Vitali sets and Hamel bases that are Marczewski measurable

by

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Abstract. We give examples of a Vitali set and a Hamel basis which are Marczewski measurable and perfectly dense. The Vitali set example answers a question posed by Jack Brown. We also show there is a Marczewski null Hamel basis for the reals, although a Vitali set cannot be Marczewski null. The proof of the existence of a Marczewski null Hamel basis for the plane is easier than for the reals and we give it first. We show that there is no easy way to get a Marczewski null Hamel basis for the reals from one for the plane by showing that there is no one-to-one additive Borel map from the plane to the reals.

Basic definitions. A subset $A$ of a complete separable metric space $X$ is called Marczewski measurable if for every perfect set $P \subseteq X$ either $P \cap A$ or $P \setminus A$ contains a perfect set. Recall that a perfect set is a non-empty closed set without isolated points, and a Cantor set is a homeomorphic copy of the Cantor middle-third set. If every perfect set $P$ contains a perfect subset which misses $A$, then $A$ is called Marczewski null. The class of Marczewski measurable sets, denoted by $(s)$, and the class of Marczewski null sets, denoted by $(s^0)$, were defined by Marczewski [10], where it was shown that $(s)$ is a $\sigma$-algebra, i.e. $X \in (s)$ and $(s)$ is closed under complements and countable unions, and $(s^0)$ is a $\sigma$-ideal in $(s)$, i.e. $(s^0)$ is closed under countable unions and subsets. Several equivalent definitions and important properties of $(s)$ and $(s^0)$ were proved in [10], for example every analytic set is Marczewski measurable, the properties $(s)$ and $(s^0)$ are preserved under “generalized homeomorphisms” (also called Borel bijections), i.e. one-to-one onto functions $f$ such that both $f$ and $f^{-1}$ are Borel measurable (i.e. pre-images of open sets are Borel), a countable product is in

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(s) if and only if each factor is in (s), and a finite product is in \((s^0)\) if and only if each factor is in \((s^0)\).

The perfect kernel of a closed set \(F\) is the set of all \(a \in F\) such that \(U \cap F\) is uncountable for every neighborhood \(U\) of \(a\).

A set is totally imperfect if it contains no perfect subset. A totally imperfect set of reals cannot contain uncountable closed set, so it must have inner Lebesgue measure zero. A set \(B\) is called Bernstein set if every perfect set intersects both \(B\) and the complement of \(B\), or, equivalently, both \(B\) and its complement are totally imperfect. Clearly, no Bernstein set can be Marczewski measurable.

A set \(A\) is perfectly dense if its intersection with every non-empty open set contains a perfect set.

Let \(\mathbb{R}\) denote the set of all real numbers and \(\mathbb{Q}\) denote the set of all rational numbers. We use \(c\) to denote the cardinality of the continuum.

The linear closure (or span) over \(\mathbb{Q}\) of a non-empty set \(A \subseteq \mathbb{R}\) is the set \([\text{span}(A)] = \{q_1 a_1 + \ldots + q_n a_n : n < \omega, q_j \in \mathbb{Q}, a_j \in A\}\) and \([\text{span}(\emptyset)] = \{0\}\). A is called linearly independent over \(\mathbb{Q}\) if \(q_1 a_1 + \ldots + q_n a_n \neq 0\) whenever \(n < \omega\), \(q_j \in \mathbb{Q}\) for \(1 \leq j \leq n\) with \(q_j \neq 0\) for at least one \(j\), and \(a_1, \ldots, a_n\) are different points from \(A\). A linearly independent set \(H\) such that \(\mathbb{R} = \text{span}(H)\) is called a Hamel basis. Note a Hamel basis must have cardinality \(c\). The inner Lebesgue measure of any Hamel basis \(H\) is zero (Sierpiński [8], see also Erdős [2]). A Hamel basis can have Lebesgue measure 0 (see Jones [4], or Kurczma [6], Chapter 11).

