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The topology of the Banach–Mazur compactum

by

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Abstract. Let $J(n)$ be the hyperspace of all centrally symmetric compact convex bodies $A \subseteq \mathbb{R}^n$, $n \geq 2$, for which the ordinary Euclidean unit ball is the ellipsoid of maximal volume contained in A (the John ellipsoid). Let $J_0(n)$ be the complement of the unique $O(n)$-fixed point in $J(n)$. We prove that: (1) the Banach–Mazur compactum $BM(n)$ is homeomorphic to the orbit space $J(n)/O(n)$ of the natural action of the orthogonal group $O(n)$ on $J(n)$; (2) $J(n)$ is an $O(n)$-AR; (3) $J_0(2)/SO(2)$ is an Eilenberg–MacLane space $\mathbf{K}(\mathbb{Q}, 2)$; (4) $BM_0(2) = J_0(2)/O(2)$ is noncontractible; (5) $BM(2)$ is a nonhomogeneous absolute retract. Other models for $BM(n)$ are established.

0. Introduction. In [30, Chapter 30, Problem 899, ANR 11] the following problems of A. Pełczyński were posed:

(a) Are the Banach–Mazur compacta $BM(n)$ AR's?
(b) Are they Hilbert cubes?

Recall that the *Banach–Mazur compactum* $BM(n)$ is the set of isometry classes of n-dimensional Banach spaces topologized by the metric

$$d(E, F) = \ln \inf\{\|T\| \cdot \|T^{-1}\| : \ T : E \to F \text{ is a linear isomorphism}\}.$$

In what follows we will use the three representations of $BM(n)$ stated below.

We always denote by $\|\cdot\|$ the ordinary Euclidean norm on the n-dimensional linear coordinate space $\mathbb{R}^n$, $n \geq 2$, i.e., $\|x\|^2 = \sum_{i=1}^n x_i^2$ for any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. By B we will denote the unit ball of $\mathbb{R}^n$, i.e., $B = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$.

As usual $GL(n)$ denotes the full linear group, i.e., $GL(n)$ is the Lie group of all linear invertible operators $T : \mathbb{R}^n \to \mathbb{R}^n$. Consider the space $C(\mathbb{R}^n)$ of all continuous functions $f : \mathbb{R}^n \to \mathbb{R}$ endowed with the compact-open topol-

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ogy. It is well known that $C(\mathbb{R}^n)$ is a locally convex, complete, separable, metrizable topological vector space. One easily verifies that $C(\mathbb{R}^n)$ becomes a $GL(n)$-space if we define a $GL(n)$-action $GL(n) \times C(\mathbb{R}^n) \to C(\mathbb{R}^n)$ by $(gf)(x) = f(g^{-1}x)$, where $g \in GL(n)$, $f \in C(\mathbb{R}^n)$, $x \in \mathbb{R}^n$. This action is continuous and linear (i.e., $g(af + bh) = a(gf) + b(gh)$ for all $g \in GL(n)$, $f, h \in C(\mathbb{R}^n)$, $a, b \in \mathbb{R}$). Denote by $\mathcal{N}(n)$ the subset of $C(\mathbb{R}^n)$ consisting of all norms $\varphi : \mathbb{R}^n \to \mathbb{R}$. Clearly $\mathcal{N}(n)$ is a convex, invariant subspace of $C(\mathbb{R}^n)$. Its closure $\overline{\mathcal{N}(n)}$ coincides with the convex invariant subspace of $C(\mathbb{R}^n)$ consisting of all pseudonorms $\psi : \mathbb{R}^n \to \mathbb{R}$.

It is not hard to see that the Banach–Mazur compactum $BM(n)$ is homeomorphic to the $GL(n)$-orbit space $\mathcal{N}(n)/GL(n)$.

