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Vitali sets and Hamel bases that are Marczewski measurable

by

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Abstract. We give examples of a Vitali set and a Hamel basis which are Marczewski measurable and perfectly dense. The Vitali set example answers a question posed by Jack Brown. We also show there is a Marczewski null Hamel basis for the reals, although a Vitali set cannot be Marczewski null. The proof of the existence of a Marczewski null Hamel basis for the plane is easier than for the reals and we give it first. We show that there is no easy way to get a Marczewski null Hamel basis for the reals from one for the plane by showing that there is no one-to-one additive Borel map from the plane to the reals.

Basic definitions. A subset A of a complete separable metric space X is called *Marczewski measurable* if for every perfect set $P \subseteq X$ either $P \cap A$ or $P \setminus A$ contains a perfect set. Recall that a *perfect set* is a non-empty closed set without isolated points, and a *Cantor set* is a homeomorphic copy of the Cantor middle-third set. If every perfect set P contains a perfect subset which misses A, then A is called *Marczewski null*. The class of Marczewski measurable sets, denoted by (s), and the class of Marczewski null sets, denoted by (s^0), were defined by Marczewski [10], where it was shown that (s) is a σ-algebra, i.e. $X \in (s)$ and (s) is closed under complements and countable unions, and (s^0) is a σ-ideal in (s), i.e. (s^0) is closed under countable unions and subsets. Several equivalent definitions and important properties of (s) and (s^0) were proved in [10], for example every analytic set is Marczewski measurable, the properties (s) and (s^0) are preserved under "generalized homeomorphisms" (also called Borel bijections), i.e. one-to-one onto functions f such that both f and f^{-1} are Borel measurable (i.e. pre-images of open sets are Borel), a countable product is in

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(s) if and only if each factor is in (s), and a finite product is in (s^0) if and only if each factor is in (s^0).

The *perfect kernel* of a closed set F is the set of all $a \in F$ such that $U \cap F$ is uncountable for every neighborhood U of a.

A set is *totally imperfect* if it contains no perfect subset. A totally imperfect set of reals cannot contain uncountable closed set, so it must have inner Lebesgue measure zero. A set B is called *Bernstein set* if every perfect set intersects both B and the complement of B, or, equivalently, both B and its complement are totally imperfect. Clearly, no Bernstein set can be Marczewski measurable.

A set A is *perfectly dense* if its intersection with every non-empty open set contains a perfect set.

Let $\mathbb{R}$ denote the set of all *real numbers* and $\mathbb{Q}$ denote the set of all *rational numbers.* We use $\mathfrak{c}$ to denote the cardinality of the continuum.

The *linear closure* (or *span*) *over* $\mathbb{Q}$ of a non-empty set $A \subseteq \mathbb{R}$ is the set

$$\operatorname{span}(A) = \{q_1 a_1 + \ldots + q_n a_n : n < \omega,\ q_j \in \mathbb{Q},\ a_j \in A\}$$

and $\operatorname{span}(\emptyset) = \{0\}$. A is called *linearly independent* over $\mathbb{Q}$ if $q_1 a_1 + \ldots + q_n a_n \neq 0$ whenever $n < \omega$, $q_j \in \mathbb{Q}$ for $1 \leq j \leq n$ with $q_j \neq 0$ for at least one j, and $a_1, \ldots, a_n$ are different points from A. A linearly independent set H such that $\mathbb{R} = \operatorname{span}(H)$ is called a *Hamel basis.* Note a Hamel basis must have cardinality $\mathfrak{c}$. The inner Lebesgue measure of any Hamel basis H is zero (Sierpiński [8], see also Erdős [2]). A Hamel basis can have Lebesgue measure 0 (see Jones [4], or Kuczma [6], Chapter 11).

