise connected subset T such that both T and $M-T$ are everywhere dense in M.

Let $cb=ab$. In section VI a study is made of the relation of $(a)_b=(a)$, $(b)_a=(b)$, $(c)_b$ and $(b)_c$. It is shown that in order that M be decomposable it is necessary and sufficient that $(a)_b=(a)_c$ for every point c of (b). If M is bounded and g is a point of M it is well known that M contains a bg which has a set $(b)_g$. In this section the relation of the possible (b_g)'s to $(b)_a$ is studied.

In section VII a definition of end point of a continuum, due to H. M. Gehman [1], is generalized thus defining what is called an n-point end-set of M where n is any positive integer. In this section irreducible continua are studied from the standpoint of two-point end-sets. The following useful theorem is obtained: if $c+d$ and $m+n$ are two-point end-sets of M then one of the sets $c+m$, $c+n$, $d+m$, and $d+n$ is a two-point end-set of M; and further if either every K_{cd} contains a point of $m+n$ or every K_{mn} contains a point of $c+d$ then either $c+m$ and $d+n$ or $c+n$ and $d+m$ are two-point end-sets of M. It is further proved that if $p+q$ is a two-point end-set of M then M contains at most one pq. Also if $p+q$ is a two-point end-set of M then in order that M be indecomposable it is necessary and sufficient that for every point x of M either $x+p$ or $x+q$ is a two-point end-set of M; this is a generalization of the previous theorem that in order that M be indecomposable it is necessary and sufficient that $M=(a)+(b)$. In the remainder of this section, among other things, a study is made of the sets $c+m$, $c+n$, $d+m$, and $d+n$, where $c+d$ and $m+n$ are two-point end-sets of M, with respect to the indecomposable subsets of M.

In the next section properties of the connected subsets of the set of two-point end-sets of M are obtained. Among other things it is proven that if M is bounded, (p) and (q) are maximal connected subsets of the set of two-point end-sets of M such that (p) contains $p+z$ and (q) contains $q+w$, and $(p)'$ and $(q)'$ are decomposable, then $z+w$ is a two-point end-set of M.

In section IX it is shown that if N is an n-point end-set of M then N contains a two-point end-set of M. And in the final section a slight study is made with respect to the points of M between which there exist one and only one irreducible subcontinuum of M.

It is known that if K is any subcontinuum of M that either $(M-K)'=au+vb$, $(M-K)'=au$, or $(M-K)'=bv$ [1]. When there exists the possibility of all three of these in the proof of a theorem of this paper if $(M-K)'=au+vb$ is typical this case only is given [2].

[1]) Trans. Amer. Math. Soc. Vol. 30 (1928), p. 64.

II. a and b co-end-sets.

Definition. A b *co-end-point* of M is a point p such that $ap=ab$. The b *co-end-set* of M is the set of b co-end-points of M. The b co-end-set will be represented by $(b)_a$ or by (b). The a co-end-set by $(a)_b$ or (a).

C. Kuratowski in his paper, *Théorie des continus irréductibles entre deux points* [3]), gives some of the properties of the set $(b)_a$, which he calls the set $I(a,M)=I$. In this section some simple additional properties of this set will be stated.

Theorem 1. *If p is a point of (b), and there exist a K_{pq} such that $K_{pq}\times b=0$, then $(b)\times q=q$.*

Proof: Let K_{aq} be any subcontinuum of M containing $a+q$. Then $K_{aq}+K_{pq}$ is a subcontinuum of M containing $a+p$. But as $ap=M$, $K_{aq}+K_{pq}=M$. But $K_{pq}\times b=0$. Therefore $K_{aq}=M$ since K_{aq} contains $a+b$. Thus $M=aq$ and so $(b)\times q=q$.

Theorem 2. *If M is bounded, then (b) is a proper connected subset of M.*

Proof: Since $(b)\times a=0$, (b) is a proper subset of M.

Let p be any point of (b). Let T be the maximal connected

[1]) C. Kuratowski, Fund. Math. III., p. 202—205.

[2]) By $W=U+V$ separate will be meant that W is the sum of two non-vacuous mutually exclusive subsets U and V neither of which contains a limit point of the other. The set W is connected W if does not contain two subsets U and V such that $W=U+V$ separate. And W is closed if it contains all its own limit points; and it is a *continuum* if it is closed and connected. The continuum W is *indecomposable* if it is not the sum of two proper subcontinua. And if a continuum in not indecomposable it is *decomposable*.

[3]) Fund. Math., Tom X., p 230. See also corollary to theorem I of S. Mazurkiewicz's paper, *Un théorème sur les continus indécomposables*, Fund. Math. I, p. 39.

subset of (b) which contains p. If T does not contain b, then T' [1]) does not contain b, since (b) contains b. Assume that T does not contain b. Then there exists a region R containing b such that $R' \times T' = 0$. Then $M - p - R'$ contains a connected subset H such that $H' \times p = p$ and $H' \times R' \times M \neq 0$. [2]). Since $R \times b = b$, $H' \times b = 0$. Since H' is a bounded continuum every point of H' can be joined to p by an irreducible continuum [3]) which does not contain b. Therefore by theorem 1 every point of H' is contained in (b). Then T contains H'. But $H' \times R' \neq 0$. Thus $R' \times T' \neq 0$ which is a contradiction. Therefore $T \times b = b$. Thus every point of (b) is contained in a connected subset of (b) which also contains b. Therefore (b) is connected.

Theorem 3. *If (b) is closed and T is the maximal connected subset of (b) which contains b, then $(a) \times T = 0$.*

Proof: Assume that $(a) \times T$ contains p. Then the subcontinuum T of (b) contains $p + b$ and so $T = bp = M$. But $T \times a = 0$, since $(b) \times a = 0$. Therefore $T \neq M$. Thus the assumption that $(a) \times T \neq 0$ is incorrect.

Corollary 1. *In order that $(a) \times (b) = 0$ it is sufficient that either (a) or (b) be closed, if M is bounded.*

Proof: This follows directly from theorem 2 and 3.

Theorem 4. *In order that M be an indecomposable continuum* [4]) *it is necessary and sufficient that $(a) \times (b) \neq 0$.*

Proof: The condition is sufficient. For let p be a point of $(a) \times (b)$. Then $ap = M = bp$ and so M is irreducible between any two of the points a, b, and p. Thus M is indecomposable [5]).

[1]) If T is a point set, then T' is composed of T and of the limit points of T.

[2]) Anna M. Mulliken, *Certain theorems relating to plane connected point sets*, Amer. Math. Soc. Trans., XXIV. (1923), p. 144—162, theorem I.

[3]) That every two points of a bounded continuum W can be joined by an irreducible continuum was proven by Janiszewski and Mazurkiewicz in 1910 (Comptes Rendus, Paris). See also Mazurkiewicz, Bull. Acad. Polonaise, 1919, p. 44.

[4]) A continuum is indecomposable when it is not the sum of two proper subcontinua. The first examples of indecomposable continua were given by Brouwer, *Zur Analysis Situs*, Math. Ann. 68 (1910), p. 426. For other examples see Z. Janiszewski, *Sur les continus irréductibles entre deux points*, Jour. de l'Ecole Polytechnique, II series, 16-ème Cahier, 1912, p. 79—170. See also B. Knaster, *Un continu dont tout sous-continu est indécomposable*, Fund. Math. III., p. 247—286.

[5]) See Z. Janiszewski and C. Kuratowski, *Sur les continus indécomposables*, Fund. Math. I., p. 215, theorem IV.

The condition is necessary. For there exist three points, $p_i (i = 1, 2, 3)$ since M is indecomposable [1]) such that $p_1 p_2 = p_1 p_3 = p_2 p_3 = M$. But $M = K_{ap_i} + K_{p_i b}$. And as M is indecomposable either $ap_i = M$. or $bp_i = M$ and so each p_i is contained in either (a) or (b). Thus at least two of these points is contained in one of these sets. Say for example that $p_1 + p_2$ is contained in (a). But $K_{p_1 a} + K_{p_2 a}$ is a subcontinuum of M containing $p_1 p_2 = M$. Therefore either $K_{p_1 a}$ or $K_{p_2 a}$ equals the indecomposable continuum M and so either p_1 or p_2 is contained in (b). Thus one of these points is contained in $(a) \times (b) \neq 0$.

Theorem 5. *In order that M be indecomposable it is necessary and sufficient that $(a) + (b) = M$.*

Proof: The condition is necessary. For let p be a point of M. Since $K_{ap} + K_{pb} = M$ either K_{ap} or K_{bp} equals the indecomposable continuum M and so p must be contained in either (a) or (b).

The condition is sufficient. For assume that M is decomposable. Let then $M = W + U$ where W and U are proper subcontinua of M. Thus neither W nor U can contain $a + b$ but each must contain one of these points. Consider for example the case where W contains a and U contains b. As M is connected and W and U are closed, $W \times U \neq 0$. Let p be a point of $W \times U$. Since $(a) + (b) = M$ either (a) or (b) contains p. Say for example that (b) contains p. Therefore $pa = M$ is contained in $W \neq M$ which is a contradiction. Therefore M is indecomposable.

Corollary 2. *In order that M be indecomposable each one of the following conditions is necessary and sufficient:*

(1) (a) *contains* $(b)' - (b) \neq 0$.

(2) $(a) \times ((b)' - (b)) \neq 0$.

Proof: By theorem 5 $(a) + (b) = M$. Therefore (a) contains $(b)' - (b)$ since M is closed. Thus condition (1) is necessary since $(b) \times a = 0$ but $(b)' \times a = a$ [2]) and so $(b)' - (b) \neq 0$. It then follows that condition (2) is necessary.

The conditions are sufficient. For let p be a point of $(a) \times ((b') - (b))$. Assume that M is decomposable. Then $M = K + K_a$ where $K \neq M \neq K_a$. As K_a contains a, $K_a \times (b) = 0$. Therefore K contains (b) and so $K \times (a) = 0$. Thus $K \times p = 0$. But as K con-

[1]) Z. Janiszewski and C. Kuratowski, loc. cit.

[2]) C. Kuratowski, Loc. cit., Fund. Math. X, theorem 1, p. 235.

tains (b) K contains $(b)'$ and so contains p. Thus a contradiction is obtained. Therefore condition (2) and so condition (1) is sufficient.

Theorem 6. *In order that M be indecomposable it is necessary that neither (a) nor (b) be closed.*

Proof: Assume that $(a)=(a)'$. Thus since M is indecomposable $(a)'=M$ and so (a) contains b which is impossible.

Theorem 7. *In order that M be indecomposable it is necessary and sufficient that $M=((a)'-(a))+(a)\times(b)+((b)'-(b))$.*

Proof: The condition is necessary. For as $M=(a)'=(b)'$ every point of M which is not contained in (a) is contained in $(a)'-(a)$ And by corollary 2 (b) contains $(a)'-(a)$. Therefore $(b)=(a)'-(a)+(a)\times(b)$ and similarly $(a)=(b)'-(b)+(a)\times(b)$. Therefore as $M=(a)+(b)$ by theorem 5, $M=((a)'-(a))+(a)\times(b)+((b)'-(b))$.

The condition is sufficient. For as $M=((a)'-(a))+(a)\times(b)+((b)'-(b))$ and as the closed set $(a)'$ contains $((a)'-(a))+(a)\times(b)$, which must contain (b), $(a)'$ must also contain $(b)'$ and so $(a)'=M$. Thus M is indecomposable [1]).

III. Remainder sets of M.

Definition. The set $p+q$ is a *remainder set* of a point set W if W contains a subcontinuum K_{pq} such that $K_{pq}\neq W$.

Yoneyama proved [2]) that if both $a+b$ and $c+d$ are non-remainder sets of a point set C then either $a+c$ or $a+d$ is a non-remainder set of C. In this section further properties of irreducible continua from the stand point of these sets are obtained.