The second, geometric representation of $BM(n)$ (see [30, p. 544] or [19, p. 1191]) is based on the one-to-one correspondence between norms and convex bodies. Namely, denote by $\mathcal{B}(n)$ the hyperspace of all compact convex bodies of $\mathbb{R}^n$ with odd symmetry about the origin. We consider the Hausdorff metric topology on $\mathcal{B}(n)$ and the natural $GL(n)$-action on it defined as follows: $gA = \{ga : a \in A\}$ for all $g \in GL(n)$, $A \in \mathcal{B}(n)$. The two $GL(n)$-spaces $\mathcal{N}(n)$ and $\mathcal{B}(n)$ are $GL(n)$-homeomorphic via the homeomorphism $h : \mathcal{N}(n) \to \mathcal{B}(n)$ defined by the classical rule $h(\varphi) = \{x \in \mathbb{R}^n : \varphi(x) \le 1\}$ for all $\varphi \in \mathcal{N}(n)$. By Kolmogorov's normability criterion, h is one-to-one and the inverse map h^{-1} is defined by the Minkowski functional (see for example [11, p. 85, 1.10.7]). The proof of continuity of h and h^{-1} as well as of the equivariance of h are a simple verification. Consequently, $BM(n) = \mathcal{B}(n)/GL(n)$.

The third representation of $BM(n)$ (Corollary 1), which is the crucial tool in our approach to Pełczyński's Problem (b), is new. It is based on the (mutually dual) classical notions of maximal volume (respectively, minimal volume) ellipsoid contained in (respectively, containing) a given body $A \in \mathcal{B}(n)$. According to a theorem of F. John [18], for any $A \in \mathcal{B}(n)$ there is a unique maximal volume ellipsoid $j(A)$, called the *John ellipsoid* of A (respectively, minimal volume ellipsoid $l(A)$, usually called the *Löwner ellipsoid* of A). This result is responsible for compactness, contractibility and local contractibility of $BM(n)$ [30, p. 544].

In Section 2 we first represent $BM(n)$ as orbit space of an action of the orthogonal group $O(n)$. This step in our proof was prompted by the following result of H. Abels reducing the investigation of proper G-actions (in the sense of R. Palais [21]) to that of compact subgroups of G.

THEOREM 1 [1, Main Theorem]. *Let G be a locally compact group having a compact space of connected components, K be its maximal compact subgroup and let X be a proper G-space. If the orbit space X/G is paracompact then X admits a global K-slice S.*

The definition of a slice is recalled in the Preliminaries. Of course here the global K-slice S is not unique. As is proved in [1, Lemma 2.3] for any two global K-slices S_1 and S_2 of X there is a G-homeomorphism $h : X \to X$ such that $h(S_1) = S_2$. Furthermore for any global K-slice S, the inclusion $S \hookrightarrow X$ induces a homeomorphism $S/K = X/G$ [1, p. 9].

In our case $G = GL(n)$, $K = O(n)$ the orthogonal group, and $X = \mathcal{N}(n) = \mathcal{B}(n)$, $n \geq 2$. That $\mathcal{N}(n)$ (equivalently, $\mathcal{B}(n)$) is a proper $GL(n)$-space is stated in [6]. Thus, we get $BM(n) = S/O(n)$, where S is any global $O(n)$-slice of $\mathcal{B}(n)$. However in $\mathcal{B}(n)$ there are two *special* global $O(n)$-slices. Namely, using the above mentioned result of F. John, we prove (Theorem 4) that the subspace $J(n)$ of $\mathcal{B}(n)$ consisting of all bodies $A \in \mathcal{B}(n)$ for which the ordinary Euclidean unit ball is the maximal volume ellipsoid *contained* in A, is a *global* $O(n)$-slice for $\mathcal{B}(n)$. So $BM(n) = J(n)/O(n)$. Analogously, the subspace $L(n)$ of $\mathcal{B}(n)$ consisting of all bodies $A \in \mathcal{B}(n)$ for which the ordinary Euclidean unit ball is the minimal volume ellipsoid *containing* A, is a *global* $O(n)$-slice for $\mathcal{B}(n)$. So $BM(n) = L(n)/O(n)$ (Remark 1). In combination with another result of H. Abels (see Theorem 3(1) below) this implies that the hyperspace $\mathcal{B}(n)$ (or equivalently, the space of all norms $\mathcal{N}(n)$) is homeomorphic to $\mathbb{R}^k \times J(n)$ with $k = n(n+1)/2$. Moreover, this homeomorphism can be made $O(n)$-equivariant (Corollary 8).