A Hamel basis H which intersects every perfect set is called a *Burstin set* [1]. Every Burstin set H is also a Bernstein set, otherwise if $P \subseteq H$ for some perfect set P, by the linear independence of H it follows that $H \cap 2P = \emptyset$ (where $2P = \{2p : p \in P\}$), a contradiction since $2P$ is a perfect set.

A Burstin set can be constructed as follows. List all perfect subsets of $\mathbb{R}$ as $\{P_\alpha : \alpha < \mathfrak{c}\}$, pick a non-zero $p_0 \in P_0$ and, using the facts that

$$|\operatorname{span}(A)| \leq |A| + \omega < \mathfrak{c} \quad \text{if } |A| < \mathfrak{c}$$

and $|P_\alpha| = \mathfrak{c}$ for each α, pick by induction

$$p_\alpha \in P_\alpha \setminus \operatorname{span}(\{p_\beta : \beta < \alpha\})$$

and let $H_{\mathfrak{c}} = \{p_\alpha : \alpha < \mathfrak{c}\}$. If H is a maximal linearly independent set with $H_{\mathfrak{c}} \subseteq H$, then H is a Burstin set.

A set $V \subseteq \mathbb{R}$ is called a *Vitali set* if V is a complete set of representatives (or a transversal) for the equivalence relation defined by $x \sim y$ iff $x - y \in \mathbb{Q}$, i.e. for each $x \in \mathbb{R}$ there exists a unique $v \in V$ such that $x - v \in \mathbb{Q}$. No Vitali set is Lebesgue measurable or has the Baire property. One may construct a Vitali set which is a Bernstein set.

Perfectly dense Marczewski measurable Vitali set. Recall that an equivalence relation on a space X is called *Borel* if it is a Borel subset of $X \times X$. The Vitali equivalence $\sim$ as defined above is Borel. We first show that a Vitali set cannot be Marczewski null.

THEOREM 1. *Suppose X is an uncountable separable completely metrizable space with a Borel equivalence relation*, $\equiv$, *on it with every equivalence class countable. Then*, *if $V \subseteq X$ meets each equivalence class in exactly one element*, *V cannot be Marczewski null.*

Proof. By a theorem of Feldman and Moore [3] every such Borel equivalence relation is induced by a Borel action of a countable group. This implies that there are countably many Borel bijections $f_n : X \to X$ for $n \in \omega$ such that $x \equiv y$ iff $f_n(x) = y$ for some n. If V were Marczewski null, then so would $X = \bigcup_{n<\omega} f_n(V)$. ■

To obtain a Marczewski measurable Vitali set we will use the following theorem:

THEOREM 2 (Silver [9]). *If E is a coanalytic equivalence relation on the space of all real numbers and E has uncountably many equivalence classes*, *then there is a perfect set of mutually E-inequivalent reals* (*in other words*, *an E-independent perfect set*). *In the case of a Borel equivalence relation E*, *one can drop the requirement that the field of the equivalence be the whole set of reals.*

If $E \subseteq X \times X$ is a Borel equivalence relation, where X is an uncountable separable completely metrizable space, and B is a Borel subset of X, then the saturation of B, $[B]_E = \bigcup_{x \in B} [x]_E$, is analytic since it is the projection onto the second coordinate of the Borel set $(B \times X) \cap E$. The saturation need not be Borel, for example let B be a Borel subset of $X = \mathbb{R}^2$ whose projection $\pi_1(B)$ into the first coordinate is not Borel. Define $(x, y)E(u, v)$ iff $x = u$ (i.e. two points are equivalent if they lie on the same vertical line). Then $[B]_E = \pi_1(B) \times \mathbb{R}$ is not Borel. On the other hand, if E is a Borel equivalence with each equivalence class countable, and f_n are as in the proof of Theorem 1, then the saturation $[B]_E = \bigcup_{n<\omega} f_n(B)$ of every Borel set B is Borel.