A Hamel basis \(H\) which intersects every perfect set is called a Burstin set [1]. Every Burstin set \(H\) is also a Bernstein set, otherwise if \(P \subseteq H\) for some perfect set \(P\), by the linear independence of \(H\) it follows that \(H \cap 2P = \emptyset\) (where \(2P = \{2p : p \in P\}\)), a contradiction since \(2P\) is a perfect set.

A Burstin set can be constructed as follows. List all perfect subsets of \(\mathbb{R}\) as \(\{P_\alpha : \alpha < c\}\), pick a non-zero \(p_0 \in P_0\) and, using the facts that

\[|\text{span}(A)| \leq |A| + \omega < c\quad \text{if } |A| < c\]

and \(|P_\alpha| = c\) for each \(\alpha\), pick by induction \(p_\alpha \in P_\alpha \setminus \text{span}\{p_\beta : \beta < \alpha\}\)

and let \(H_c = \{p_\alpha : \alpha < c\}\). If \(H\) is a maximal linearly independent set with \(H_c \subseteq H\), then \(H\) is a Burstin set.

A set \(V \subseteq \mathbb{R}\) is called a Vitali set if \(V\) is a complete set of representatives (or a transversal) for the equivalence relation defined by \(x \sim y\) iff \(x - y \in \mathbb{Q}\), i.e. for each \(x \in \mathbb{R}\) there exists a unique \(v \in V\) such that \(x - v \in \mathbb{Q}\). No Vitali set is Lebesgue measurable or has the Baire property. One may construct a Vitali set which is a Bernstein set.
Perfectly dense Marczewski measurable Vitali set. Recall that an equivalence relation on a space $X$ is called Borel if it is a Borel subset of $X \times X$. The Vitali equivalence $\sim$ as defined above is Borel. We first show that a Vitali set cannot be Marczewski null.

**Theorem 1.** Suppose $X$ is an uncountable separable completely metrizable space with a Borel equivalence relation, $\equiv$, on it with every equivalence class countable. Then, if $V \subseteq X$ meets each equivalence class in exactly one element, $V$ cannot be Marczewski null.

**Proof.** By a theorem of Feldman and Moore [3] every such Borel equivalence relation is induced by a Borel action of a countable group. This implies that there are countably many Borel bijections $f_n : X \to X$ for $n \in \omega$ such that $x \equiv y$ iff $f_n(x) = y$ for some $n$. If $V$ were Marczewski null, then so would $X = \bigcup_{n<\omega} f_n(V)$. ■

To obtain a Marczewski measurable Vitali set we will use the following theorem:

**Theorem 2 (Silver [9]).** If $E$ is a coanalytic equivalence relation on the space of all real numbers and $E$ has uncountably many equivalence classes, then there is a perfect set of mutually $E$-independent reals (in other words, an $E$-independent perfect set). In the case of a Borel equivalence relation $E$, one can drop the requirement that the field of the equivalence be the whole set of reals.

If $E \subseteq X \times X$ is a Borel equivalence relation, where $X$ is an uncountable separable completely metrizable space, and $B$ is a Borel subset of $X$, then the saturation of $B$, $[B]_E = \bigcup_{x \in B} [x]_E$, is analytic since it is the projection onto the second coordinate of the Borel set $(B \times X) \cap E$. The saturation need not be Borel, for example let $B$ be a Borel subset of $X = \mathbb{R}^2$ whose projection $\pi_1(B)$ into the first coordinate is not Borel. Define $(x, y)E(u, v)$ iff $x = u$ (i.e. two points are equivalent if they lie on the same vertical line). Then $[B]_E = \pi_1(B) \times \mathbb{R}$ is not Borel. On the other hand, if $E$ is a Borel equivalence with each equivalence class countable, and $f_n$ are as in the proof of Theorem 1, then the saturation $[B]_E = \bigcup_{n<\omega} f_n(B)$ of every Borel set $B$ is Borel.

**Theorem 3.** Suppose $X$ is an uncountable separable completely metrizable space with a Borel equivalence relation $E$. Then there exists Marczewski measurable $V \subseteq X$ which meets each equivalence class in exactly one element.