In Section 2 we also prove that $J(n) \in O(n)$-AR and $\Phi(n) \in$ AR, where $\Phi(n)$ is the fixed point set of the induced $\mathbb{Z}_2 = O(n)/SO(n)$-space $J(n)/SO(n)$ and $SO(n)$ denotes as usual the special orthogonal group.

The same John ellipsoid trick allowed P. Fabel [12] to observe that $BM(n)$ is a retract of the $O(n)$-orbit space $\mathcal{B}(n)/O(n)$. This made it possible [6] to obtain an affirmative answer to Pełczyński's Problem (a) as a consequence of the following result, which we will also need in what follows:

Theorem 2 [4, Theorem 8]. *Let G be a compact metric group, $N \subseteq G$ be a closed normal subgroup and X be a G-ANR* (*resp.*, *a G-AR*). *Then the N-orbit space X/N is a G/N-ANR* (*resp.*, *a G/N-AR*). *In particular*, *X/G is an ANR* (*resp.*, *an AR*).

In this paper for $n = 2$ we solve Pełczyński's Problem (b) in the negative (Corollary 6). We follow the general idea of a profound paper by H. Toruńczyk and J. E. West [26]. Establishing Q-manifold hyperspace localization of the integers, they proved [26, Corollary 4] that the orbit space $(\exp S^1)/S^1$ of the natural action of the circle group S^1 on the hyperspace of all nonvoid closed subsets of S^1 is not a Hilbert cube. More precisely, [26, Theorem 4] asserts that the orbit space $((\exp S^1)\backslash\{S^1\})/S^1$ is a Q-manifold Eilenberg–MacLane space of type $\mathbf{K}(\mathbb{Q}, 2)$ for the group $\mathbb{Q}$ of rationals, forming a Q-manifold realization on π_2 of the localization of integers.

In the case of the Banach–Mazur compactum the situation is quite similar. Let $J_0(n)$ be the complement of the unique $O(n)$-fixed point in $J(n)$. Our main conjecture is that $BM_0(n) = J_0(n)/O(n)$ is not contractible for arbitrary $n \geq 2$. Here we prove it for $n = 2$.

In Section 3 we first show that $BM_0(2)$ is homotopically equivalent to the orbit space of the so-called *standard* $O(2)$-action (see [22]) on the Hilbert cube without a point. Namely, let $(H_1), (H_2), \ldots$ be the sequence of all orbit types occurring in $J_0(n)$. Let $\Pi(n)$ be the product $\prod_{i=1}^{\infty}(\mathrm{con}(O(n)/H_i))^{\infty}$ equipped with the diagonal $O(n)$-action and let $\Pi_0(n) = \Pi(n) \setminus \{a\}$, where a is the unique $O(n)$-fixed point of $\Pi(n)$. Theorem 5 asserts that $J_0(2)$ and $\Pi_0(2)$ have the same $O(2)$-equivariant homotopy type, and hence, $BM_0(2)$ and $\Pi_0(2)/O(2)$ have the same ordinary homotopy type. This result provides new possibilities to attack Pełczyński's Problem (b).

Let $D_1, D_2, \ldots$ be the unit discs of mutually inequivalent representations of $O(2)$ on $\mathbb{R}^2$ and let $\Psi(2)$ be the product $\prod_{k=1}^{\infty} D_k^{\infty}$ endowed with the diagonal $O(2)$-action. If $\Psi_0(2)$ denotes the complement of the unique $O(2)$-fixed point in $\Psi(2)$, then Theorems 5 and 6 imply that $J_0(2)$ and $\Psi_0(2)$ have the same $O(2)$-equivariant homotopy type. This result allows us to prove in Section 4 that the $SO(2)$-orbit space $\Psi_0(2)/SO(2)$, and hence $J_0(2)/SO(2)$, is an Eilenberg–MacLane space $\mathbf{K}(\mathbb{Q}, 2)$ (Theorem 8). Corollary 5 provides different homotopical models for $BM_0(2)$.