THEOREM 3. *Suppose X is an uncountable separable completely metrizable space with a Borel equivalence relation E. Then there exists Marczewski measurable $V \subseteq X$ which meets each equivalence class in exactly one element.*

Proof. Let $\{P_\alpha : \alpha < \mathfrak{c}\}$ list all perfect subsets of X. We will describe how to construct disjoint C_α, each C_α either countable (possibly finite or empty) or a Cantor set such that the set $V_\alpha = \bigcup_{\beta<\alpha} C_\beta$ is E-independent. Then extend the set $V_\mathfrak{c} = \bigcup_{\alpha<\mathfrak{c}} C_\alpha$ to a maximal E-independent set V.

Case (a). If $P_\alpha \cap [C_\beta]_E$ is uncountable for some $\beta < \alpha$, then let $C_\alpha = \emptyset$.

Subcase (a1): $|P_\alpha \cap C_\beta| > \omega$. Then the perfect kernel of $P_\alpha \cap C_\beta$ is contained in both P_α and V_α (and hence in V).

Subcase (a2): $|P_\alpha \cap C_\beta| = \omega$. Then, since $P_\alpha \cap [C_\beta]_E \setminus C_\beta$ is uncountable analytic, it contains a perfect set Q which misses V.

Case (b): Not Case (a). Then

$$|P_\alpha \cap [V_\alpha]_E| = \Big|P_\alpha \cap \bigcup_{\beta<\alpha} [C_\beta]_E\Big| \le |\alpha|\omega < \mathfrak{c},$$

and hence $P_\alpha \setminus [V_\alpha]_E$ contains a Cantor set P.

Subcase (b1): The restriction of E to P has only countably many classes. Let C_α be a countable E-independent subset of P with $P \subseteq [C_\alpha]_E$. Then $P \setminus C_\alpha$ contains a perfect set which misses V.

Subcase (b2): Case (b) but not case (b1). Then, by the above theorem of Silver, there is a perfect E-independent set $C_\alpha \subseteq P$ (and $C_\alpha \subseteq V$). ∎

Remark 4. The Vitali equivalence shows that a Borel equivalence need not have a transversal that is Lebesgue measurable or has the Baire property. See Kechris [5], 18.D, for more on transversals of Borel equivalences.

Theorem 5. *There exists a Vitali set which is Marczewski measurable and its intersection with each non-empty open set contains a perfect set.*

Proof. By Theorem 3 there is a Marczewski measurable Vitali set V, and by Theorem 1, V contains a perfect set C. Split C into countably many Cantor sets $C_0, C_1, \ldots$, fix a basis $\{B_n : n < \omega\}$ for the topology of $\mathbb{R}$ and pick rational numbers q_n so that the set $q_n + C_n = \{q_n + c : c \in C_n\}$ intersects B_n for each n. Then

$$V' = (V \setminus C) \cup \bigcup\{(q_n + C_n) : n < \omega\}$$

is a perfectly dense Marczewski measurable Vitali set. ∎

Remark 6. A Vitali set V cannot have the stronger property that its intersection with every perfect set contains a perfect set. This is because if V contains a perfect set P, then the perfect set

$$P' = P + 1 = \{p + 1 : p \in P\}$$

does not intersect V. Similarly, if H is a Hamel basis that contains a perfect set P, then

$$2P = \{2p : p \in P\}$$

is a perfect set which misses H.

Marczewski null Hamel bases

REMARK 7 (Erdős [2]). Under CH there is a Hamel basis H which is a Lusin set (and hence Marczewski null). To see this, note that by a result of Sierpiński there is a Lusin set X such that $X+X=\{x+y : x,y\in X\}=\mathbb{R}$ (see e.g. [7]). Let H be any maximal linearly independent subset of X; then clearly $\operatorname{span}(H)=\operatorname{span}(X)=\mathbb{R}$.

Our construction (without CH) of a Marczewski null Hamel basis is slightly simpler for the plane, so we do it first.