Theorem 8. *If M is decomposable, $p\times(b)=p$, and $q\times(a)=q$, then $p+q$ is a non-remainder set of M.*

Proof: Assume that $p+q$ is a remainder set of M. Then $K_{qp}\neq M$ and so K_{pq} does not contain a for if it did it would contain $pa=M$; $(M-K_{pq})'$ contains a subcontinuum A [3]) containing a such that $A\times K_{pq}\neq 0$. Therefore $A+K_{pq}$ contains $a+p$ and so contains $pa=M$. As $K_{pq}\times b=0$, otherwise K_{pq} contains $bq=M$, $A\times b=b$ and so $A=M$. Thus $(M-K_{pq})'=M$ and so K_{pq} is con-

[1]) C. Kuratowski, loc. cit.

[2]) Yoneyama, *Theory of continuous sets of points*, Tôhoku Math. Journ., Sendai 1917, p. 48, theorem 3. See also Kuratowski, Fund. Math. III., p. 208.

[3]) Kuratowski, Fund. Math. III, p. 212.

tained in $(p)_a\times(q)_b$ [1]). Thus q is contained in $(p)a$ and so $qa=pa=M=qb$. M is then indecomposable [2]) which is a contradiction. Thus $p+q$ must be a non-remainder set of M.

Theorem 9. *In order that $p+q$ be a non-remainder set of M it is necessary that (a) contain one point of $p+q$ and that (b) contain the other.*

Proof: Both (a) and (b) contain points of $p+q$ [3]). Consider for example the case where (a) contains p. If also (a) contains q then the theorem is true since $(b)\times(p+q)\neq 0$. Consider then the case where $q\times(a)=0$. It is then required to prove that $q\times(b)=q$. As $q\times(a)=0$, $K_{qb}\neq M$. Thus $K_{bq}\times p=0$. But $K_{bq}+K_{qa}=M$. Therefore $K_{qa}=M$ and so $q\times(b)=q$.

Theorem 10. *If $p+q$ is a remainder set of M, (a) contains p, (b) contains q, then $(a)\times(b)$ contains $p+q$.*

Proof: By hypothesis $bq=M=aq\neq K_{pq}$. Thus $K_{pq}\times(a+b)=0$ for if K_{pq} contains either a or b $K_{pq}=M$. As $K_{ap}+K_{pq}$ contains $aq=M$, K_{ap} contains b and so equals M. Therefore p is contained in $(a)\times(b)$. Similarly $K_{bq}+K_{pq}=M$ and so $K_{bq}=M$ and thus q is contained in $(a)\times(b)$.

Theorem 11. *If M is decomposable then in order that $p+q$ be a non-remainder set of M it is necessary and sufficient that $(a)\times(p+q)$ and $(b)\times(p+q)$ each contain one and only one point.*

Proof: The condition is necessary. By theorem 9 both (a) and (b) contains a point of $p+q$. If either contain two points then either p or q would be contained in $(a)\times(b)$ and so M would be indecomposable which is contrary to hypothesis.

The condition is sufficient. Neither p nor q is contained in $(a)\times(b)$ for M is decomposable. Consider for example the case where (a) contains p and (b) contains q. Thus by theorem 8 $p+q$ is a non-remainder set of M.

Corollary 3. *In order that M be decomposable it is necessary and sufficient that $(a)\times(p+q)$ and $(b)\times(p+q)$ each contain one and only one point for each non-remainder set $p+q$ of M.*

Proof: The condition is necessary for since M is decomposable, if $p+q$ is a non-remainder set of M, then by theorem 11 $(a)\times(p+q)$ and $(b)\times(p+q)$ each contain one and only one point.

[1]) Kuratowski, Fund. Math. X., p. 233.

[2]) Janiszewski et Kuratowski, Fund. Math. I., p. 215.

[3]) Yoneyama, loc. cit.

The condition is sufficient. For assume that M is indecomposable. Then by theorem 4 $(a) \times (b)$ contains a point p. Since p is contained in (b) $a+p$ is a non-remainder set of M. Thus by hypothesis $(a) \times \times (a+p)$ contains but one point. However it is seen that it contains two points. Therefore M is decomposable.

IV. Frontier sets.

Definition. The set K is a *frontier set* of the point set C if $(C-K)' = C$. If C is a continuum and K is a subcontinuum of C then, if K is a frontier set of C, K is a *continuum of condensation.*

C. Kuratowski proved that if K is a subcontinuum of M containing a then a necessary and sufficient condition that K be contained in (a) is that $(M-K)' = M$ [1]; also if M is bounded then $(M-(a))' = M$. A few additional properties of frontier sets are given in this section.

Theorem 12. *If $p \times (b) = p$, and there exist in M a decomposable pb, then $(M-pb)' = M$.*

Proof: It is seen that $(M-pb)' \neq 0$ for if it did $pb = M = ap$ and so pb is indecomposable [2]) which is a contradiction. Assume that $(M-pb)' \neq M$. Then $(M-pb)'$ is a continuum [3]) containing a. Since $(M-pb)' \neq M$, $(M-pb)' \times (p+b) = 0$. Let y be a point of $(M-pb)' \times pb$. Then $(M-(M-pb)')' = py = by = pb$ [4]) and so bp is indecomposable which is a contradiction. Hence $(M-pb)' = M$.

Corollary 4. *If $p \times (b) = p$, and there exist in M a decomposable pb, then $pb \times (b) = pb$.*

Proof: This follows directly from theorem 12 and from the theorem by Kuratowski.

Theorem 13. *If M is bounded then in orher that $(b)' - (b)$ be closed it is necessary and sufficient that $(b)' - (b) = 0$.*

Proof: That the condition is sufficient is evident.

The condition is necessary. Since $(M-(b))' = M$ every point of (b) is a limit point of $M-(b)$. Consider any point q of (b). For any region K containing q there exists a region R containing q such that K contains R' and $R' \times ((b)' - (b)) = 0$. Since q is a limit point of $(M-(b))$ R contains points of $M-(b)$ and so of $M-(b)- -((b)'-(b)) = M-(b)'$. The set (b) is connected by theorem 2 and so $(b)'$ is a subcontinuum of M. Thus $(b)'$ is contained in (b) [1]). Therefore $(b)' - (b) = 0$.

Corollary 5. *In order that M be decomposable it is sufficient that $(b)' - (b)$ be closed, if M is bounded.*

Proof: Since $(b)' - (b)$ is closed by theorem 13 $(b)' - (b) = 0$. Thus (b) is closed. Therefore by corollary 1 $(a) \times (b) = 0$. And so by theorem 4 M is decomposable.

V. Basic-wise connected subsets of M.

Definition. Two points x and y of a point set W are *basic-wise connected in W* if W contains an irreducible continuum $xy \neq W$. A *basic-wise connected subset of W* is a subset every two points of which are basic-wise connected in W. A point set W is *strongly connected* if every two points of W are contained in a subcontinuum of W. A *maximal strongly connected proper subset of W* is a strongly connected subset which does not contain W and which is contained in no strongly connected subset of W which it itself does not contain. A maximal strongly connected proper subset of W is a maximal basic-wise connected subset of W if W is bounded, since there exists an irreducible continuum between every two points of a bounded continuum. The set $\mathfrak{P}(a, C)$, for a continuum C containing a, of Janiszewski and Kuratowski's paper *Sur les continus indécomposables* [2]), is a maximal strongly connected proper subset of C if it is a proper subset of C.

If C is an arc xy and a is any point of C except x and y, then $\mathfrak{P}(a, C) = C$ while both $C-x$ and $C-y$ are maximal strongly connected proper subsets of C.

Theorem 14. *If $(a)' - (a)$ contains p, T is a maximal strongly connected proper subset of $(a)'$ containing p, then $(a)' - (a)$ contains T.*

Proof: Since $(a)'$ contains T, if $(a)' - (a)$ does not contain T, then $T \times (a) \neq 0$. Let a_1 be a point of $T \times (a)$. The set K_{pa_1} of T

[1]) Loc. cit., Fund. Math. X., p. 233–236, theorem III, and corollary and lemma 2.

[2]) C. Kuratowski and Z. Janiszewski, Fund. Math. I., theorem IV. p. 215.

[3]) C. Kuratowski, Fund. Math. III, p. 203, theorem III.

[4]) Ibid., p. 205, th. VI.

[1]) C. Kuratowski, loc. cit., Fund. Math. X., lemma 2.

[2]) Fund. Math. I., p. 215.

does not contain (a) since $K_{pa_1} \neq (a)'$ as T is a proper subset of $(a)'$. Let a_2 be a point of $(a) - K_{pa_1}$. Since $(a) = (a_2)_b$, by theorem 1 K_{pa_1} is contained in (a_2) and so in (a). Therefore p is not contained in $(a)' - (a)$ which is a contradiction. Thus $(a)' - (a)$ must contain T.

Theorem 15. *If M is indecomposable then $(a)' - (a)$ is a maximal strongly connected proper subset of $(a)'$.*

Proof: If $(a)' - (a)$ contains less than two points it is evidently a strongly connected proper subset of $(a)'$. Let then $p + q$ be any two points of $(a)' - (a)$. There exists K_{pq} of $(a)' = M$[1]. Either $K_{pq} = M$ or $K_{pq} \neq M$. The case where $K_{pq} = M$ is impossible since by theorem 9 p or q is contained in (a) instead of in $(a)' - (a)$. Thus $K_{pq} \neq M$. And as p is contained in $(a)' - (a)$, $K_{pb} \neq M$. Assume that $K_{pq} \times (a)$ contains a_1. Then $K_{pq} + K_{pb}$ contains $a_1 + b$ and so contains $a_1 b \neq M$. This is impossible since $K_{pq} \neq M \neq K_{pb}$. Thus $K_{pq} \times (a) = 0$. Therefore K_{pq} is contained in $(a)' - (a)$ and so $(a)' - (a)$ is a strongly connected proper subset of $(a)' = M$.

Assume that W is a strongly connected proper subset of $(a)'$ containing $(a)' - (a)$. If $W \neq (a') - (a)$ then $W \times (a) \neq 0$. Let a_2 be contained in $W \times (a)$ and let K_{pa_2} be a subcontinuum of W. As $K_{pa_2} + K_{pb} = M$, $K_{pa_2} = M$ since M is indecomposable and $K_{pb} \neq M$. Thus $W = M$ and so W is not a proper subset of $(a)'$. Therefore $(a)' - (a)$ is a maximal strongly connected proper subset of $(a)'$.

Corollary 6. *If M is bounded then in order that M be indecomposable it is necessary and sufficient that $(a)' - (a)$ be a maximal basic-wise connected subset of M everywhere dense in M.*

Proof: The condition is necessary. For since M is bounded it follows from theorem 15 that $(a)' - (a)$ is a maximal basic-wise connected subset of M. And $(a)' - (a)$ is not closed since if it were then by corollary 5 M would be decomposable. As $(a)'$ contains $(a)' - (a)$, (a) must then contain a limit point, a_1 say, of $(a)' - (a)$. Therefore $((a)' - (a))'$ is a continuum containing a_1 and a point p of $(a)' - (a)$ and so containing a K_{pa_1}. But $K_{pb} + K_{pa_1}$ is a subcontinuum of M containing $a_1 + b$ and so containing M. As $K_{pb} \neq M$, $K_{pa_1} = M$. Therefore $((a)' - (a))' = M$ and so $(a)' - (a)$ is everywhere dense in M.

The condition is sufficient. For as $((a)' - (a))' = M$, $(a)' = M$ and so M is indecomposable.

[1]) C. Kuratowski, Fund. Math. X. p. 235.