The passage from $J_0(2)/SO(2)$ to $BM_0(2) = J_0(2)/O(2)$ is a straightforward consequence of the following two well known results. The first one (see [7, p. 142, Theorem 7.2], or [8, p. 139]) asserts that if G is a finite group acting on a paracompact space X, then for every $k \geq 1$ the rational singular cohomology module $H^k(X/G)$ of the orbit space X/G is isomorphic to the submodule $H^k(X)^G$ of $H^k(X)$ consisting of all elements fixed under the induced action of G on $H^k(X)$. In our case $X = J_0(2)/SO(2)$ and $G = \mathbb{Z}_2 = O(2)/SO(2)$. The second result, due to D. Sullivan [24, p. 91, Theorem], states that $H^*(\mathbf{K}(\mathbb{Q}, 2), \mathbb{Q}) = \mathbb{Q}[x, 2]$, the graded polynomial algebra over $\mathbb{Q}$ in one indeterminate x of degree 2. It then follows that $H^{4k}(BM_0(2)) \approx \mathbb{Q}$, $k \geq 0$, implying the noncontractibility of $BM_0(2)$. Thus $BM(2)$ is not a Hilbert cube (Corollary 6).

In Section 5 we establish some further relevant results. Namely, Corollary 7 states that the absolute retract $BM(2)$ is nonhomogeneous. Using the Toruńczyk–West Theorem [26], we show that $\Psi_0(2)/SO(2)$ is homeomorphic to $(\exp_0 S^1)/SO(2)$, where $\exp_0 S^1 = (\exp S^1) \setminus \{S^1\}$ (Corollaries 9 and 10).

The results of Section 5 are based on Theorem A1 from the Appendix which may also be of independent interest. In a particular case it asserts the following: Let X_i, $i = 1, 2, \ldots$, be a sequence of compact G-AR's with G a compact Lie group. Assume that if an orbit type (N) occurs in some

X_i then it also occurs in infinitely many spaces X_j. Then the orbit space $((\prod_{i=1}^{\infty} X_i) \setminus \{a\})/G$ is a Hilbert cube manifold, where $a \in \prod_{i=1}^{\infty} X_i$ is a G-fixed point.

In the final Section 7 some related conjectures are formulated.

Note that in [30, p. 544] the notation "$Q(n)$" is suggested for the Banach–Mazur compactum. As the latter is not a Hilbert cube, here we denote it by $BM(n)$ (think of "Banach–Mazur"), reserving the letter "Q" for the Hilbert cube.

1. Preliminaries. For a given topological group G we denote by G-A(N)R (resp., by G-A(N)E) the class of all G-equivariant absolute (neighborhood) retracts (resp., extensors) for all metrizable G-spaces. These concepts are straightforward extensions to the case of G-spaces of the corresponding concepts of ordinary A(N)R's (resp., A(N)E's; see, for example [2-4]). We refer to the monographs [7] and [20] for basic notions of the theory of G-spaces.

If G is a topological group and X is a G-space, for any $x \in X$ we denote by $G_x = \{g \in G : gx = x\}$ the *stabilizer* (or stationary subgroup) of x. When for all $x \in X$, G_x contains only the unity of G, we say that the G-action is *free* or X is a free G-space.

For each subgroup $H \subseteq G$ the *H-fixed point set* $X[H]$ is defined to be the set $\{x \in X : H \subseteq G_x\}$.

For a subset $S \subseteq X$, $H(S)$ denotes the *H-saturation* of S, i.e., $H(S) = \{hs : h \in H,\ s \in S\}$. In particular $H(x)$ denotes the H-orbit $\{hx \in X : h \in H\}$ of x. The H-orbit space is denoted by X/H. By G/H we will denote the G-space of cosets $\{gH : g \in G\}$ under the action induced by left translations. When H is a normal subgroup of G, X/H admits a natural action of the group G/H defined by $(gH) * H(x) = H(gx)$. In particular X/G denotes the orbit space of X.