THEOREM 8. *There exists a Hamel basis*, H, *for* $\mathbb{R}\times\mathbb{R}$, *i.e. a basis for the plane as a vector space over* $\mathbb{Q}$, *which is a Marczewski null set*, *i.e.*, *every perfect set contains a perfect subset disjoint from* H.

LEMMA 9. *Suppose* V *with* $|V|<\mathfrak{c}$ *is a subspace of* $\mathbb{R}\times\mathbb{R}$ *as a vector space over* $\mathbb{Q}$ (*not necessarily closed*), $p\in\mathbb{R}\times\mathbb{R}$, $y\in\mathbb{R}$, *and*

$$U\subseteq U_y=(\{y\}\times\mathbb{R})\cup(\mathbb{R}\times\{y\})$$

with $|U|<\mathfrak{c}$. *Then there exists a finite* $F\subseteq(U_y\setminus U)$ *with* $p\in\operatorname{span}(F\cup V)$ *and such that* F *is linearly independent over* $\mathbb{Q}$ *and independent over* V, *i.e.*, $\operatorname{span}(F)$ *meets* V *only in the zero vector.*

Proof. CASE 1: $p=(u,0)$. Let y_1 and y_2 be so that

$$y_2-y_1=u,\quad (y_1,y)\notin U\quad\text{and}\quad (y_2,y)\notin U.$$

Clearly, $p\in\operatorname{span}(\{(y_1,y),(y_2,y)\})$. Let

$$F\subseteq\{(y_1,y),(y_2,y)\}\subseteq U_y\setminus U$$

be minimal such that $p\in\operatorname{span}(V\cup F)$. Then F works.

CASE 2: $p=(0,v)$. Obviously, this case is symmetric.

CASE 3: $p=(u,v)$. Apply Case 1 to $(u,0)$ obtaining F_1. Let

$$V'=\operatorname{span}(V\cup F_1)$$

and apply Case 2 to V' obtaining F_2 (and let $F=F_1\cup F_2$) so that

$$(u,0),(0,v)\in\operatorname{span}(V\cup F_1\cup F_2).\ \blacksquare$$

Proof of Theorem 8. The theorem is proved from the lemma as follows. Let $\{B_\alpha : \alpha<\mathfrak{c}\}$ list all uncountable Borel subsets of $\mathbb{R}\times\mathbb{R}$ which have the property that for every y the set $B_\alpha\cap U_y$ is countable. Let also $\{p_\alpha : \alpha<\mathfrak{c}\}=\mathbb{R}\times\mathbb{R}$ and $\{y_\alpha : \alpha<\mathfrak{c}\}=\mathbb{R}$. Construct an increasing sequence $H_\alpha\subseteq\mathbb{R}\times\mathbb{R}$ for $\alpha<\mathfrak{c}$ so that

1. H_α are linearly independent over the rationals,
2. $\beta<\alpha$ implies $H_\beta\subseteq H_\alpha$,
3. $H_\lambda=\bigcup_{\alpha<\lambda}H_\alpha$ at limit ordinals λ,
4. $H_{\alpha+1}\setminus H_\alpha\subseteq U_{y_\alpha}$ is finite,

5. $p_\alpha \in \mathrm{span}(H_{\alpha+1})$,
6. $H_\alpha \cap B_\beta \subseteq H_{\beta+1}$ whenever $\beta < \alpha$,
7. $H_\alpha \cap U_{y_\beta} \subseteq H_{\beta+1}$ whenever $\beta < \alpha$.

At successor ordinals $\alpha + 1$ apply the lemma with $p = p_\alpha$, $V = \mathrm{span}(H_\alpha)$, and

$$U = \{p \in U_{y_\alpha} : \exists \beta < \alpha \ (p \in B_\beta \text{ or } p \in U_{y_\beta})\}.$$

Then let $H_{\alpha+1} = H_\alpha \cup F$.