Theorem 16. *If M is bounded and $(a)' - (a)$ contains p, then the maximal basic-wise connected subset of $(a)'$ containing p is everywhere dense in $(a)'$.*

Proof: Let R_i $(i = 1, 2, \ldots)$ be a set of regions having only a in common, such that $R_i' \times p = 0$ and R_i contains R_{i+1}. Since $(a)'$ is a bounded continuum, by theorem 2, there exists a connected subset[1]) X_1 of $(a)' - R_1' - p$ such that $X_1' \times p = p$ and $X_1' \times R_1' \neq 0$. Similarly there exists a connected subset X_i of $(a)' - R_i' - (X_1' + X_2' + \ldots + X_{i-1}')$ such that $X_i' \times (X_1' + \ldots + X_{i-1}') \neq 0$ and $X_i' \times R_i' \neq 0$. Thus $X_i' \times a = 0$. Let $T = p + X_1' + X_2' + \ldots$. The set T' contains a for $R_1, R_2, \ldots$ have only this point in common. The set T is a basic-wise connected subset of $(a)'$ since for every two points x and y of T there exists an m such that $(X_1' + \ldots + X_m')$ contains $x + y$ and so contains an xy of $(a)'$ which does not contain a. As p is not contained in (a) $pb \neq M$. Thus $pb \times (a) = 0$. But $pb + T'$ is a subcontinuum containing $a + b$ and so containing M. Therefore T' contains $(a)'$ and so the maximal basic-wise connected subset of $(a)'$ containing p, which also contains T, is everywhere dense in $(a)'$.

Corollary 7. *If M is bounded then in order that $(a)' - (a) \neq 0$ it is necessary and sufficient that $((a)' - (a))' = (a)''$.*

Proof. The condition is necessary. For by theorem 14 there exists in $(a)' - (a)$ a maximal basic-wise connected subset of $(a)'$ which, by theorem 16, is everywhere dense in $(a)'$. Thus $((a)' - (a))'$ contains $(a)'$ and as $(a)'$ contains $((a)' - (a))'$, $(a)' = ((a)' - (a))'$.

It is evident that the condition is sufficient.

Theorem 17. *If M is bounded and $((a)' - (a)) \times ((b)' - (b))$ contains p, then there exists a basic-wise connected subset T of $(a)' - (a) + (b)' - (b)$ which contains p and $T' = M$ is decomposable.*

Proof. By theorem 14 the maximal basic-wise connected subset, W say, of $(a)'$ containing p is contained in $(a)' - (a)$; and the maximal basic-wise connected subset, U say, of $(b)'$ containing p is contained in $(b)' - (b)$. Thus $W' + U'$ is a subcontinuum of M which by theorem 16 contains $a + b$ and so contains M.

As $((a)' - (a)) \times ((b)' - (b))$ contains p, $((a) + (b)) \times p = 0$. Therefore by theorem 5 M is decomposable.

Assume that $W + U$ is an irreducible continuum. Then $W + U =$

[1]) Anna M. Mulliken, loc. cit.

$= W' + U' = M$. But as $W \times (a) = 0$, $U \times (a) = (a)$. Thus $(b)'$ contains (a) and so contains $a + b$. Therefore $(b)' = M$ and so M is indecomposable [1]) which is a contradiction. Therefore $W + U = T$ is a basic-wise connected subset of $(a)' - (a) + (b)' - (b)$.

In corollary 6 it was shown that if M is bounded then in order that M be indecomposable it is necessary that $(a)' - (a)$ be a maximal basic-wise connected subset of M everywhere dense in M. And $(a)' - (a)$ contains b. It is known that for any point c of an indecomposable continuum M there exists a point d such that $cd = M$ [2]). Thus for a bounded continuum every point c is contained in a maximal basic-wise connected subset of M everywhere dense in M. It has been shown in theorem 7 that any indecomposable continuum is the sum of the three point sets $(a)' - (a)$, $(a) \times (b)$, and $(b)' - (b)$ and these three point sets are distinct. Thus it is seen that if a maximal basic-wise connected subset contains a point of one of these sets it is contained in that set.

The *composant du point c dans M* [3]) of Janiszewski and Kuratowski is a maximal basic-wise connected subset of M if M is a bounded indecomposable continuum [4]). They point out [5]) that there are an uncountable number of these distinct composants in an indecomposable continuum. We thus have that *a bounded indecomposable continuum M is the sum of an uncountable number of distinct maximal basic-wise connected subsets of M each of which is everywhere dense in M*. One of these maximal basic-wise connected subsets is the set $(a)' - (a)$; another is the set $(b)' - (b)$; and $(a) \times (b)$ contains the remaining uncountable number of such sets. If c is contained in one such maximal basic-wise connected subsets and d is contained in another then $cd = M$.

Theorem 18 [6]). *If M is bounded then in order that M be indecomposable it is necessary and sufficient that M contain a basic-wise*

1) C. Kuratowski, Fund. Math. X., theorem 1, p. 235.

2) Janiszewski and Kuratowski, loc. cit., theorem IV., p. 215.

3) Loc. cit., Fund. Math. I., p. 218.

4) A maximal basic-wise connected subset of an irreducible continuum is not necessarily a composant of that continuum. But a composant of a bounded continuum is always a maximal basic-wise connected subset of that continuum.

5) Loc. cit., p. 218—219.

6) See P. Urysohn, *Mémoire sur les multiplicités Cantoriennes*, Fund. Math. VIII., p. 226.

connected subset T everywhere dense in M such that $M - T$ is everywhere dense in M and $T \times ((a) + (b)) \neq 0$.

Proof: As shown above the condition is necessary.

The condition is sufficient. Consider for example the case where $T \times (a)$ contains c. Then $cb = M$. Since every point of T can be joined to c by an irreducible continuum of T, every point of T is contained in $(c)_b$ [1]), since $(M - T)' = M$. Thus $T' = M$ is contained in $(c)'_b$ and as then $(c)'_b = cb = M$ the set M is indecomposable.

Corollary 8. *If M is bounded then in order that M be indecomposable it is necessary and sufficient that M contain two distinct basic-wise connected subsets T_i $(i = 1, 2)$ each of which is dense in M and $T_1 \times ((a) + (b)) \neq 0$.*

Proof. That the condition is necessary is shown above and that it is sufficient follows from theorem 18.

Corollary 9. *If M is bounded then in order that M be indecomposable it is necessary and sufficient that M be the sum of more than two distinct basic-wise connected subsets of M each of which is everywhere dense in M.*

Proof. That the condition is necessary is shown above. It is sufficient by theorem 18 since one of these distinct basic-wise connected subsets must contain a.

Corollary 10. *If M is bounded then in order that M be indecomposable it is necessary and sufficient that M contain a basic-wise connected subset T which is everywhere dense in M and T contains a but does not contain (a).*

Proof: As shown above the condition is necessary. The condition is sufficient. Let c be contained in $(a) - T$ since $T \times (a) \neq (a)$. Then $cb = M = ab$ and so $(c)_b$ contains a. Let T contain p. The set T contains a pa which does not contain c. Thus by theorem 1 $(c)_b$ contains pa. Therefore $(c)_b$ contains T and so $(c)_b'$ contains $T' = M$. Thus M is indecomposable.

Lemma 1. *If K is not a continuum of condensation and $K \times (a + b) = 0$, then there exist an $au + vb = (M - K)'$ and a $uv = (M - (M - K)')'$ in K such that $au + uv + vb = M$. If K is not a continuum of condensation and $K \times (a + b) = a$, then there exist an $au = (M - (M - K)')'$ and a $ub = (M - K)'$ such that $M = au + ub$.*

1) C. Kuratowski, Fund. Math. X., lemma 2.

Proof: Consider the case where $K \times (a + b) = 0$. It is evident that K contains the continuum[1] $(M - (M - K)')'$. Since $(M - K)' = (M - (M - (M - K)')')'$[2] and since $(M - K)' \neq M$, $(M - (M - (M - K)')')' \neq M$ and so $(M - (M - K)')'$ is not a continuum of condensation. Let $(M - K)' = (M - (M - (M - K)')')' = A + B$, where A contains a and B contains b. Let $A \times (M - (M - K)')'$ contain u and $B \times (M - (M - K)')'$ contain v. Thus $A = au$[3]) and $B = by$ and so $M = au + (M - (M - K)')' + vb$. As $(M - K)' = (M - (M - (M - K)')')'$, $(M - (M - K)')' = (M - (M - (M - (M - K)')')')'$. Therefore $(M - (M - K)')' = uv$[4]. Thus $M = au + uv + vb$ where K contains uv and $(M - K)'$ contains $au + vb$.

The case where $(a + b) \times K = a$ is treated in the same manner.

Theorem 19. *If T is a basic-wise connected subset of M such that T and $N - T$ are both everywhere dense in M and $T \times ((a) + (b)) = 0$ then for any point p of $M - (a) - (b)$ either there exist a pa and an $(M - pa)'$ or a pb and an $(M - pb)'$ which are indecomposable proper subcontinua of M.*

Proof: Either there exist pa or pb[5]. Take for example the case where there exist pa. Since $M - (a) - (b)$ contains p, $pa \neq M \neq K_{pb}$. Thus $(M - pa)' \neq M$ and so pa is not a continuum of condensation. Therefore by lemma 1 there exists an $au = (M - (M - K)')'$ and a $bu = (M - K)'$, where $K = ap$, such that $M = au + ub$. Assume that $au \times T = 0$. Then bu contains T and so contains $T' = M$. Thus $(M - pa)' = M$ which is a contradiction. Therefore $au \times T \neq 0$ and similarly $bu \times T \neq 0$. Let g be any point of $au \times T$. Then T contains a gu_1, joining g and bu, since T is basic-wise connected. Thus $M = gu_1 + ub + K_{ag}$, where au contains K_{ag}. But $M - T$ is dense in M and so $(M - T)' = M$ contains T in which case $K_{ag} + ub$ contains gu_1. Thus $K_{ag} + ub = M$ and so $(M - ub)' = (M - (M - ap)')' = K_{ag} = au$. Thus $(u)_a$ contains g and so contains $T \times au$. Then $(u)'_a$ contains $(T \times au)' = T' \times au = au$ as T is dense in M. Therefore au, and similarly bu, is indecomposable. And both au and bu are proper subcontinua of M. If $au \times p = p$, then $au = ap$ and $bu = (M - ap)'$. If $bu \times p = p$, then $ap \times (M - ap)'$ contains p and so $bu = (M - ap)' = bp$[1]) and $au = (M - bp)'$.

[1]) C. Kuratowski, Fund. Math. III., theorem V., p. 205.

[2]) Ibid. theorem 6., p. 183.

[3]) Ibid. theorem IV., p. 204.

[4]) Ibid. theorem VI., p. 205.

[5]) Ibid. p. 219.

Theorem 20. *In order that the bounded decomposable continuum M be the sum of two proper indecomposable subcontinua it is necessary and sufficient that M contain a basic-wise connected subset T, such that T and $(M - T)$ are both everywhere dense in M.*

Proof: The condition is sufficient. For $T \times ((a) + (b)) = 0$ for if not then by theorem 18 M is indecomposable. Therefore by theorem 19 M is the sum of two indecomposable proper subcontinua of M.

The condition is necessary. Let ap and bq be two indecomposable proper subcontinua of $ap + qb = M$. Then by lemma 1 there exists a point u such that $au + ub = M$ where ap contains au and $(M - ap)' = bu$. But every point of an indecomposable continuum is contained in a maximal basic-wise connected subset which is everywhere dense in that continuum. Thus say W of ap contains u and that V of bq contains u.

Let $T = W + V$. Then T is everywhere dense in M, since W is in ap and V is in bq and so $W + V = T$ is in $ap + bq = M$. As $bq \neq M$ $bq \times a = 0$ and so $V \times a = 0$. There exists a region R containing a such that $R' \times bq = 0$. Let $R \times (ap - W)$ contain y. Thus $W \times a = 0$ for if not W contains au but $au + bq = M$ does not then contain y. Therefore $T = W + V$ does not then contain a and so T is a basic-wise connected subset of M.

Let K be the basic-wise connected subset of ap which contains a and let N be that of bq which contains b. Let $R \times W$ of ap contain x. Then $K \times x = 0$, as $K \times W = 0$ and $bq \times x = 0$ as $R' \times bq = 0$. Thus $K \times bq = 0$ for if not $K + bq = M$ does not contain x. Similarly $N \times ap = 0$. Thus as K is dense in ap and N in bq, $ap + bq - T$ is everywhere dense in M.