**Proof.** Let $\{P_\alpha : \alpha < c\}$ list all perfect subsets of $X$. We will describe how to construct disjoint $C_\alpha$, each $C_\alpha$ either countable (possibly finite or
empty) or a Cantor set such that the set \( V_\alpha = \bigcup_{\beta < \alpha} C_\beta \) is \( E \)-independent. Then extend the set \( V_\alpha = \bigcup_{\beta < \alpha} C_\beta \) to a maximal \( E \)-independent set \( V_\alpha \).

**Case (a).** If \( P_\beta \cap [C_\beta]_E \) is uncountable for some \( \beta < \alpha \), then let \( C_\alpha = \emptyset \).

**Subcase (a1):** \( |P_\alpha \cap C_\beta| > \omega \). Then the perfect kernel of \( P_\alpha \cap C_\beta \) is contained in both \( P_\alpha \) and \( V_\alpha \) (and hence in \( V \)).

**Subcase (a2):** \( |P_\alpha \cap C_\beta| = \omega \). Then, since \( P_\alpha \cap [C_\beta]_E \) is uncountable analytic, it contains a perfect set \( Q \) which misses \( V \).

**Case (b):** Not Case (a). Then

\[
|P_\alpha \cap [V_\alpha]_E| = |P_\alpha \cap \bigcup_{\beta < \alpha} [C_\beta]_E| \leq |\alpha| < \omega,
\]

and hence \( P_\alpha \setminus [V_\alpha]_E \) contains a Cantor set \( P \).

**Subcase (b1):** The restriction of \( E \) to \( P \) has only countably many classes. Let \( C_\alpha \) be a countable \( E \)-independent subset of \( P \) with \( P \subseteq [C_\alpha]_E \). Then \( P \setminus C_\alpha \) contains a perfect set which misses \( V \).

**Subcase (b2):** Case (b) but not case (b1). Then, by the above theorem of Silver, there is a perfect \( E \)-independent set \( C_\alpha \subseteq P \) (and \( C_\alpha \subseteq V \)).

**Remark 4.** The Vitali equivalence shows that a Borel equivalence need not have a transversal that is Lebesgue measurable or has the Baire property. See Kechris [5], 18.D, for more on transversals of Borel equivalences.

**Theorem 5.** There exists a Vitali set which is Marczewski measurable and its intersection with each non-empty open set contains a perfect set.

**Proof.** By Theorem 3 there is a Marczewski measurable Vitali set \( V \), and by Theorem 1, \( V \) contains a perfect set \( C \). Split \( C \) into countably many Cantor sets \( C_0, C_1, \ldots \), fix a basis \( \{B_n : n < \omega\} \) for the topology of \( \mathbb{R} \) and pick rational numbers \( q_n \) so that the set \( q_n + C_n = \{q_n + c : c \in C_n\} \) intersects \( B_n \) for each \( n \). Then

\[
V' = (V \setminus C) \cup \bigcup \{(q_n + C_n) : n < \omega\}
\]

is a perfectly dense Marczewski measurable Vitali set.

**Remark 6.** A Vitali set \( V \) cannot have the stronger property that its intersection with every perfect set contains a perfect set. This is because if \( V \) contains a perfect set \( P \), then the perfect set

\[
P' = P + 1 = \{p + 1 : p \in P\}
\]

does not intersect \( V \). Similarly, if \( H \) is a Hamel basis that contains a perfect set \( P \), then

\[
2P = \{2p : p \in P\}
\]

is a perfect set which misses \( H \).
Vitali sets and Hamel bases

Remark 7 (Erdős [2]). Under CH there is a Hamel basis $H$ which is a Lusin set (and hence Marczewski null). To see this, note that by a result of Sierpiński there is a Lusin set $X$ such that $X + X = \{x + y : x, y \in X\} = \mathbb{R}$ (see e.g. [7]). Let $H$ be any maximal linearly independent subset of $X$; then clearly $\text{span}(H) = \text{span}(X) = \mathbb{R}$.