The set $H = \bigcup_{\alpha<\mathfrak{c}} H_\alpha$ is a Hamel basis; note that for every $y_\alpha \in \mathbb{R}$ we have $H \cap U_{y_\alpha} \subseteq H_{\alpha+1}$ and so

$$|H \cap U_{y_\alpha}| < \mathfrak{c}$$

and similarly for every α we have

$$|H \cap B_\alpha| < \mathfrak{c}.$$

To see that H is Marczewski null, suppose that P is any perfect subset of the plane. If $P \cap U_y$ is uncountable and closed for some $y \in \mathbb{R}$, then since $|H \cap U_y| < \mathfrak{c}$ and every perfect set can be split into continuum many perfect subsets, there exists a perfect set $P' \subseteq P \cap U_y$ disjoint from H.

On the other hand, if there is no such y then $P = B_\alpha$ for some α and therefore $|P \cap H| < \mathfrak{c}$. Thus again by splitting P into continuum many pairwise disjoint perfect subsets, there must be a perfect subset of P disjoint from H. ■

Theorem 10. *There exists a Hamel basis*, H, *for the reals which is a Marczewski null set.*

Obviously, this implies Theorem 8, since $(H \times \{0\}) \cup (\{0\} \times H)$ is a Marczewski null Hamel basis for the plane. But the proof is a little messier so we chose to do the one for the plane first.

For $p, q \in {}^\omega 2$ define

$$\sigma(p,q) = \sum_{n=0}^{\infty} \frac{p(n)}{2^{2n+1}} + \sum_{n=0}^{\infty} \frac{q(n)}{2^{2n+2}}.$$

So we are basically looking at the even and odd digits in the binary expansion. The function $\sigma(p,q)$ maps ${}^\omega 2 \times {}^\omega 2$ onto the unit interval $[0,1]$. For any $p \in {}^\omega 2$ define

$$U_p = \{\sigma(p,q) : q \in {}^\omega 2\}$$

The following is the analogue of Lemma 9.

Lemma 11. *Suppose we have a subspace*, $V \subseteq \mathbb{R}$, *with* $|V| < \mathfrak{c}$ *and* $1 \in V$, $p \in {}^\omega 2$, $U \subseteq U_p$ *with* $|U| < \mathfrak{c}$, *and* $z \in \mathbb{R}$. *Then there exists a finite* $F \subseteq U_p \setminus U$ *such that*

$$z \in \mathrm{span}(V \cup F) \quad \textit{and} \quad \mathrm{span}(F) \cap V \textit{ is trivial.}$$

P r o o f. CASE 1: $z = \sigma(\underline{0}, q)$ (where $\underline{0} \in {}^{\omega}2$ is the constantly zero function).

We may assume that there are infinitely many n such that $q(n) = 0$, because otherwise $z \in \mathbb{Q}$ and so we may take F to be empty. Let

$$A = \{n : q(n) = 0\}.$$

For any $B \subseteq A$ define the pair $q_B, q'_B \in {}^{\omega}2$ as follows:

$$q_B(n) = \begin{cases} q(n) & \text{if } n \notin B, \\ 1 & \text{if } n \in B, \end{cases} \qquad q'_B(n) = \begin{cases} 0 & \text{if } n \notin B, \\ 1 & \text{if } n \in B. \end{cases}$$

Since $q(n) = 0$ for each $n \in B$, it follows that $q(n) = q_B(n) - q'_B(n)$ for every n. Since we never do any "borrowing" we have

$$z = \sigma(\underline{0}, q) = \sigma(p, q_B) - \sigma(p, q'_B).$$

Since $|U| < \mathfrak{c}$ there are continuum many $B \subseteq A$ such that neither $\sigma(p, q_B)$ nor $\sigma(p, q'_B)$ are in U. Fix one of these B's and let

$$F \subseteq \{\sigma(p, q_B), \sigma(p, q'_B)\} \subseteq U_p \setminus U$$

be minimal such that $z \in \operatorname{span}(V \cup F)$.