The family of all subgroups of G which are conjugate to H is denoted by (H), i.e., $(H) = \{gHg^{-1} : g \in G\}$. The set (H) is called a *G-orbit type* (or simply an orbit type). For two orbit types (H_1) and (H_2) one says that $(H_1) \preceq (H_2)$ iff $H_1 \subseteq gH_2g^{-1}$ for some $g \in G$. The relation $\preceq$ is a partial ordering on the set of all G-orbit types. Since $G_{gx} = gG_xg^{-1}$ for any $x \in X,\ g \in G$, we have $(G_x) = \{G_{gx} : g \in G\}$. By $X^{(H)}$ we denote the G-equivariant subset $\{x \in X : (G_x) \preceq (H)\}$ of X. It follows immediately from the slice existence theorem [20, p. 37] that $X^{(H)}$ is an open subset of X whenever G is a compact Lie group and X is a completely regular Hausdorff G-space. If H is a normal subgroup then (H) is a singleton $\{H\}$, and we write X^H instead of $X^{(H)}$.

An equivariant map $f : X \to Y$ of G-spaces is said to be *isovariant* or (G-isovariant) if $G_x = G_{f(x)}$ for all $x \in X$.

If X is metrizable and ϱ is a G-invariant metric for X then $\varrho^*(G(x), G(y)) = \inf\{\varrho(x', y') : x' \in G(x), y' \in G(y)\}$ is a metric for X/G, provided G is compact.

The hyperspace $\mathcal{B}(n)$ of all compact convex symmetric bodies of $\mathbb{R}^n$ is topologized by the Hausdorff metric $d_{\mathrm{H}}(A, C) = \max\{\sup_{x\in C} \operatorname{dist}(x, A), \sup_{y\in A} \operatorname{dist}(y, C)\}$. For a convex body $A \in \mathcal{B}(n)$ we denote by ∂A the boundary of A in $\mathbb{R}^n$.

Let us recall the well known and important definition of a slice [20, p. 27].

Definition 1. Let G be a topological group, $H \subseteq G$ be a closed subgroup and X be a G-space. A subset $S \subseteq X$ is called an *H-slice* in X if

(1) S is H-invariant, i.e., $H(S) = S$,
(2) the saturation $G(S)$ is open in X,
(3) if $g \in G \setminus H$ then $gS \cap S = \emptyset$,
(4) S is closed in $G(S)$.

The saturation $G(S)$ will be said to be *tubular*. If in addition $G(S) = X$ then we say that S is a *global H-slice* of X.

Following R. Palais [21, Definition 1.2.2] we call a G-space X *proper* if: (1) G is a locally compact Hausdorff topological group, (2) X is a completely regular Hausdorff space and (3) every point of X has a neighborhood V such that for every point of X there is a neighborhood U with the property that the set $\langle U, V\rangle = \{g \in G : gU \cap V \neq \emptyset\}$ has compact closure in G. Clearly if G is compact then every G-space is proper. Each locally compact group G with the compact space of connected components has a maximal compact subgroup K, i.e., every compact subgroup of G is conjugate to a subgroup of K [1, Theorem A.5]. The corresponding classical theorem on Lie groups can be found in [14, Ch. XV, Theorem 3.1].

Theorem 3 [1, p. 9]. *Let G, K, X and S be as in Theorem* 1. *Then*

(1) *X is K-equivariantly homeomorphic to the product $G/K \times S$ of K-spaces, where K acts on G/K by left translations*;

(2) *there is a K-equivariant retraction $\alpha : X \to S$ such that $\alpha(x)$ belongs to the G-orbit $G(x)$ for every $x \in X$*;

(3) *the K-orbit space S/K is homeomorphic to the G-orbit space X/G.*

2. The Banach–Mazur compactum as orbit space of an $O(n)$-action. It is well known that $BM(n) = \mathcal{B}(n)/GL(n)$ is a compact metrizable space (see, e.g., [10]).

In what follows we will need another model of $BM(n)$ which is crucial in our approach to Pełczyński's Problem (b).

Throughout we denote by $J(n)$ the subspace of $\mathcal{B}(n)$ consisting of all convex bodies A such that the ordinary Euclidean unit ball B is the John ellipsoid of A, i.e., $j(A) = B$ (see the introduction). By $J_0(n)$ we denote the complement $J(n)\setminus\{B\}$.