VI. Co-end-sets of M.

If c is contained in (b) then $(c)_a = (b)_a$ for if d is contained in $(c)_a$ then $da = ca = ba$ and so $(b)_a$ contains d. Thus it only remains to determine the relation between $(a)_c$ and $(a)_b$. In the case of an

[1]) C. Kuratowski, Fund. Math. III., theorem IV., p. 204.

indecomposable continuum, if T_b is the maximal strongly connected proper subset of M which contains b and T_c is the corresponding set containing c, then, if $T_b = T_c$, $(a)_b = M - T_b = M - T_c = (a)_c$; if $T_b \neq T_c$ then, if $N = M - T_b - T_c$, $(a)_b = N + T_c$ and $(a)_c = N + T_b$.

Theorem 21. *In order that M be decomposable it is necessary and sufficient that $(a)_b = (a)_c$ for every point c of (b).*

Proof: The condition is necessary. Let $(a)_c$ contain g. Then $ac = ab = gc$. Thus by theorem 11 $(a)_b \times (g + c)$ contains one point. It does not contain c for if it did $(a) \times (b)$ contains c and so M would be indecomposable. Therefore $(a)_b$ contains g and so contains $(a)_c$. Since, if (b) contains c then $(c)_a$ contains b, it also follows from the above reasoning that $(a)_c$ contains $(a)_b$. Therefore $(a)_b = (a)_c$.

The condition is sufficient. Assume that M is indecomposable. Then $(a) \times (b)$ contains a point c. Thus $ac = ab$. The set $(a)_c$ does not contain c although $(a)_b$ does. Therefore $(a)_c \neq (a)_b$ and so M must be decomposable.

Corollary 11. *If (b) contains e then in order that $(a)_b$ does not contain e it is necessary and sufficient that $(a)_b = (a)_e$.*

Proof: The condition is necessary. If M is decomposable then by theorem 21 $(a)_b = (a)_e$. Consider now the case where M is indecomposable. As shown above if T_b contains e then $(a)_b = (a)_e$. If T_b does not contain e then $M - T_b = (a)_b$ contains e which is a contradiction. Thus this case does not exist. Therefore the theorem is true for all possible cases.

The condition is sufficient. Assume that $(a)_b$ contains e. Thus $(a)_b = (a)_e$ contains e which is impossible. Therefore $(a)_b$ does not contain e.

Corollary 12. *If (b) contains e then in order that $(a)_b$ contain e it is necessary and sufficient that $(a)_e$ contain b.*

Proof: The condition is necessary. As (b) contains e $ea = ba$ and as (a) contains e $eb = ab$. Thus $ea = eb$ and so $(a)_e$ contains b. In a similar manner it is shown that the condition is sufficient.

Relations between co-end-sets of various subcontinua of M are also of interest. In the following theorems properties of (b) and the co-end-sets of certain subcontinua of M are obtained.

Theorem 22. *If $F = (b)' - (b)$ contains a point g then $bg = (b)'$ and $(b)_g$ of bg contains (b) and $(g)_b$ contains F.*

Proof: As $(b) \times g = 0$, $K_{ag} \times (b) = 0$. As $(b)'$ contains $b + g$ it contains a K_{bg} and $K_{ag} + K_{gb} = M$. Thus $K_{bg} \times (b) = (b)$ and so K_{bg} contains $(b)'$. Therefore $K_{bg} = (b)' = bg$. Since g is any point of F, $(g)_b$ contains F. Consider now any point b_1 of (b). Then $(b)'$ contains a $K_{b_1 g}$ and $K_{ag} + K_{b_1 g} = M$. Hence $K_{b_1 g} = (b)' = b_1 g = bg$. Therefore $(b)_g$ contains (b).

If g is contained in F there exist but one bg. If $M - (b)'$ contains g the following corollary is also true.

Corollary 13. *If M is bounded and (b) does not contain the point g of M then the sum of the $(b)_g$'s contain (b).*

Proof: Let h be any point of (b). For every hg and ga, $hg + ga = M = ha$. Since $ga \neq M$, $ga \times (b) = 0$. Thus every hg contains $(b)'$. Then every hg contains a bg. But for every bg, $bg + ga = M$, and so bg contains $(b)'$ and so contains h. Hence every bg contains an hg and so a $bg = hg$. Therefore a $(b)_g$ contains h. Thus the sum of the $(b)_g$'s contains (b).

That (b) does not always contain the sum of the $(b)_g$'s is seen from the following example. Let N be an indecomposable continuum containing the three distinct maximal basic-wise connected subsets, T_i $(i = 1, 2, 3)$, where T_1 contains b. Let ax be an irreducible continuum such that $(x)_a \times N = x_2 + x_3$ where T_i contains x_i but ax contains no other point of N. Then $ab = N + xa$. Hence $(b) = N - T_2 - T_3$. Let T_2 contain g. Then $bg = N$ and $(b)_g = N - T_2$. Hence T_3 is contained in $(b)_g$ but not in (b).

Theorem 23. *If M is bounded and (b) does not contain a point g of M then either the sum of the $(b)_g$'s $= (b)$ or an indecomposable subcontinuum of M contains (b).*

Proof: Assume that (b) does not contain some point e of a $(b)_g$. Then $eg = bg$ but $ea \neq M$. Assume that eg contains a. Then $eg = bg = M$. Hence $(e + g) \times (b) \neq 0$ by theorem 9. As this is a contradiction $eg \times a = 0$. There exists then by lemma 1 an $ax + gb = M = ax + xb$. And every ag contains ax, so $ax \times (b) = 0$ as $ag \neq M$,

Consider the case where bx of bg contains g. Then bx contains $bg = eg$ and so contains e. Thus if $bx \times g \neq 0$, $bx \times e \neq 0$. Consider now the case where $bx \times e \neq 0$ but $bx \times g = 0$. Let gy [1]) of bg join g and bx. Then if ze of bx joins gy and e, $ez + gy = bg$ and so

[1]) Anna M. Mulliken, loc. cit.

$ze \times b = b$ as ax contains gy. Hence ze contains $b + x$ and so contains bx. Thus $ez = bx = bz$. Then if $ax + ze = ae$, $ae = M$ which is a contradiction. Thus ae contains an ew joining ax and e such that $ew \neq bz$. Then $ew \times z = 0$. Assume $eb \neq bz$. Then $eb \times z = 0$. Therefore $be + ew$ contains bx but does not contain z which is a contradiction. Hence $ez = eb = bz = bx$ and so bx is indecomposable and contains (b). The following two cases remain to be considered.

(1) Consider the case where $bx \times (e + g) = 0$. Let ey_1 join e and bx in bg and gy_2 join g and bx. But $ey_1 \times gy_2 = 0$ for if not $ey_1 + y_2 g = eg = bg$ and so ax contains b. Then $ey_1 + y_1 y_2 + y_2 g = eg = bg$ and so $y_1 y_2$ contains b. Hence $bx = by_1 = y_1 y_2 = y_2 b$ and so bx is indecomposable and contains (b).

(2) Consider the case where bx contains $e + g$. Let ex_1 of bx join e and ax and let gx_2 join g and ax. Then $ex_1 \times gx_2 = 0$ for if not $gx_2 + x_1 e = ge = gb$ and so gx_2 or ex_1 contains b and so ag or ae contains b. Hence $ex_1 + x_1 x_2 + x_2 g = eg = bg$ and so $x_1 x_2$ contains b. Thus $bx_1 = x_1 x_2 = x_2 b$ is indecomposable. And as shown above bx contains (b).

Corollary 14. *If M is bounded, (b) does not contain g, and there exists but one hg where h is any point of $(b)_g$, then $(b) = (b)_g$.*

Proof: As $ga \neq M$, $ga \times (b) = 0$. But $ah + ag$ contains $hg = bg$. Thus ah contains b and so $ah = M$ and (b) contains h. Thus (b) contains $(b)_g$ and so by corollary 13 $(b) = (b)_g$.

Theorem 24. *If M is bounded and $(h + g) \times (b) = 0$ but $h \times (b)_g = h$, where $(b)_g$ is any $(b)_g$, then any hb of bg is indecomposable.*

Proof: By hypothesis $hg = bg$. Either $hb = bg$ in which case hb is indecomposable or $hb \neq bg$, where hb is contained in bg. Consider the latter case. Then $hb \times g = 0$. Let $hb + wg = bg$ where $wg = (bg - hb)'$. Either (1) $wg = bg$ or (2) $wg \neq bg$.

(1) $wg = bg$. Let $bh + za = M$, where ha contains za. Then za contains $M - bh$ and so contains $(bg - bh)' = wg = bg$. Thus za contains b and so $za = M$, which is a contradiction as za is contained in $ha \neq M$.

(2) $wg \neq bg$. Thus $wg \times (b + h) = 0$. Let $wb + wh = bh$. Then $wg + wb = bg = hg$. Thus $wb \times h = h$. Also $wg + wh = gh = bg$. Thus $wh \times b = b$. Hence $wb = bh = wh$ is indecomposable.

Theorem 25. *In order that M be indecomposable it is necessary and sufficient that there exists a point g such that a $(b)_g$ contains a.*

Proof: The condition is necessary. By theorem 4 $(a) \times (b)$ contains a point g. Then $ga = gb = M$ and so $(b)_g \times a = a$.

The condition is sufficient. For $ag = bg$. Thus ag contains b and so $ag = M = bg$. Thus M is indecomposable.

Definition. If N is any subcontinuum, containing a point p, of a continuum W and if p is not a limit point of any connected subset of $W - N$, then p will be said to be an *end-point* of W[1]).

Theorem 26. *If b is a non-end-point of M and p is a point such that p and b are arc-wise connected in M, then (b) contains p.*

Proof: Assume that (b) does not contain p. Then $K_{pa} \neq M$ and so $K_{pa} \times (b) = 0$. Let pb be an arc of M. Let bq of pb join b and K_{pa}. Then $K_{pa} + bq = M$ and so K_{pa} contains $(M - bq)'$. Thus $(M - bq)' \times b = 0$. Let N be any proper subcontinuum of M containing b. Then $bq \times N \neq N$ or $bq \times N = N$.

(1) Consider the case where $bq \times N = N$. Then there exists a $bx = N$ and so $K_{pa} + qx + xb = M$ and $K_{pa} + xq$ contains $(M - xb)'$ which cannot contain b for then xq contains b. Thus $(M - xb)' = (M - N)'$ does not contain b. (2) Consider the case where bq does not contain N. Then $(M - bq)' \times N \neq 0$ and so $(M - bq)' + N = M$. Hence $(M - bq)'$ contains $(M - N)'$ which then cannot contain b. Therefore in every case b must be an end-point of M which is a contradiction. Hence $(b) \times p = p$.

Theorem 27. *In order that b be an end-point of M it is sufficient that $M - N$ be arc-wise connected, if N is a subcontinuum of M such that* $N \times b = 0$.

Proof: Assume that b is a non-end-point of M. Then by theorem 26 (b) contains $M - N$. Then $M - N$ does not contain a and so N contains a. Since N is closed $N \times (M - N)' \neq 0$. Assume that N contains a point p of (b). Then N contains $pa = M$ which contains b. As this is a contradiction $N \times (b) = 0$ and so $M - N = (b)$. Thus $(M - N)'$ is indecomposable[2]) and as $M - N$ is basic-wise connected $((M - N)' - (M - N))' = (M - N)'$. Therefore N contains $(M - N)'$ and so contains $M - N$ which is a contradiction. Hence b must be an end-point of M.

[1]) H. M. Gehman, *Concerning end points of continuous curves and other continua*, Trans. Amer. Math. Soc., Vol. 30 (1928), p. 64,

[2]) C. Kuratowski, Fund. Math. X. theorem II.