Our construction (without CH) of a Marczewski null Hamel basis is slightly simpler for the plane, so we do it first.

Theorem 8. There exists a Hamel basis $H$, for $\mathbb{R} \times \mathbb{R}$, i.e. a basis for the plane as a vector space over $\mathbb{Q}$, which is a Marczewski null set, i.e., every perfect set contains a perfect subset disjoint from $H$.

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Lemma 9. Suppose $V$ with $|V| < \mathfrak{c}$ is a subspace of $\mathbb{R} \times \mathbb{R}$ as a vector space over $\mathbb{Q}$ (not necessarily closed), $p \in \mathbb{R} \times \mathbb{R}$, $y \in \mathbb{R}$, and

$$U \subseteq U_y = \{(y) \times \mathbb{R} \cup (\mathbb{R} \times \{y\})$$

with $|U| < \mathfrak{c}$. Then there exists a finite $F \subseteq \{(y_1, y), (y_2, y)\}$ with $p \in \text{span}(F \cup V)$ and such that $F$ is linearly independent over $\mathbb{Q}$ and independent over $V$, i.e., $\text{span}(F)$ meets $V$ only in the zero vector.

Proof. Case 1: $p = (u, 0)$. Let $y_1$ and $y_2$ be so that

$$y_2 - y_1 = u, \quad (y_1, y) \notin U \quad \text{and} \quad (y_2, y) \notin U.$$

Clearly, $p \in \text{span}(\{(y_1, y), (y_2, y)\})$. Let

$$F \subseteq \{(y_1, y), (y_2, y)\} \subseteq U_y \setminus U$$

be minimal such that $p \in \text{span}(F \cup V)$. Then $F$ works.

Case 2: $p = (0, v)$. Obviously, this case is symmetric.

Case 3: $p = (u, v)$. Apply Case 1 to $(u, 0)$ obtaining $F_1$. Let

$$V' = \text{span}(V \cup F_1)$$

and apply Case 2 to $V'$ obtaining $F_2$ (and let $F = F_1 \cup F_2$) so that

$$(u, 0), (0, v) \in \text{span}(V \cup F_1 \cup F_2).$$

Proof of Theorem 8. The theorem is proved from the lemma as follows.

Let $\{B_\alpha : \alpha < \mathfrak{c}\}$ list all uncountable Borel subsets of $\mathbb{R} \times \mathbb{R}$ which have the property that for every $y$ the set $B_\alpha \cap U_y$ is countable. Let also $\{p_\alpha : \alpha < \mathfrak{c}\} = \mathbb{R} \times \mathbb{R}$ and $\{y_\alpha : \alpha < \mathfrak{c}\} = \mathbb{R}$. Construct an increasing sequence $H_\alpha \subseteq \mathbb{R} \times \mathbb{R}$ for $\alpha < \mathfrak{c}$ so that

1. $H_\alpha$ are linearly independent over the rationals,
2. $\beta < \alpha$ implies $H_\beta \subseteq H_\alpha$,
3. $H_\lambda = \bigcup_{\alpha < \lambda} H_\alpha$ at limit ordinals $\lambda$,
4. $H_{\alpha+1} \setminus H_\alpha \subseteq U_{y_\alpha}$ is finite,
5. $p_\alpha \in \text{span}(H_{\alpha+1})$,
6. $H_\alpha \cap B_\beta \subseteq H_{\beta+1}$ whenever $\beta < \alpha$,
7. $H_\alpha \cap U_{y_\beta} \subseteq H_{\beta+1}$ whenever $\beta < \alpha$.

At successor ordinals $\alpha + 1$ apply the lemma with $p = p_\alpha$, $V = \text{span}(H_\alpha)$, and

$$U = \{ p \in U_{y_\alpha} : \exists \beta < \alpha \ (p \in B_\beta \text{ or } p \in U_{y_\beta}) \}.$$

Then let $H_{\alpha+1} = H_\alpha \cup F$.