CASE 2: $z = \sigma(q, \underline{0})$. Since

$$\tfrac{1}{2}z = \tfrac{1}{2}\sigma(q, \underline{0}) = \sigma(\underline{0}, q)$$

this follows easily from Case 1.

To prove the result for general $z \in \mathbb{R} \setminus \mathbb{Q}$ first we may assume that $z = \sigma(q_1, q_2)$ for some $q_1, q_2 \in {}^{\omega}2$ since a rational multiple of z is in $[0, 1]$. Next we may apply Case 1 to $\sigma(\underline{0}, q_2)$ and then iteratively (as in the proof of Lemma 9) to $\sigma(q_1, \underline{0})$. Then since $z = \sigma(q_1, \underline{0}) + \sigma(\underline{0}, q_2)$ the lemma is proved. ∎

Proof of Theorem 10. For any distinct $p_1, p_2 \in {}^{\omega}2$ if neither is eventually one, then U_{p_1} and U_{p_2} are disjoint. The proof is now similar to that of Theorem 8, using the family of U_p for $p \in {}^{\omega}2$ which are not eventually one. ∎

REMARK 12. Similar proofs can be given to produce Marczewski null Hamel bases for $\mathbb{R}^n$, $\mathbb{Q}^{\omega}$, and $\mathbb{R}^{\omega}$. For $\mathbb{R}^n$ one can either modify the proofs of Theorem 8 and Lemma 9, or else observe (for example when $n = 3$) that if H is a Marczewski null Hamel basis for $\mathbb{R}$, then

$$(H \times \{0\} \times \{0\}) \cup (\{0\} \times H \times \{0\}) \cup (\{0\} \times \{0\} \times H)$$

is a Marczewski null Hamel basis for $\mathbb{R}^3$. If $X = \mathbb{Q}^{\omega}$ or $X = \mathbb{R}^{\omega}$ then X is isomorphic to $X \times X$ and the proofs are similar to the proof for the plane.

CONJECTURE 13. *Suppose X is an uncountable completely metrizable separable metric space which is also a vector space over a field $\mathbb{F}$ and scalar* *multiplication and vector sum are Borel maps. Then there exists a basis H for X over $\mathbb{F}$ such that H is Marczewski null.*

Note that our conjecture reduces to the case where the field $\mathbb{F}$ is either $\mathbb{Q}$ or $\mathbb{Z}_p$ for some prime p. This is because if $\mathbb{K}$ is a subfield of $\mathbb{F}$ and and H is a Marczewski null basis for X over $\mathbb{K}$, then some maximal linearly independent (over $\mathbb{F}$) subset of H is a Marczewski null basis for X over $\mathbb{F}$.

F. B. Jones [4] constructed a Hamel basis containing a perfect set and attributed the construction of what might be called Vitali-independent perfect set to R. L. Swain.

THEOREM 14. *There is a Hamel basis for $\mathbb{R}$ which is Marczewski measurable and perfectly dense.*

Proof. Let C be a linearly independent Cantor set and H_0 a Marczewski null Hamel basis. Split C into countably many Cantor sets $C_0, C_1, \ldots,$ fix a basis $\{B_n : n < \omega\}$ for the topology of the real line and for each n pick a non-zero rational q_n such that $q_n C_n$ intersects B_n. Note that

$$C' = \bigcup \{q_n C_n : n < \omega\}$$

is still linearly independent (though not a Cantor set) and for all open sets U there exists a perfect $P \subseteq C' \cap U$. Let $H_1 \subseteq H_0$ be maximal such that

$$H = C' \cup H_1$$

is linearly independent. It is easy to see that H works. ∎

Borel additive mappings. We might hope to obtain Theorem 10 as a corollary to Theorem 8 getting a Borel linear isomorphism between $\mathbb{R} \times \mathbb{R}$ and $\mathbb{R}$. Since a Borel bijection preserves the Marczewski null sets, we would be able to obtain a Marczewski null Hamel basis for the reals from one for the plane.