VII. Two-point end-sets of M.

Definition. An *n-point end-set* of a continuum W is a set of n points of W such that for every connected subset T of $W-N$, where N is any subcontinuum of W containing these n points, T' does not contain any of these n points[1].

Definition. The point q is a p *joined-point of order k in W* if there exist in W k and only k irreducible continua pq no two of which are equal. The p joined *set of order k in W* is the set of p joined-points of order k in W.

In this section the following problem is considered: if $p_i + q_i$ $(i=1, 2)$ is a two-point end-set of M, then under what conditions are p_1+p_2, p_1+q_2, q_1+p_2, and q_1+q_2 two-point end-sets of M.

Theorem 28. *If $p+q$ is a two-point end-set of M, then q is a p joined-point of order at most one in M.*

Proof: Assume that q is a p joined-point of order greater than one. Then there exist $pxq \neq pyq$ and so $pxq \neq M \neq pyq$. Consider for example the case where $(pxq+pyq)\times(a+b)=0$. As $pxq+pyq$ is not a continuum of condensation there exist $au+bv=(M-(pxq+pyq))'$[2] such that $M=au+(pxq+pyq)+vb$. Hence $(au+vb)\times(p+q)=0$. Consider for example the case where $au\times pyq=0$. Thus $au\times pxq\neq 0$ while $bv\times pxq=0$ and $bv\times pyq\neq 0$. The set pxq contains $(M-(au+pyq+vb))=U$. Assume that $U=U_1+U_2$ separate. But $M=au+U_1+U_2+pyq+vb$ is connected. Therefore either U_1 or U_2 has a limit point in both au and $(pyq+vb)$. Say U_1 does. Then $au+U_1'+pyq+vb=M$ which is impossible. Thus U is connected, and U' joins au and $pyq+vb$. Similarly pyq contains a connected set V such that V' joins $au+pxq$ and vb. Hence $(U'+V')\times(p+q)=0$ and so $(au+U')\times(bv+V')=0$. Therefore pxq contains a subcontinuum X and pyq contains a subcontinuum Y each of which joins $au+U'$ and $bv+V'$. Thus $au+U'+X+V'+vb=M=au+U'+Y+V'+vb$ and so $X=Y$. But $X\times Y$ contains $p+q$ and so $pxq=X=Y=pyq$ which is a contradiction. Therefore q is a p joined-point of order at most one in M.

[1] For other theorems concerning two-point end-sets see P. M. Swingle, *A certain type of continuous curve and related point sets.* (Submitted for publication to the Trans. Amer. Math. Soc.).

[2] C. Kuratowski, Fund. Math. III. theorem *iv*. p. 204.

Corollary 15. *If M is bounded and $p+q$ is a two-point end-set of M, then q is a p joined-point of order one in M.*

Proof: Since M is bounded there exist a pq in M and by theorem 28 there exist only one pq in M. Thus q is a p joined-point of order one in M.

Theorem 29. *If M is indecomposable then in order that $p+q$ be a two-point end-set of M it is necessary and sufficient that $M=pq$.*

Proof: The condition is necessary. For assume that $pq\neq M$. Then since M is indecomposable. $p+q$ is not a two-point end-set of M[1]. That the condition is sufficient is evident.

Theorem 30. *If $p+q$ is a two-point end-set of M then in order that M be indecomposable it is necessary and sufficient that for every point x of M either $x+p$ or $x+q$ is a two-point end-set of M.*

Proof: The condition is necessary. For by theorem 29 $pq=M$ and so by theorem 5 $(p)_q+(q)_p=M$. Thus either $(p)_q$ or $(q)_p$ contains x. Therefore either $x+p$ or $x+q$ is a two-point end-set of M.

The condition is sufficient. For there exist, under the assumption that M is decomposable, two proper subcontinua, U and V say, of M such that $M=U+V$. It is seen then that $M=(M-U)'+(M-(M-U)')'$[2] and so $(M-U)'\times(M-(M-U)')'$ contains a point z. But z is then a point of $(M-(M-U)')'\times(M-(M-(M-U')')')$[3] and so a limit point of a connected subset of both $M-(M-U)'$ and of $M-(M-(M-U)')'$. By hypothesis either $z+p$ or $z+q$ is a two-point end-set of M. Consider for example the case where $z+p$ is. As either $(M-U)'$ or $(M-(M-U)')'$ must contain p, one of these set must contain $z+p$. However as its complement is a connected subset of M having z as a limit point $p+z$ cannot be a two-point end-set of M which is a contradiction. Therefore M is indecomposable.

Theorem 31. *In order that M be indecomposable it is necessary that there exist a two-point end-set $p+q$ of M such thata $+p$, $a+q$, $b+p$, and $b+q$ are each two-point end-sets of M.*

Proof: Since M is indecomposable it is composed of an uncountable number of maximal strongly connected proper subsets

[1] Z. Janiszewski and C. Kuratowski, loc, cit., theorem II., p. 212.

[2] C. Kuratowski, Fund. Math, III., theorem III., p. 203.

[3] C. Kuratowski, Fund. Math. III., theorem 6., p. 183.

of M[1]); let p be a point of one of these sets which contains neither a nor b and q a point of one which contains no point of $a+b+p$. The truth of the theorem is then evident.

Theorem 32. *If M is bounded. $p+q$ is a two-point end-set of M, and $p+z$ and $q+z$ are non-remainder sets of M, then pq is indecomposable.*

Proof: As $qz=M=pz$ if $pq=M$ then M is indecomposable. Consider then the case where $pq \neq M$. Then there exists by lemma 1 a zv such that $zv+pq=M=zq$. Let $pv+vq=pq$. Then $zv+vp=zp=zq=zv+vq$. As $p+q$ is a two-point end-set of M $zv \times (p+q)=0$. Thus vp contains q and vq contains p. Therefore $vp=pq=vq$ and so pq is indecomposable.

Corollary 16. *If M is bounded, $p+q$ and $z+v$ are two-point end-sets of M, and $p+z$, $q+z$, and $p+v$ are non-remainder sets of M then pq and zv are indecomposable.*

Proof: This follows directly from theorem 32.

Theorem 33. *If $b+p$ and $b+q$ are two-point end-sets of $M = ab = pq$ then in order that M be decomposable it is necessary and sufficient that at least two of the sets $a+p$, $a+q$, $b+p$, and $b+q$ be remainder sets of M.*

Proof: The condition is necessary. For assume that there does not exist at least two of the sets $a+p$, $a+q$, $b+p$, and $b+q$ which are remainder sets of M. Then at least three of these sets are non-remainder sets of M. Consider for example the case where $a+p$ and $a+q$ are. Then $ap=aq=pq$ and so M is indecomposable which is a contradiction. Therefore the assumption is incorrect.

The condition is sufficient. For assume that M is indecomposable. Since $pq=M$, by corollary 3, (a) contains one point of $p+q$ and (b) contains the other. Consider for example the case where (a) contains p and (b) contains q. Then $pb=M=qa$ and by hypothesis $K_{pa} \neq M \neq K_{qb}$. Either $K_{pa} \times K_{qb} \neq 0$ or $K_{pa} \times K_{qb} = 0$.

Consider the case where $K_{pa} \times K_{qb} \neq 0$. Then $K_{pa}+K_{qb}=M$ and as M is indecomposable either $K_{pa}=M$ or $K_{qb}=M$ which is a contradiction.

Consider now the case where $K_{pa} \times K_{qb} = 0$. Then $b+q$ is not a two-point end-set of M[2]). Thus M must be decomposable.

[1]) Z. Janiszewski and C. Kuratowski, Fund. Math. I., p. 218—219.

[2]) Z. Janiszewski and C. Kuratowski, Fund. Math. I., theorem II. p. 212.

Lemma 2. *If $M=au+uv+vb$ where $au \times vb=0$ and $(M-(au+vb))'=uv$, and N is a subcontinuum of M such that $N \times u=u$ and $N \times vb=0$, then $uv-N$ is connected and if x is a point of $(uv-N)' \times N$ then $(uv-N)'=vx$.*

Proof: Assume that $uv-N=X+Y$ separate. Both $(au+N)+X+vb=X_1+X_2$ separate and $(au+N)+Y+vb=Y_1+Y_2$ separate for since each of these sets is closed if they were connected they would equal M which is impossible. Consider for example the case where X_1 contains $au+N$ which is also contained in Y_1. Assume that X_1 also contains bv. Then $M=(au+N+vb+Y+X_1)+X_2$ separate which is a contradiction. Hence both X_2 and Y_2 must contain vb. Thus $M=(au+N+X_1+Y_1)+(X_2+Y_2+vb)$ separate which is a contradiction. Therefore $uv-N$ is connected and contains v. Let K_{vx} be any subcontinuum of $(uv-N)'$ containing $v+x$. Then $au+N+K_{vx}+vb=M$ and so K_{vx} contains $(uv-N)'$ and so equals it.

Theorem 34. *If $c+d$ and $m+n$ are two-point end-sets of M and three of the sets $c+m$, $c+n$, $d+m$, and $d+n$ are not then the fourth is and every subcontinuum of M containing these two points contains $c+d+m+n$.*

Proof: Consider for example the case where $c+m$, $c+n$, and $d+m$ are not two-point end-sets of M. Then there exist subcontinua K_{cm}, K_{cn}, and K_{dm} of M whose complements in M contain limit points in $c+m$, $c+n$, and $d+m$ respectively. Let K_{dn} be any subcontinuum of M containing $d+n$. The set $K=K_{cm}+K_{cn}+K_{dm}$ is a subcontinuum of M containing $c+d+m+n$ and so $(M-K)' \times (c+d+m+n)=0$. Let $M=au+uv+vb$ where $(M-K)'=au+vb$. Consider for example that K_{cm} contains u. As $M-K_{cm}$ contains a connected subset having a limit point in $c+m$, so also does $M-K_{cm}-au-vb$ for $uv-K_{cm}$ is connected or vacuous by lemma 2 and $(au+vb) \times (c+m)=0$. Thus $uv-K_{cm}$ must be a non-vacuous connected set. Hence none of the sets K_{cm}, K_{cn}, or K_{dm} contains a point of both au and vb.

As one of the sets must contain u consider for example the case where K_{cm} does. Then as $uv-K_{cm}$ is a connected set having a limit point in $c+m$, c say. $(uv-K_{cm})'=vc$ by lemma 2. Let $uz+vc=uv$ where $uz=(uv-vc)'$. As $(M-(au+uz))$ has c as a limit point vc must contain d as $c+d$ is a two-point end-set of M. Consider now the set K_{dm} since K_{cm} contains m and vc contains d.

However $vc - K_{dm}$ does not contain a connected set having d as a limit point since $vc - K_{dm}$ is contained in $M - (K_{cm} + K_{dm})$ and $K_{cm} + K_{dm}$ is a subcontinuum of M containing the two-point end-set $c + d$. And for $vc - K_{dm}$ to have m as a limit point vc must contain m and so, as K_{cm} contains m, $vc = vm$. So by reasoning similar to the above vc must contain n and the connected set $vc - K_{cn}$ does not have n as a limit point nor could it then have c as a limit point. And as vc contains $c + d + m + n$, $uv - vc$ does not have a limit point in $c + d + m + n$ and so $uz - K_{cn}$ does not have a limit point in $c + n$. Thus if vc contains m, $uv - K_{cn}$ does not have a limit point in $c + n$ and so $M - K_{cn}$ does not which is a contradiction. Therefore $vc \times m = 0$ and so $vc - K_{dm}$ does not have a limit point in $d + m$. And vc contains $c + d$ while $(uv - vc)' = uz$ contains m.