THEOREM 4. *$J(n)$ is a global $O(n)$-slice for the $GL(n)$-space $\mathcal{B}(n)$.*

Proof. We verify conditions (1)–(4) of Definition 1 above.

(1) is obvious.

(2) We prove that $GL(n)(J(n)) = \mathcal{B}(n)$. Let $A \in \mathcal{B}(n)$ and let $g \in GL(n)$ be a linear operator mapping the John ellipsoid $j(A)$ onto the unit ball B. We claim that

$$(*) \qquad j(gA) = gj(A) \quad \text{for all } g \in GL(n),\ A \in \mathcal{B}(n).$$

Indeed, suppose the contrary. Let $j(gA) = D \neq gj(A)$. Since $B = gj(A)$ is an ellipsoid contained in gA, and since the John ellipsoid is unique, we infer that $\mathrm{vol}(D) > \mathrm{vol}(gj(A))$. By the same argument $\mathrm{vol}(g^{-1}D) < \mathrm{vol}(j(A))$. Now we apply the well known fact that each linear operator preserves the ratio of volumes of any pair of compact convex bodies in a Euclidean space. Thus we obtain

$$\frac{\mathrm{vol}(j(A))}{\mathrm{vol}(A)} = \frac{\mathrm{vol}(gj(A))}{\mathrm{vol}(gA)} < \frac{\mathrm{vol}(D)}{\mathrm{vol}(gA)} = \frac{\mathrm{vol}(g^{-1}D)}{\mathrm{vol}(A)} < \frac{\mathrm{vol}(j(A))}{\mathrm{vol}(A)}.$$

This contradiction proves $(*)$. Now as $gj(A) = B$, the condition $(*)$ yields $j(gA) = B$, i.e., $gA \in J(n)$. As $A = g^{-1}(gA)$ we see that $A \in GL(n)(J(n))$, proving (2).

(3) If $gA \in J(n)$ for some $A \in J(n)$ and $g \in GL(n)$ then $j(A) = B$ and $j(gA) = gj(A)$. So $gB = B$, and therefore, $g \in O(n)$.

(4) Let $\{A_k\} \subseteq J(n)$ be a sequence with limit $A \in \mathcal{B}(n)$. We prove that $j(A) = B$. Suppose the contrary is true. As $B \subseteq A_k$ for all $k \geq 1$, we see that $B \subseteq A$. Hence, by uniqueness of the John ellipsoid we have $\mathrm{vol}(j(A)) > \mathrm{vol}(B)$. Choose an ellipsoid L which is concentric and homothetic to $j(A)$ with ratio < 1 and $\mathrm{vol}(L) > \mathrm{vol}(B)$. As L is contained in the interior of $j(A)$, the distance ε between ∂L and ∂A is positive. Consider the ε-neighborhood U of ∂A in $\mathbb{R}^n$. Since $\{A_k\}$ converges to A and all the sets A_k are convex, the sequence $\{\partial A_k\}$ of boundaries converges to ∂A, and therefore there exists $k_0 \geq 1$ such that $\partial A_{k_0} \subseteq U$. By convexity it follows that $L \subseteq A_{k_0}$, and hence, $\mathrm{vol}(j(A_{k_0})) \geq \mathrm{vol}(L)$. On the other hand, $j(A_{k_0}) = B$ and $\mathrm{vol}(B) < \mathrm{vol}(L)$. This contradiction proves that $j(A) = B$, i.e., $A \in J(n)$. ■

The following corollary is immediate from Theorems 3(3) and 4 and provides the desired model for $BM(n)$:

COROLLARY 1. *The Banach–Mazur compactum $BM(n)$ is homeomorphic to the $O(n)$-orbit space $J(n)/O(n)$.*

COROLLARY 2. *$J(n)$ is a compact $O(n)$-AR.*

Proof. By Corollary 1, $J(n)/O(n) = BM(n)$, which is compact by [10]. Since the orbit map $J(n) \to J(n)/O(n)$ is perfect (by compactness of $O(n)$), we conclude that $J(n)$ is also compact.