This will not work because of the following result. A mapping is called *additive* iff $f(x+y) = f(x) + f(y)$ for any x and y. Note that if f is additive, then $f(rx) = rf(x)$ for any rational r.

THEOREM 15. *Any additive Borel map $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ fails to be one-to-one.*

LEMMA 16. *Suppose $g : \mathbb{R} \to \mathbb{R}$ is an additive Borel map. Then there exists a comeager $G \subseteq \mathbb{R}$ and a real a such that $g(x) = ax$ for every $x \in G$.*

Proof. This is due to F. Burton Jones [4]. Since g is additive it is not hard to prove that $g(ax) = ag(x)$ for every rational $a \in \mathbb{Q}$ and real x. Also, since g is Borel there exists a comeager G such that the restriction of g to G is continuous. Since aG is comeager for any non-zero a we may without loss assume that $aG \subseteq G$ for every non-zero rational a. Let x_0 be any fixed non-zero element of G. For any $a \in \mathbb{Q}$ we have $g(ax_0) = ag(x_0)$ and $ax_0 \in G$. So by the continuity of g we get $g(yx_0) = yg(x_0)$ for any y with $yx_0 \in G$. Now for any $x \in G$,

$$g(x) = g\left(\frac{x}{x_0}x_0\right) = \frac{x}{x_0}g(x_0) = x\frac{g(x_0)}{x_0}$$

and so $a = g(x_0)/x_0$ works. ∎

Proof of Theorem 15. Assume that f is an additive map. By the lemma there exist comeager G_i and reals a_i, $i = 0, 1$, such that for every $x \in G_0$ we have $f(x, 0) = a_0x$ and for every $y \in G_1$ we have $f(0, y) = a_1y$. Since f is additive it follows that for every $x, y \in G = G_0 \cap G_1$,

$$f(x, y) = a_0x + a_1y.$$

If either a_i is zero, then of course f is not one-to-one. So assume both are non-zero. Let x and x' be arbitrary distinct elements of G and define

$$z = -\frac{a_0}{a_1}(x - x')$$

Since G is comeager, so is $G + z$ and hence we can choose y in both G and $G + z$. If we let y' be so that $y = y' + z$, then $y' = y - z \in G$ and

$$f(x, y) = a_0x + a_1y = a_0x + a_1y' - a_0(x - x') = a_0x' + a_1y' = f(x', y')$$

and f is not one-to-one. ∎

We use similar Baire category arguments to prove the following theorem:

THEOREM 17. *There is no Borel* (*or even Baire*) 1-1 *additive function f of the following form for any $n = 1, 2, \ldots$:*

(1) $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$,
(2) $f : \mathbb{R}^n \to \mathbb{Q}^\omega$, *or* $f : \mathbb{R}^n \to \mathbb{Z}^\omega$ (*even for* any 1-1 *additive* f),
(3) $f : \mathbb{Q}^\omega \to \mathbb{R}^n$, *or* $f : \mathbb{Z}^\omega \to \mathbb{R}^n$.

Proof. (1) This argument is a generalization of Theorem 15. There exists a comeager $G \subseteq \mathbb{R}$ and a linear transformation $L : \mathbb{R}^{n+1} \to \mathbb{R}^n$ with the property that

$$f(x_1, \ldots, x_{n+1}) = L(x_1, \ldots, x_{n+1}) \quad \text{for any } x_1, \ldots, x_{n+1} \in G.$$

Since L cannot be 1-1 there must be distinct vectors u and v with $L(u) = L(v)$. Since G is comeager there exists a vector w such that $u_i + w_i$, $v_i + w_i \in G$ for all coordinates $i = 1, \ldots, n + 1$ (choose $w_i \in (G - u_i) \cap (G - v_i)$). But then

$$f(u + w) = L(u + w) = L(u) + L(w) = L(v) + L(w) = L(v + w) = f(v + w)$$

implies that f is not 1-1.