It is then necessary that $uz - K_{md}$ have a limit point in $m + d$ since $uv - K_{dm}$ does. But $(uv - vc)' = uz$ does not contain d. Thus m is a limit point of $uz - K_{dm}$ and so $(uz - K_{dm})' = um$. Hence um contains n and so contains $m + n$ but $um \times (c + d) = 0$; and $vc \times (m + n) = 0$ while vc contains $c + d$. The set $uz - K_{nd}$ does not have n as a limit point since $uz - K_{dn}$ is contained in um and so in $uz - (K_{dn} + K_{dm})$ which does not have a limit point in $m + n$. Then $vb - K_{dn}$ would have to have d as a limit point for some K_{dn} if $d + n$ is not a two-point end-set of M. It would then be necessary that there exist a vd but this must contain c and so $vd = vc = (uv - (K_{dn} + K_{cm}))'$. As K_{dn} and K_{cm} both have points of uz in common, $K_{dn} + K_{cm}$ is a subcontinuum of M containing $n + m$ and so $uv - (K_{dn} + K_{cm})$ does not have a limit point in $c + d + m + n$ which is a contradiction if $d + n$ is not a two-point end-set of M. Therefore $d + n$ must be a two-point end-set of M. And as any K_{dn} joins vc and um it must contain also $c + m$ for $(vc - K_{nd})'$ does not contain d and so does not contain c; also $(um - K_{nd})'$ does not contain n and so does not contain m.

Theorem 35. *If $c + d$, $m + n$, and $c + m$ are two-point end-sets of M and either every subcontinuum K_{cd} contains a point of $m + n$ or every subcontinuum K_{mn} contains a point of $c + d$, then*

(1) either $c + n$ or $d + n$ is a two-point end-set of M.

(2) either $d + m$ or $d + n$ is a two-point end-set of M.

Proof: Since the proof of (1) and (2) are similar only the proof of (1) will be given. Assume that $c + n$ and $d + n$ are each not two-point end-sets of M. Then there exist K_{cn} and K_{dn} whose complements in M have a limit point in $c + n$ and $d + n$ respectively. Let $K = K_{cn} + K_{dn}$. The set K is a subcontinuum of M containing $c + d$ and so $(M - K)' \times (c + d) = 0$. Let $M = au + uv + vb$ where $(M - K)' = au + vb$. Either $(au + vb) \times n = n$ or $(au + vb) \times n = 0$.

(i) Consider the case where $(au + vb) \times n = n$. Then either K_{cn} or K_{dn} contains a point of both au and vb and so contains uv. Consider for example the case where K_{cn} does. Then K_{cn} contains d and so is a subcontinuum of M containing $c + d$. Since $au + vb$ contains n consider for example the case where $au \times n = n$. Thus $au = an$, an contains m and so au contains $n + m$. But $au \times uv$ does not contain c or d since $c + d$ is a two-point end-set of M. Also $uv \times (m + n) = 0$ since $m + n$ is a two-point end-set of M. Thus as a contradiction with the hypothesis is obtained this case can not exist under our assumption.

(ii) Consider then the case where $(au + vb) \times n = 0$. Neither K_{cn} nor K_{dn} contain a point of both au and vb, for say that K_{cn} did. Then $(M - K_{cn})$ which is contained in $au + vb$ must contain a point of $c + n$ but this is impossible. Consider then for example the case where $K_{cn} \times u = u$ and $K_{dn} \times v + v$. As $(uv - K_{cn})'$ contains either c or n it equals either vc or vn. Similarly $(uv - K_{dn})'$ equals either ud or un.

Consider the case where one of these sets equal vc and the other ud. Then vc contains d and so $ud + vc = uv$ and so one of the set ud or vc contains n and thus either $ud = un$ or $vc = vn$. Thus either $au + K_{cn}$ contains $c + d + m + n$, since $au + ud$ does, or $bv + K_{dn}$ does since $bv + vc$ does. As this is impossible under our assumption this case does not exist.

Consider then the case where $(uv - K_{cn})' = vc$ and $un = (uv - K_{dn})' \neq ud$. Then vc contains $c + d$ and $au + un$ contains $m + n$. However $vc \times n = 0$ for if not $bv + vc$ and so $bv + K_{dn}$ contains $c + d + m + n$ which is impossible. Thus vc contains $c + d$ but does not contain n while $au + un$ contains $m + n$ but does not contain d. As $(uv - vc)'$ contains $u + n$ and so contains un and as $(uv - vc)'$ cannot contain a point of $c + d$ and so $au + un$ does not contain either c or d. And as $au + un$ contains m, $(uv - un)'$ contains $c + d$ but does not contain a point of $m + n$.

As this is contrary to our hypothesis this case cannot exist under our assumption.

Consider then the case where $(uv-K_{cn})'=vn\neq vc$ and $(uv-K_{dn})'=ud$. Hence ud contains $c+d$ but does not contain n for if it did $au+K_{cn}$ would contain $c+d+m+n$ which is impossible. And $bv+vn$ contains $m+n$ but does not contain c. Assume that vn contains d. But $(vn-ud)'$ contains $v+n$ and so equals vn. Thus every point of $ud\times vn$, and so d, is a limit point of the connected set $vn-ud$ which is impossible as $c+d$ is a two-point end-set of M. Hence $vn\times d=0$. Assume now that ud contains m. But $(ud-vn)'$ contains $u+d$ and so equals ud. Thus every point of $ud\times vn$, and so m, is a limit point of the connected set $ud-vn$ which is impossible. Hence ud contains $c+d$ but $ud\times(m+n)=0$; and $vn+vb$ contains $n+m$ but $(bv+vn)\times\times(c+d)=0$. As this is contrary to hypothesis this case does not exist.

The case where $(uv-K_{cn})'=vn$ and $(uv-K_{dn})'=un$ remains to be considered. This case cannot exist for then $bv+vn$, and so $bv+K_{dn}$, contains $m+n$ and so $(uv-K_{dn})'\times n=0$. Therefore as every case is impossible under our assumption the assumption must be incorrect. Thus either $c+n$ or $d+n$ is a two-point end-set of M.

Corollary 17. *If $c+d$ and $m+n$ are two-point end-sets of M and either every subcontinuum K_{cd} contains a point of $m+n$ or every subcontinuum K_{mn} contains a point of $c+d$ then either $c+m$ and $d+n$ or $c+n$ and $d+m$ are two-point end-sets of M.*

Proof: The proof follows directly from theorems 34 and 35.

Theorem 36. *If $a+p$ and $b+p$ are both two-point end-sets of M, then $K_{ap}\times K_{bp}$ is a continuum.*

Proof: Assume that $K_{ap}\times K_{bp}=K=K_1+K_2$ separate. As $K_{ap}+K_{bp}=M$, $(M-K_{ap})'=bv$, where $bv\times p=0$, and $(M-K_{bp})'==au$, where $au\times p=0$. Thus $au\times bv=0$. The set bv contains all the points of M which K_{ap} does not contain and so contains all the points of K_{bp} which K_{ap} does not contain. Hence $K_{bp}=bv++K_{ap}\times K_{bp}$. Similarly $K_{ap}=au+K_{bp}$. Thus $M=au+K_{ap}\times\times K_{bp}+vb=au+K_1+K_2+vb$. Consider for example the case where $au\times K_1\neq 0$. Then since M is irreducible between a and b $au\times K_2=0$, $bv\times K_1=0$, and $bv\times K_2\neq 0$. Therefore $M==(au+K_1)+(K_2+vb)$ separate which is a contradiction. Therefore since K is closed, K is a continuum.

Corollary 18. *If M is bounded and $a+p$ and $b+p$ are both two-point end-sets of M, then $ap\times bp$ is an indecomposable continuum.*

Proof: Let $K=ap\times bp$. As K is a continuum by theorem 36 let $au+K+vb=M=au+uv+vb$ where $(M-K)'=au++vb$. As $(au+uv)\times p=0$, uv contains p. Thus uv contains a pu and a pv. Hence it is necessary that $au+up=ap$ and $bv+vp==bp$. Then pu contains v and pv contains u and so $pu=uv=pv$ and so uv is indecomposable. And it is evident that every point of K must be contained in either pu or in pv. Therefore $K=uv$.

Theorem 37. *If there exists a pq in M where $p+q$ is a two-point end-set of M then in order that pq be indecomposable it is necessary and sufficient that every point x of $pq-(M-pq)'$ be such that either $x+p$ or $x+q$ is a two-point end-set of M.*

Proof: The condition is necessary. For assume that there exists an x such that $x+p$ and $x+q$ are each not two-point end-sets of M. Thus there exist K_{xp} and K_{xq} whose complements in M have a limit point in $x+p$ and $x+q$ respectively. Let $au+pq++vb=au+uv+vb=M$ where $(M-pq)'=au+vb$. Then $(au+vb)\times(p+q)=0$. Let $K_{px}+K_{xq}=K$. As $M-K_{px}$ has a limit point in $x+p$, $pq-K_{px}$ does also. Thus neither K_{px} nor K_{xq} contain $uv=pq$. And pq does not contain K for then $K=pq$ and so, since K is then indecomposable, either K_{px} or K_{xq} equals pq which is impossible. Assume that $K\times au=0$. Then, as $K\times v\neq 0$, by lemma 2 $(uv-K)'=uz$ and $uz\times(p+q)=0$. As $K+uz$ contains uv, $(uv-uz)'=vy$ is contained in K and so does not contain uv. Therefore $uv=uz+yv$ where $uz\neq uv\neq vy$ which is a contradiction. Hence $K\times au\neq 0\neq K\times bv$. Thus consider for example the case where $K_{px}\times u=u$ and $K_{xq}\times v=v$. Let $(pq-K_{px})'=U$ and $(pq-U)'=V$. Then K_{qx} contains U and K_{px} contains V. Thus a contradiction is obtained since either $U=pq$ or $V=pq$ as pq is indecomposable. Thus the condition is seen to be necessary.

The condition is sufficient. For assume that pq is decomposable. Let then $pq=U+V$, where $U\neq pq\neq V$ and $U\times p=p$ and $V\times q=q$. Let $au+pq+vb=M$. It is seen that $(pq-U)'++(pq-(pq-U)')'=pq$ and $(pq-U)'\times(pq-(pq-U)')'$ contains a point z. As z is contained in $(pq-(pq-U)')'\times(pq-$

$-(pq-(pq-U)')')'$[1] neither $p+z$ nor $q+z$ is a two-point end-set of M. And z is not contained in $au+vb$. Thus a contradiction is obtained and so pq must be indecomposable.

Lemma 3. *If $p+q$ is a two-point end-set of M such that M contains a pq, and K is a subcontinuum of M containing $p+q$, then K contains pq.*

Proof: Assume that K does not contain pq. Let $(M-pq)'=au+vb$, and let $M=au+pq+vb=au+uv+vb$. As pq does not contain K, $(au+vb)\times K\neq 0$, but either $K\times au=0$ or $K\times vb=0$. Consider for example the case where $K\times au=0$. Thus $(uv-K)'=uz$ by lemma 2. And $(au+uz+vb)\times(p+q)=0$. Therefore $(uv-uz)'=vy$ which must contain $p+q$ and so $vy=pq=uv$. But as K contains vy, $K\times au\neq 0$ and so K must contain pq.

Theorem 38. *If M contains a pq such that $pq\times(a+b)=b$ where $p+q$ is a two-point end-set of M, then (1) either $a+p$ or $a+q$ is a non-remainder set of M; (2) if $a+p$ and $a+q$ are both non-remainder sets of M then pq is indecomposable; (3) if every K_{pb} contains pq then $p+b$ is a two-point end-set of M; and (4) if $a+q$ is a non-remainder set of M and if a K_{pb} does not contain pq, then $a+p$ is a non-remainder set of M and $b+q$ is a two-point end-set of M.*

Proof: (1) $K_{ap}+K_{aq}$, by lemma 3, contains pq and so contains b. Therefore either K_{ap} or K_{aq} contains b and so equals M. Thus either $a+p$ or $a+q$ is a non-remainder set of M.

(2) Let $au+pq=M$ where $(M-pq)'=au$. Let pq contain K_{pu}. Then $K_{pu}+au=M$ since $a+p$ is a non-remainder set of M. Hence $au\times(p+q)=0$, $K_{pu}\times q=q$ and so $K_{pu}=pq=pu$. Similarly $K_{qu}=pq=qu$ and so pq is indecomposable.