The set $H = \bigcup_{\alpha < \xi} H_\alpha$ is a Hamel basis; note that for every $y_\alpha \in \mathbb{R}$ we have $H \cap U_{y_\alpha} \subseteq H_{\alpha+1}$ and so

$$|H \cap U_{y_\alpha}| < \xi$$

and similarly for every $\alpha$ we have

$$|H \cap B_\alpha| < \xi.$$

To see that $H$ is Marczewski null, suppose that $P$ is any perfect subset of the plane. If $P \cap U_y$ is uncountable and closed for some $y \in \mathbb{R}$, then since $|H \cap U_y| < \xi$ and every perfect set can be split into continuum many perfect subsets, there exists a perfect set $P' \subseteq P \cap U_y$ disjoint from $H$.

On the other hand, if there is no such $y$ then $P = B_\alpha$ for some $\alpha$ and therefore $|P \cap H| < \xi$. Thus again by splitting $P$ into continuum many pairwise disjoint perfect subsets, there must be a perfect subset of $P$ disjoint from $H$. □

**Theorem 10.** There exists a Hamel basis, $H$, for the reals which is a Marczewski null set.

Obviously, this implies Theorem 8, since $(H \times \{0\}) \cup (\{0\} \times H)$ is a Marczewski null Hamel basis for the plane. But the proof is a little messier so we chose to do the one for the plane first.

For $p, q \in \omega^2$ define

$$\sigma(p, q) = \sum_{n=0}^{\infty} p(n) \frac{2n+1}{2^{2n+1}} + \sum_{n=0}^{\infty} q(n) \frac{2n+2}{2^{2n+2}}.$$ 

So we are basically looking at the even and odd digits in the binary expansion. The function $\sigma(p, q)$ maps $\omega^2 \times \omega^2$ onto the unit interval $[0, 1]$. For any $p \in \omega^2$ define

$$U_p = \{ \sigma(p, q) : q \in \omega^2 \}$$

The following is the analogue of Lemma 9.

**Lemma 11.** Suppose we have a subspace, $V \subseteq \mathbb{R}$, with $|V| < \xi$ and $1 \in V$, $p \in \omega^2$, $U \subseteq U_p$ with $|U| < \xi$, and $z \in \mathbb{R}$. Then there exists a finite $F \subseteq U_p \setminus U$ such that

$$z \in \text{span}(V \cup F) \quad \text{and} \quad \text{span}(F) \cap V \text{ is trivial.}$$
Proof. Case 1: \( z = \sigma(\mathbb{0}, q) \) (where \( \mathbb{0} \in \omega^2 \) is the constantly zero function).

We may assume that there are infinitely many \( n \) such that \( q(n) = 0 \), because otherwise \( z \in \mathbb{Q} \) and so we may take \( F \) to be empty. Let \( A = \{ n : q(n) = 0 \} \).

For any \( B \subseteq A \) define the pair \( q_B, q'_B \in \omega^2 \) as follows:

\[
q_B(n) = \begin{cases} 
q(n) & \text{if } n \notin B, \\
1 & \text{if } n \in B,
\end{cases} \quad q'_B(n) = \begin{cases} 
0 & \text{if } n \notin B, \\
1 & \text{if } n \in B.
\end{cases}
\]

Since \( q(n) = 0 \) for each \( n \in B \), it follows that \( q(n) = q_B(n) = q'_B(n) \) for every \( n \). Since we never do any “borrowing” we have

\[
z = \sigma(\mathbb{0}, q) = \sigma(p, q_B) - \sigma(p, q'_B).
\]

Since \( |U| < \aleph \) there are continuum many \( B \subseteq A \) such that neither \( \sigma(p, q_B) \) nor \( \sigma(p, q'_B) \) are in \( U \). Fix one of these \( B \)'s and let

\[
F = \{ \sigma(p, q_B), \sigma(p, q'_B) \} \subseteq U_p \setminus U
\]

be minimal such that \( z \in \text{span}(V \cup F) \).

Case 2: \( z = \sigma(q, \mathbb{0}) \). Since

\[
\frac{1}{2}z = \frac{1}{2}\sigma(q, \mathbb{0}) = \sigma(\mathbb{0}, q)
\]

this follows easily from Case 1.