(2) It is enough to prove this for the case $f : \mathbb{R}^1 \to \mathbb{Q}^\omega$, since there are such maps from $\mathbb{R}^1$ into $\mathbb{R}^n$ and from $\mathbb{Z}^\omega$ into $\mathbb{Q}^\omega$. Let $f(x)(m) \in \mathbb{Q}$ refer to

the mth coordinate of $f(x)$. If f is 1-1 and additive, then for each non-zero $x \in \mathbb{R}$ there is some m such that $f(x)(m) \neq 0$. By Baire category there must exist some $q_0 \in \mathbb{Q}$ with $q_0 \neq 0$, coordinate m, open interval I and $H \subseteq I$ comeager in I such that

$$f(x)(m) = q_0 \quad \text{for every } x \in H.$$

But this is impossible because we can find $\varepsilon \in \mathbb{Q}$ with ε close to 1 but different from 1 and some x such that $x, \varepsilon x \in H$ but

$$f(x) + f(\varepsilon x) = f(x + \varepsilon x) = f((1+\varepsilon)x) = (1+\varepsilon)f(x).$$

Since both x and εx are in H we have $f(x)(m) = f(\varepsilon x)(m) = q_0$, contradicting $2q_0 \neq (1+\varepsilon)q_0$.

(3) We show there is no such map $f : \mathbb{Z}^\omega \to \mathbb{R}^n$. Since there is a 1-1 additive Borel map (inclusion) from $\mathbb{Z}^\omega$ into $\mathbb{Q}^\omega$, this suffices. We start by giving the proof for $n = 1$. Assume for contradiction that $G \subseteq \mathbb{Z}^\omega$ is a comeager G_δ-set and $f \restriction G$ is continuous on G.

The topology on $\mathbb{Z}^\omega$ is determined by the basic open sets

$$[s] = \{x \in \mathbb{Z}^\omega : s \subseteq x\}$$

where $s \in \mathbb{Z}^{<\omega}$ is the set of finite sequences from $\mathbb{Z}$.

Claim. *For any $N \in \omega$ and any $s \in \mathbb{Z}^{<\omega}$ there exists $t \in \mathbb{Z}^{<\omega}$ with $s \subseteq t$ and for every $x \in G \cap [t]$ we have $f(x) > N$.*

Proof. Let $m = |s|$ be the length of s (so $s = \langle s(0), \ldots, s(m-1)\rangle$). For each $k \in \mathbb{Z}$ let $x_k \in \mathbb{Z}^\omega$ be the sequence which is all zeros except on the mth coordinate where it is k. Since f is additive and 1-1 we must have either $\lim_{k\to\infty} f(x_k) = \infty$ or $\lim_{k\to-\infty} f(x_k) = \infty$. Since G is comeager there exists $u \in [s]$ such that $u + x_k \in G$ for every $k \in \mathbb{Z}$ (i.e., choose $u \in \bigcap_{k\in\mathbb{Z}}(-x_k + G)$). Note that $u + x_k \in [s]$ for every k and $f(u + x_k) = f(u) + f(x_k)$, hence for some $k \in \mathbb{Z}$ we have $f(u + x_k) > N$. Since f is continuous on G we can find the t as required. This proves the Claim.

According to the Claim for each N there exists a dense open set D_N such that for every $x \in D_N \cap G$ we have $f(x) > N$. But this is a contradiction since it implies

$$G \cap \bigcap_{N\in\omega} D_N = \emptyset.$$

For the case of $f : \mathbb{Z}^\omega \to \mathbb{R}^n$ the argument is similar, we just prove a claim that says: For any $N \in \omega$ and any $s \in \mathbb{Z}^{<\omega}$ there exists $t \in \mathbb{Z}^{<\omega}$ with $s \subseteq t$ and for every $x \in G \cap [t]$ we have $f(x)(i) > N$ for some coordinate $i < n$. ∎

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