(3) As $(M-K_{pb})'\times(b+p+q)=0$, $p+b$ is a two-point end-set of M.

(4) It is seen that $K_{ap}+K_{pb}=M$. But as K_{pb} does not contain pq, by lemma 3 $K_{pb}\times q=0$. Thus K_{ap} contains q and so contains pq which contains b. Hence $K_{ap}=M$. Therefore by (2) pq is indecomposable. But $K_{pb}+K_{bq}$ contains pq and so K_{bq} contains pq as K_{pb} does not since pq is indecomposable. Thus $(M-K_{bq})'\times(p+q+b)=0$ and so $b+q$ is a two-point end-set of M.

[1]) C. Kuratowski, Fund. Math. III., theorem 6, p. 183.

Theorem 39. *If M contains a pq such that $pq\times(a+b)=b$ where $p+q$ is a two-point end-set of M and $a+p$ is a two-point end-set of $M=aq\neq K_{ap}$, then pq is decomposable.*

Proof: As $K_{ap}\neq M$, $K_{ap}\times(b+q)=0$. Let $M=au+pq$ where $(M-pq)'=au$. By lemma 2 $(pq-K_{ap})'=qz$ and $(pq-qz)'=py$. As $a+p$ is a two-point end-set of M qz contains neither a nor p and as K_{ap} contains py, $py\neq pq\neq qz$ although $py+zq=pq$. Thus pq is decomposable.

Theorem 40. *If M contains a pq such that $pq\times(a+b)=b$ where $p+q$, $a+p$, and $a+q$ are two-point end-sets of M, then in order that pq be indecomposable it is necessary and sufficient that both $a+p$ and $a+q$ be non-remainder sets of M.*

Proof: That the condition is sufficient follows from (2) of theorem 38.

The condition is necessary. For by (1) of theorem 38 either $ap=M$ or $aq=M$. Consider for example the case where $aq=M$ and assume that $K_{ap}\neq M$. Then by theorem 39 pq is decomposable which is a contradiction. Thus $aq=M=ap$.

Theorem 41. *If M contains a pq such that $pq\times(a+b)=0$ where $p+q$ is a two-point end-set of M then (1) if $a+p$ is a non-remainder subset of a K_{aq} then $a+p$ is a two-point end-set of M; (2) if there exist an aq of which $a+p$ is a non-remainder subset then $a+p$ and $a+q$ are two-point end-sets of M and pq is indecomposable; and (3) if for every K_{aq} $a+p$ is a remainder set of K_{aq} then $a+q$ and $b+p$ are two-point end-sets of M and if M is bounded and $a+p$ is a two-point end-set then $ap\times pq$ is indecomposable.*

Proof: (1) As K_{aq} contains ap it contains $p+q$ and so contains pq. And as every $(M-K_{ap})'$ is contained in $(M-ap)'$, $a+p$ is a two-point end-set of M.

(2) By hypothesis $ap=qa$. Thus by (1) $a+p$ and $a+q$ are two-point end-sets of M. And since $p+q$ is a two-point end-set of aq and $pq+(a+q)=q$, by (2) of theorem 38 pq is indecomposable.

(3) As every K_{aq} contains $p+q$, $(M-K_{aq})'\times(p+q)=0$. Hence $a+q$ is a two-point end-set of M. Assume that there exist a K_{bp} such that $(M-K_{bp})'=ap$. Then ap must contain q and so is a K_{aq} of which $a+p$ is not a remainder set. Therefore $b+p$ must be a two-point end-set of M.

If M is bounded then by corollary 15 there exists one and only one aq in M. And if $a+p$ is a two-point end-set of M it is also one of aq as is also $p+q$. Then by corollary 18 $ap \times pq$ is indecomposable.

VIII. Connected subsets of the set of two-point end-sets of M.

If $(a)_b$ contains a point p then $p+b$ is a two-point end-set of M and so if M is bounded $(a)_b$ is a connected subset of the set of two-point end-sets of M. Thus in the preceding sections, for the case where M is bounded, properties of these connected subsets have been obtained. In this section further properties of these connected subsets, which need not contain a or b, are obtained.

Theorem 42. *If M is bounded and N is a maximal connected subset, containing a point x, of the set of two-point end-sets of M such that $a+x$ and $b+x$ are both two-point end-sets of M, then N' is indecomposable.*

Proof: By corollary 18 $ax \times bx$ is indecomposable. Let $T = ax \times bx$ and $M = au + T + vb = au + uv + vb$ where $(M - T)' = au + vb$. As $T + vb$ contains $b+x$, which is a two-point end-set of M, $au \times x = 0$. Similarly $bv \times x = 0$. Thus $au \times vb = 0$ Hence $uv = T$ as $T + vb = bx$ and $T + au = ax$. Therefore every point of $(au+vb) \times T$ is a limit point of a connected subset of $T-(au+vb)$ and so no point y of $(au+vb) \times T$ is such that either $a+y$ or $b+y$ is a two-point end-set of M. Therefore no point y of $(au+vb) \times T$ is a point of the set of two-point end-sets of M, for by corollary 17 if it was then either $a+y$ or $b+y$ would be a two-point end-set of M.

And as every point z of $T-(au+vb)$ is such that either $a+z$ or $b+z$ is a two-point end-set of M, $T = N'$ and so N' is indecomposable.

Theorem 43. *If M is bounded and N is a maximal connected ubset, containing $p+q$, of the set of two-point end-sets of M suchs sthat $a+p$ and $b+q$ are both two-point end-sets of M, then N' i indecomposable.*

Proof: Assume that N' is decomposable. If N contains a point x then by corollary 17 either $a+x$ or $b+x$ is a two-point end-set of M. And by theorem 42 there does not exist in N a point x such that $a+x$ and $b+x$ are both two-point end-sets of M. Thus $N = Z + W$ where every point z of Z is such that $a+z$ is a two-point end-set of M and every point w of W is such that $b+w$ is a two-point end-set of M and $Z \times W = 0$.

Let z be a point of Z. Assume that az does not contain N. Then az contains a limit point y of $N - az$. As $M - az$ is connected, y is a limit point of the connected set $M - az$ and so Z does not contain y. As $a+z$ is a two-point end-set of M, $by \times z = 0$, where $(M - az)' = by$. Therefore by contains a limit point x of $N - by$. Thus W does not contain x. And as az contains x as does also $(M-az)'$ Z does not contain x. Therefore N does not contain x which is a contradiction. Hence az must contain N as must bw where w is any point of W.

By hypothesis there exists a point z of Z and a point w of W. Thus $a+z$ and $b+w$ are two-point end-sets of M. But az contains N and so contains w. Hence by corollary 17 either $z+w$ or $z+b$ is a two point end-set of M. But by theorem 42, since $a+z$ is a two-point end-set of M, $b+z$ cannot be. Thus $z+w$ must be a two-point end set of M. Let $au + zw + vb = M$ where $(M - zw)' = au + vb$. Then $au \times vb = 0$.

For every point c of zw either the irreducible continuum joining c and bv or that joining c and au contains points of N and so either ac or bc contains N and so contains zw. If ac contains zw, then $ac = au + zw$ and $a+c$ is a two-point end-set of M if $c \times bv = 0$. Similarly if bc contains zw $b+c$ is a two-point end-set of M if $c \times au = 0$. Thus $zw - (au+vb)$ is contained in N and as au cannot contain a point of N without containing N, zw contains N. Therefore $zw = N'$. Since $zw = N'$ is decomposable by assumption, by theorem 37, every point of $zw - (au+vb)$ is not contained in N. As this is a contradiction N' is indecomposable.

Theorem 44. *If M is bounded, N is a maximal connected subset of the set of two point end-sets of M, and every point p of N is such that $a+p$ is a two-point end-set of M, then in order that N be closed it is necessary and sufficient that N contain b.*

Proof: The condition is necessary. For assume that $N \times b = 0$. As $a+a$ is not a two-point end-set, $a \times N = 0$. Let u be a point of $(M-N)' \times N$. Hence $(M-N)'$ contains an au. And $(M - (au+N))' = bv$. Thus $au + N + vb = M$. For any point p of N as N is closed $au + N$ is a subcontinuum of M containing $a+p$ and so $bv \times N = 0$. Therefore $au \times vb \neq 0$ and so au must contain N. Hence u is any point of N and so $(u)_a$ contains N. Also every point of $(u)_a - bv$ must be contained in N. Therefore $(u)_a - bv = N$ which is closed. And $(u)_a$ contains the closed set $au \times vb$. Thus $(u)_a$ is the sum of two distinct closed sets which, since by theorem 2 $(u)_a$ is connected, gives a contradiction. Therefore $N \times b = b$.

The condition is sufficient. Let p be any point of N. As ap contains N, ap contains b and so $ap = M$. Hence $N = (b)$. Assume that N is not closed. Then N' is indecomposable [1]). Let then $(M - N')' = au$, where au is vacuous if $M = N'$ but letting then $u = a$. Hence $N' - au$ is contained in N. But there must then exist a basic-wise connected subset of N' containing u, a point p of which is contained in N but $a + p$ is not a two-point end-set of M. Thus a contradiction is obtained and so $N = N'$.

Theorem 45. *If M is bounded, (p) and (q) are maximal connected subsets of the two-point end-sets of M such that (p) contains $p + z$ and (q) contains $q + w$, and $(p)'$ and $(q)'$ are each decomposable, then $z + w$ is a two-point end-set of M.*

Proof: By corollary 17 either $p + a$ or $p + b$ is a two-point end-set of M but not both by theorem 42. Consider for example the case where $p + b$ is a two-point end-set of M. Then by theorem 43 $z + b$ is a two-point end-set of M. But zb contains (p) and so contains p. Therefore by corollary 17 either $z + p$ or $z + q$ is a two-point end-set of M. But if $z + p$ is then by corollary 17 either $p + a$ or $z + a$ is. But neither is by theorem 42. Therefore $z + q$ must be a two-point end-set of M.

As $(z) = (p)$, just as $z + q$ was shown to be a two-point end-set of M so $z + w$ can be shown to be.

Theorem 46. *If M is bounded, $p + q$ is a two-point end-set of M, (p) and (q) are the maximal connected subsets of the set of two-point end-sets of M such that (p) contains p and (q) contains q, and $(p)'$ and $(q)'$ are decomposable, then $(p)' \times (q)' = 0$.*

Proof: As $p + q$ is a two-point end-set of M, pq contains $(p) + (q)$. Assume that $(p)' \times (q)' \neq 0$. Then $(p)' + (q)' = pq$. Similarly by theorem 45 $zq = (p)' + (q)'$, if (p) contains z. Thus $(p)_q$ contains (p) and $(q)_p$ contains (q).

Let $(p)' \times (q)'$ contain x and let xp be contained in $(p)'$. Then $(q)' + xp = pq$. Hence xp contains (p) and so $xp = (p)'$ for $(q)' \times (p) = 0$ for otherwise $(q)' = pq = (q)'_p$ and so $(q)'$ is indecomposable. Furthermore $(p)_x$ contains (p) since $xp = xz = (p)'$ where z is any point of (p). Thus $(p)' = (p)'_x$ and so $(p)'$ is indecomposable. As this is a contradiction $(p)' \times (q)' = 0$.

[1]) C. Kuratowski, Fund. Math. X., theorem 2, p. 235.

IX. n-Point end-sets of M.

In the previous sections we have dealt with n-point end-sets where $n = 2$. Here a few theorems are given concerning n-point end-sets where n is greater than two.