To prove the result for general \( z \in \mathbb{R} \setminus \mathbb{Q} \) first we may assume that \( z = \sigma(q_1, q_2) \) for some \( q_1, q_2 \in \omega^2 \) since a rational multiple of \( z \) is in \([0, 1]\). Next we may apply Case 1 to \( \sigma(\mathbb{0}, q_2) \) and then iteratively (as in the proof of Lemma 9) to \( \sigma(q_1, \mathbb{0}) \). Then since \( z = \sigma(q_1, \mathbb{0}) + \sigma(\mathbb{0}, q_2) \) the lemma is proved. \( \blacksquare \)

Proof of Theorem 10. For any distinct \( p_1, p_2 \in \omega^2 \) if neither is eventually one, then \( U_{p_1} \) and \( U_{p_2} \) are disjoint. The proof is now similar to that of Theorem 8, using the family of \( U_p \) for \( p \in \omega^2 \) which are not eventually one.

Remark 12. Similar proofs can be given to produce Marczewski null Hamel bases for \( \mathbb{R}^n \), \( \mathbb{Q}^\omega \), and \( \mathbb{R}^\omega \). For \( \mathbb{R}^n \) one can either modify the proofs of Theorem 8 and Lemma 9, or else observe (for example when \( n = 3 \)) that if \( H \) is a Marczewski null Hamel basis for \( \mathbb{R} \), then

\[
(H \times \{0\} \times \{0\}) \cup (\{0\} \times H \times \{0\}) \cup (\{0\} \times \{0\} \times H)
\]

is a Marczewski null Hamel basis for \( \mathbb{R}^3 \). If \( X = \mathbb{Q}^\omega \) or \( X = \mathbb{R}^\omega \) then \( X \) is isomorphic to \( X \times X \) and the proofs are similar to the proof for the plane.

Conjecture 13. Suppose \( X \) is an uncountable completely metrizable separable metric space which is also a vector space over a field \( \mathbb{F} \) and scalar
multiplication and vector sum are Borel maps. Then there exists a basis $H$ for $X$ over $F$ such that $H$ is Marczewski null.

Note that our conjecture reduces to the case where the field $F$ is either $\mathbb{Q}$ or $\mathbb{Z}_p$ for some prime $p$. This is because if $K$ is a subfield of $F$ and and $H$ is a Marczewski null basis for $X$ over $K$, then some maximal linearly independent (over $F$) subset of $H$ is a Marczewski null basis for $X$ over $F$.

F. B. Jones [4] constructed a Hamel basis containing a perfect set and attributed the construction of what might be called Vitali-independent perfect set to R. L. Swain.

Theorem 14. There is a Hamel basis for $\mathbb{R}$ which is Marczewski measurable and perfectly dense.

Proof. Let $C$ be a linearly independent Cantor set and $H_0$ a Marczewski null Hamel basis. Split $C$ into countably many Cantor sets $C_0, C_1, \ldots$, fix a basis $\{B_n : n < \omega\}$ for the topology of the real line and for each $n$ pick a non-zero rational $q_n$ such that $q_n C_n$ intersects $B_n$. Note that

$$C' = \bigcup \{q_n C_n : n < \omega\}$$

is still linearly independent (though not a Cantor set) and for all open sets $U$ there exists a perfect $P \subseteq C' \cap U$. Let $H_1 \subseteq H_0$ be maximal such that $H = C' \cup H_1$ is linearly independent. It is easy to see that $H$ works. \qed

Borel additive mappings. We might hope to obtain Theorem 10 as a corollary to Theorem 8 getting a Borel linear isomorphism between $\mathbb{R} \times \mathbb{R}$ and $\mathbb{R}$. Since a Borel bijection preserves the Marczewski null sets, we would be able to obtain a Marczewski null Hamel basis for the reals from one for the plane.

This will not work because of the following result. A mapping is called additive iff $f(x+y) = f(x) + f(y)$ for any $x$ and $y$. Note that if $f$ is additive, then $f(rx) = rf(x)$ for any rational $r$.

Theorem 15. Any additive Borel map $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ fails to be one-to-one.

Lemma 16. Suppose $g : \mathbb{R} \to \mathbb{R}$ is an additive Borel map. Then there exists a comeager $G \subseteq \mathbb{R}$ and a real $a$ such that $g(x) = ax$ for every $x \in G$.

Proof. This is due to F. Burton Jones [4]. Since $g$ is additive it is not hard to prove that $g(ax) = ag(x)$ for every rational $a \in \mathbb{Q}$ and real $x$. Also, since $g$ is Borel there exists a comeager $G$ such that the restriction of $g$ to $G$ is continuous. Since $aG$ is comeager for any non-zero $a$ we may without loss assume that $aG \subseteq G$ for every non-zero rational $a$. Let $x_0$ be
any fixed non-zero element of $G$. For any $a \in \mathbb{Q}$ we have $g(ax_0) = ag(x_0)$ and $ax_0 \in G$. So by the continuity of $g$ we get $g(yx_0) = yg(x_0)$ for any $y$ with $yx_0 \in G$. Now for any $x \in G$, 

$$g(x) = g\left(\frac{x}{x_0}x_0\right) = \frac{x}{x_0}g(x_0) = x \frac{g(x_0)}{x_0}$$

and so $a = g(x_0)/x_0$ works. 

**Proof of Theorem 15.** Assume that $f$ is an additive map. By the lemma there exist comeager $G_i$ and reals $a_i$, $i = 0, 1$, such that for every $x \in G_0$ we have $f(x, 0) = a_0x$ and for every $y \in G_1$ we have $f(0, y) = a_1y$. Since $f$ is additive it follows that for every $x, y \in G = G_0 \cap G_1$, 

$$f(x, y) = a_0x + a_1y.$$ 

If either $a_i$ is zero, then of course $f$ is not one-to-one. So assume both are non-zero. Let $x$ and $x'$ be arbitrary distinct elements of $G$ and define 

$$z = \frac{a_0}{a_1}(x - x')$$

Since $G$ is comeager, so is $G + z$ and hence we can choose $y$ in both $G$ and $G + z$. If we let $y'$ be so that $y = y' + z$, then $y' = y - z \in G$ and 

$$f(x, y) = a_0x + a_1y = a_0x + a_1y' - a_0(x - x') = a_0x' + a_1y' = f(x', y')$$

and $f$ is not one-to-one. 

We use similar Baire category arguments to prove the following theorem:

**Theorem 17.** There is no Borel (or even Baire) 1-1 additive function $f$ of the following form for any $n = 1, 2, \ldots$:

1. $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$,
2. $f : \mathbb{R}^n \to \mathbb{Q}^n$, or $f : \mathbb{R}^n \to \mathbb{Z}^n$ (even for any 1-1 additive $f$),
3. $f : \mathbb{Q}^n \to \mathbb{R}^n$, or $f : \mathbb{Z}^n \to \mathbb{R}^n$.

**Proof.** (1) This argument is a generalization of Theorem 15. There exists a comeager $G \subseteq \mathbb{R}$ and a linear transformation $L : \mathbb{R}^{n+1} \to \mathbb{R}^n$ with the property that 

$$f(x_1, \ldots, x_{n+1}) = L(x_1, \ldots, x_{n+1}) \quad \text{for any } x_1, \ldots, x_{n+1} \in G.$$ 

Since $L$ cannot be 1-1 there must be distinct vectors $u$ and $v$ with $L(u) = L(v)$. Since $G$ is comeager there exists a vector $w$ such that $u_i + v_i, v_i + w_i \in G$ for all coordinates $i = 1, \ldots, n + 1$ (choose $w_i \in (G - u_i) \cap (G - v_i)$). But then 

$$f(u + w) = L(u + w) = L(u) + L(w) = L(v) + L(w) = L(v + w) = f(v + w)$$

implies that $f$ is not 1-1.