Lemma 4. *If M is bounded, $p + q$ is a two-point end-set of M, and (x) is the point set composed of all points x such that $p + x + q$ is a three-point end-set of M where pq contains x and $pq \neq M$, then $p + (x) + q$ is connected but not closed.*

Proof: As $pq \neq M$ there exists an $au + pq + vb = M$. As there exist but one pq every continuum containing $p + x + q$ contains pq which in turn contains x. Then if $(p + x + q) \times (M - pq)' = 0$ $(p + x + q) \times (M - N)' = 0$ where N is any subcontinuum containing $p + x + q$. As $p + q$ is a two-point end-set of M $(p + x + q) \times (M - pq)' = 0$ if $x \times (au + vb) = 0$. Therefore $p + (x) + q = pq - au - vb$. Thus $p + (x) + q$ is connected but not closed.

Lemma 5. *If M is bounded, $p + q$ is a two-point end-set of M, and (x) is the set composed of all points x such that $p + x + q$ is a three-point end-set of M where $pq \neq M$, then $p + (x) + q$ is neither closed nor connected.*

Proof: As $pq \neq M$, there exist an $au + pq + vb = M$ where $(M - pq)' = au + vb$. Consider for example the case where au is non-vacuous. As every point of $pq - (au + vb)$ is contained in (x), $(x) \times pq \neq 0$. And if K is any subcontinuum of M containing $a + p + q$ then $(M - K)' \times (a + p + q) = 0$ and so (x) contains a. Therefore (x) is not connected since it contains a point of pq and a point not in pq. And as $(p + (x) + q) \times pq$ is not closed by lemma 4 $p + (x) + q$ is not closed.

Theorem 47. *If M is bounded, $p + q$ is a two-point end-set of M, and (x) is composed of all points x such that $p + x + q$ is a three-point end-set of M, then in order that $p + (x) + q$ be connected it is necessary and sufficient that pq contain (x).*

Proof: The condition is necessary. For assume that pq does not contain (x). Then $pq \neq M$. Thus a contradiction is obtained by lemma 5.

The condition is sufficient. If $pq \neq M$ then the theorem follows by means of lemma 4; and if $pq = M$ then $p + (x) + q = M$ and so is connected.

Lemma 6. *If $N = p_1 + p_2 + \ldots + p_n$ is an n-point end-set of M then N contains a two-point end-set of M.*

Proof: Assume that N does not contain a two-point end-set of M. Then for every i and j, where $i \neq j$ $(i, j = 1, 2, \ldots n)$ there exists a $K_{p_i p_j} = T_{p_i p_j}$ whose complement in M has a limit point in $p_i + p_j$. The set $T_{p_i p_j}$ then has the property that it does not contain N and so does not equal M. Let $T = T_{p_1 p_2} + T_{p_1 p_3} + \ldots + T_{p_1 p_n}$. Hence $(M - T)' \times N = 0$ as T contains N. Let $(M - T)' = au + vb$ and let $M = au + uv + vb$. As $(M - T_{p_1 p_l})' \times N \neq 0$ it is necessary that there exist a $T_{p_1 p_l}$, C_1 say, such that $C_1 \times au \neq 0$ and $C_1 \times vb = 0$ and there exist another, C_2 say, such that $C_2 \times au = 0$ and $C_2 \times vb \neq 0$. The set $M = au + C_1 + C_2 + vb$, where uv contains $C_1 + C_2$. Hence $(uv - C_1)' \times N \neq 0$. Say it contains q_1. Then $(uv - C_1)' = vq_1$. As C_2 contains vq_1, $vq_1 \neq vu$. As C_1 cannot contain N, $vq_1 - C_1$ must contain at least one point of N. Let $(K_{q_1 p_l})$ be the sum of a set of $K_{q_1 p_l}$'s which sum contains a $K_{q_1 p_l}$ for each p_i contained in $vq_1 - C_1$ where $K_{q_1 p_l}$ is selected so as not to contain $N \times (vq_1 - C_1)$ if such a $K_{q_1 p_l}$ exists. Let $(vq_1 - (K_{q_1 p_l}))' = vz$. As $C_1 + (K_{q_1 p_l})$ contains N, $N \times vz = 0$ but vz contains a point of a $K_{q_1 p_l}$, $K_{q_1 q_2}$ say. Then $C_1 + K_{q_1 q_2} + zv = uv$ and so every $K_{q_1 q_2}$ must contain $(vq_1 - C_1) \times N$. Therefore let zv be such that $(uv - T_{q_1 q_2})' = uq_1 + zv$. Hence $(uq_1 - T_{q_1 q_2}) \times N \neq 0$. Thus proceeding as above it is seen that there exists a $K_{q_1 p_l}$, $K_{q_1 q_3}$ say, where q_3 is contained in $uq_1 - T_{q_1 q_2}$ such that every $K_{q_1 q_3}$ contains $(uq_1 - T_{q_1 q_2}) \times N$.

There exist a $T_{q_2 q_3} = Q$. But $vq_1 + Q$ is a $K_{q_1 q_3}$ and so contains N. Thus $(uq_1 - Q) \times N = 0$. Also $uq_1 + Q$ is a $K_{q_1 q_3}$ and so contains N. Thus $(vq_1 - Q)' \times N = 0$. Therefore $(uv - Q)' \times N = 0$ which is contrary to our assumption. Hence N must contain a two-point end-set of M.

Theorem 48. *If $N = p_1 + p_2 + \ldots + p_n$ is an n-point end-set of M, then N contains a two-point end-set of M such that every subcontinuum of M containing these two points contains N.*

Proof: By lemma 6N contains a two-point end-set of M. Consider for example the case where $p_1 + p_2$ is a two-point end-set of M and assume that $K_{p_1 p_2} = P$ does not contain N. Then $(M - P)' \times (p_1 + p_2) = 0$. Let $(M - P)' = au + vb$ and let $au + uv + vb = M$. For each p_i of $au - uv$ let T_{up_i} be a subcontinuum of M joining p_i and uv and if possible let it be so taken that it does not contain $(au - uv) \times N$. Let (T_{up_i}) be the sum of such sets. Similarly there exists a (T_{vp_i}) where p_i is contained in $bv - uv$. Hence $(T_{up_i}) + uv + (T_{vp_i}) = K$ is a subcontinuum of M containing N. Let $(M - K)' = aw + zb$ and let $aw + wz + zb = M$. Thus aw contains a point of a T_{up_i}, T_{uq_1} say, and every K_{uq_1} must then contain $(au + uv) \times N$. Similarly bz contains a point of a T_{up_i}, T_{vq_2} say, and every K_{vq_2} must contain $(vb - uv) \times N$. If neither (T_{up_i}) nor (T_{vp_i}) is vacuous then it is evident that every $K_{q_1 q_2}$ contains N and so $q_1 + q_2$ is a two-point end-set of M. Consider then for example the case where $(T_{vp_i}) = 0$. Then either every K_{up_1} or every K_{up_2} contains $p_1 + p_2$ otherwise M would contain a subcontinuum not containing one of these points but containing $a + b$. Consider for example the case where every K_{up_2} contains p_1. If there exist a K_{up_2} such that $au + K_{up_2}$ does not contain N then the theorem is true by the above proof. And, if for every K_{up_2}, $au + K_{up_2}$ contains N, then $K_{uq_1} + K_{up_2}$ contains N and so every $K_{q_1 p_2}$ contains N. Hence also $q_1 + p_2$ is a two-point end-set of M. Thus in every case the theorem is true.

X. Joined sets of M.

In corollary 15 it was proven that if M is bounded and $p + q$ is a two-point end-set of M then q is a p joined point of order one in M. Kuratowski has shown [1]) that *if p is any point of M then either a or b is a p joined point of order one in M.* It follows that *if N is the joined set of order one in M then $N = M$.* In this section a few other properties of joined sets will be obtained.

Theorem 49. *If p is not an a joined point of order one in M then $b + p$ is a two-point end-set of M.*

Proof: Since either a or b is a p joined point of order one in M it follows that p is a b joined point of order one in M. If $bp = M$ the theorem is true. If $pb \neq M$ then $au + bp = M$ where $(M - bp)' = au$. If $au \times p = p$ then $au = ap$ and so p is an a joi-

[1]) Fund. Math. III., p. 219—220.

ned point of order one in M which is a contradiction. Thus $au \times \times (b+p) = 0$ and as $(M - K_{bp})'$ is contained in au $(M - K_{bp})' \times \times (b+p) = 0$. Hence $b+p$ is a two-point end-set of M.

Theorem 50. *If M is bounded and neither p nor q is a limit point of the non-cut points of $(M - pq)'$ then either q is a p joined point of order one in M or pq is indecomposable.*

Proof: If $pq = M$ the theorem is true. Consider then the case where $pq \neq M$. Then let $au + pq + vb = M = au + uv + vb$ where $(M - pq)' = au + vb$. Consider for example the case where au is non-vacuous. Let $(u) = (u)_a$ of au. The set (u) contains $au \times pq$. Either $au \times pq = u$ or $au \times pq \neq u$. Consider the case where $au \times \times pq \neq u$. No point of (u) is a cut point of au for if (u) contains x $au = ax$. Assume that x of (u) is a cut point of M. Then $M - x = = W + Z$ separate. Either W or Z contains a and so contains $au - x + pq + vb = M - x$. Thus either W or Z must be vacuous. Hence no point of (u) can be a cut point of M. Let u and x be two distinct points of (u) contained in $au \times pq \neq u$. Then there exists a region R containing x such that $R' \times u = 0$. Then x is contained in a subcontinuum of $R' \times au$ which does not contain u and so by theorem 1 this subcontinuum is contained in (u). Hence as each point of (u) is a non-cut point of M x is a limit point of non-cut points of M which are contained in au of $(M - pq)'$. Therefore $(u) \times (p+q) = 0$ and so $au \times (p+q) = 0$. Consider now the case where $au \times pq = u$. If $u \times (p+q) = 0$, $au \times (p+q) = 0$. Thus in every case $au \times (p+q) = 0$ or else either $u = p$ or $u = q$. Hence always $pq = uv$.

If there exists but one irreducible continuum of M joining p and q the theorem is true. If $pxq \neq pq$ then consider for example the case where $pu_1 \neq pq \neq qu_2$ where $pu_1 \neq qu_2$ is contained in pxq. Let uv contain pv_1 and qv_1. Then $pu_1 + pv_1 = uv = pq$. Thus $pv_1 = pq$ and similarly $qv_1 = pq$. Thus pq is indecomposable.

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Les ensembles analytiques comme criblés au moyen des ensembles fermés.

Par

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1. H étant un ensemble de points donné quelconque, situé dans le plan, nous désignerons par $\Gamma(H)$ et nous appellerons, d'après M. N. Lusin, *ensemble criblé au moyen du crible H* [1]), l'ensemble de tous les nombres réels a, tels que la droite $x = a$ rencontre l'ensemble H en un ensemble de points (non vide) qui *n'est pas bien ordonné* à l'aide de cette convention que le rang des points soit conforme à la direction positive de l'axe $0Y$.

M. Lusin a démontré que les ensembles criblés au moyen des cribles F_σ (même des cribles F_σ d'une nature particulière) coïncident avec les ensembles analytiques.

Or, nous prouverons dans ce § que *les ensembles analytiques coïncident avec les ensembles criblés au moyen des ensembles fermés* [2]).

Soit E un ensemble analytique linéaire donné. Il existe, comme on sait, un système d'intervalles fermés $\{\delta_{n_1, n_2, \ldots, n_k}\}$, tel qu'on a pour tout système fini d'indices $n_1, n_2, \ldots, n_k, n_{k+1}$:

(1) $$\delta_{n_1, n_2, \ldots, n_k, n_{k+1}} \subset \delta_{n_1, n_2, \ldots, n_k},$$

et que

(2) $$E = \Sigma\, \delta_{n_1} \delta_{n_1, n_2} \delta_{n_1, n_2, n_3} \ldots,$$

[1]) N. Lusin: *Fund. Math.* t. X, p. 10; aussi: *„Leçons sur les ensembles analytiques..."*, Paris, Gauthier-Villars 1930, p. 178 ss.

[2]) J'ai signalé ce théorème (sans le démontrer) dans ma note des *C. R.*, t. 185, p. 835 (séance du 24 octobre 1927).