(2) It is enough to prove this for the case $f : \mathbb{R}^1 \to \mathbb{Q}^n$, since there are such maps from $\mathbb{R}^1$ into $\mathbb{R}^n$ and from $\mathbb{Z}^n$ into $\mathbb{Q}^n$. Let $f(x)(m) \in \mathbb{Q}$ refer to
the $m$th coordinate of $f(x)$. If $f$ is 1-1 and additive, then for each non-zero $x \in \mathbb{R}$ there is some $m$ such that $f(x)(m) \neq 0$. By Baire category there must exist some $q_0 \in \mathbb{Q}$ with $q_0 \neq 0$, coordinate $m$, open interval $I$ and $H \subseteq I$ comeager in $f$ such that

$$f(x)(m) = q_0 \quad \text{for every } x \in H.$$ 

But this is impossible because we can find $\varepsilon \in \mathbb{Q}$ with $\varepsilon$ close to 1 but different from 1 and some $x$ such that $x, \varepsilon x \in H$ but

$$f(x) + f(\varepsilon x) = f(x + \varepsilon x) = f((1 + \varepsilon)x) = (1 + \varepsilon)f(x).$$ 

Since both $x$ and $\varepsilon x$ are in $H$ we have $f(x)(m) = f(\varepsilon x)(m) = q_0$, contradicting $2q_0 \neq (1 + \varepsilon)q_0$.

(3) We show there is no such map $f : \mathbb{Z}^\omega \to \mathbb{R}^n$. Since there is a 1-1 additive Borel map (inclusion) from $\mathbb{Z}^\omega$ into $\mathbb{Q}^\omega$, this suffices. We start by giving the proof for $n = 1$. Assume for contradiction that $G \subseteq \mathbb{Z}^\omega$ is a comeager $G_\delta$-set and $f|G$ is continuous on $G$.

The topology on $\mathbb{Z}^\omega$ is determined by the basic open sets

$$[s] = \{ x \in \mathbb{Z}^\omega : s \subseteq x \}$$

where $s \in \mathbb{Z}^{<\omega}$ is the set of finite sequences from $\mathbb{Z}$.

**Claim.** For any $N \in \omega$ and any $s \in \mathbb{Z}^{<\omega}$ there exists $t \in \mathbb{Z}^{<\omega}$ with $s \subseteq t$ and for every $x \in G \cap [t]$ we have $f(x) > N$.

**Proof.** Let $m = |s|$ be the length of $s$ (so $s = (s(0), \ldots, s(m-1))$). For each $k \in \mathbb{Z}$ let $x_k \in \mathbb{Z}^\omega$ be the sequence which is all zeros except on the $m$th coordinate where it is $k$. Since $f$ is additive and 1-1 we must have either $\lim_{k \to \infty} f(x_k) = \infty$ or $\lim_{k \to -\infty} f(x_k) = \infty$. Since $G$ is comeager there exists $u \in [s]$ such that $u + x_k \in G$ for every $k \in \mathbb{Z}$ (i.e., choose $u \in \bigcap_{k \in \mathbb{Z}} (\mathbb{Z}^\omega - (x_k + G))$). Note that $u + x_k \in [s]$ for every $k$ and $f(u + x_k) = f(u) + f(x_k)$, hence for some $k \in \mathbb{Z}$ we have $f(u + x_k) > N$. Since $f$ is continuous on $G$ we can find the $t$ as required. This proves the Claim.

According to the Claim for each $N$ there exists a dense open set $D_N$ such that for every $x \in D_N \cap G$ we have $f(x) > N$. But this is a contradiction since it implies

$$G \cap \bigcap_{N \in \omega} D_N = \emptyset.$$ 

For the case of $f : \mathbb{Z}^\omega \to \mathbb{R}^n$ the argument is similar, we just prove a claim that says: For any $N \in \omega$ and any $s \in \mathbb{Z}^{<\omega}$ there exists $t \in \mathbb{Z}^{<\omega}$ with $s \subseteq t$ and for every $x \in G \cap [t]$ we have $f(x)(i) > N$ for some coordinate $i < n$. ■
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