By the equivariant Dugundji extension theorem (see [2]) the space $\overline{\mathcal{N}(n)}$ defined in the Introduction is an $O(n)$-AR. Observe that $\mathcal{N}(n)$ is locally compact [6]. Consequently, $\mathcal{N}(n)$ is open in $\overline{\mathcal{N}(n)}$ and hence is an $O(n)$-ANR. As $\mathcal{N}(n)$ is also $O(n)$-equivariantly contractible to the point $\|\cdot\| \in \mathcal{N}(n)$, we conclude that $\mathcal{N}(n) \in O(n)$-AR [2, Proposition 2]. As $\mathcal{B}(n)$ is $O(n)$-homeomorphic to $\mathcal{N}(n)$, we see that $\mathcal{B}(n) \in O(n)$-AR. Now the result is immediate from Theorems 4 and 3(2). ∎

Consider the induced action of the group $\mathbb{Z}_2 = O(n)/SO(n)$ on the $SO(n)$-orbit space $J(n)/SO(n)$. Theorem 2 and Corollary 2 have the following immediate

COROLLARY 3. *$J(n)/SO(n)$ is a $\mathbb{Z}_2$-AR.*

COROLLARY 4. *Let $\Phi(n)$ be the set of $\mathbb{Z}_2$-fixed points in the $\mathbb{Z}_2$-space $J(n)/SO(n)$. Then $\Phi(n)$ is an AR.*

Proof. Immediate from Corollary 3 and [2, Theorem 7]. ∎

REMARK 1. Yet another concrete global $O(n)$-slice is provided by the subset $L(n)$ of $\mathcal{B}(n)$ consisting of all bodies A for which the Euclidean unit ball B is the minimal volume ellipsoid containing A. Acting exactly in the same way as in the proof of Theorem 4, it can be proved that $L(n)$ is a global $O(n)$-slice for $\mathcal{B}(n)$. This implies that $BM(n) = L(n)/O(n)$, $L(n) \in O(n)$-AR, $L(n)/SO(n) \in \mathbb{Z}_2$-AR. Moreover, it follows from the result of H. Abels [1, Lemma 2.3] mentioned in the Introduction that the two global $O(n)$-slices $J(n)$ and $L(n)$ of $\mathcal{B}(n)$ are $O(n)$-equivariantly homeomorphic. It is not hard to see that a geometric $O(n)$-homeomorphism of $J(n)$ onto $L(n)$ is provided by the so-called *polar map*, which assigns to each $A \in J(n)$ its polar $A^\diamond = \{y \in \mathbb{R}^n : |(x,y)| \le 1 \text{ for all } x \in A\}$ (see, e.g., [19, p. 1154], [28, §2.8]). Consequently, all the subsequent results on $J(n)$ have also their analogies on $L(n)$, which can be proved by trivial modifications of our proofs.

It follows from Corollary 2 that $J(n)$ is $O(n)$-contractible; however, we will need the following special $O(n)$-contraction of $J(n)$.

LEMMA 1. *There is an $O(n)$-equivariant strong deformation retraction (f_t) of $J(n)$ to its point B such that $f_t : J(n) \to J(n)$ is an $O(n)$-isovariant map for all $0 < t \le 1$.*

Proof. For each $A \in J(n)$ and $0 \le t \le 1$ write $f_t(A) = (1-t)B + tA$. Here for convex bodies $X, Y \in \mathcal{B}(n)$ and nonnegative numbers a, b with $a + b = 1$ we denote by $aX + bY$ the convex body $\{ax + by : x \in X,\ y \in Y\} \in \mathcal{B}(n)$. ∎

LEMMA 2. *For each finite subgroup $K \subseteq O(2)$ and each $\varepsilon > 0$ there is a K-equivariant map $h_\varepsilon : J(2) \to J_0(2)$, ε-close to the identity map of $J(2)$. In particular $h_\varepsilon(J(2)[K]) \subseteq J_0(2)[K]$.*

Proof. Since $J(2)$ is a global $O(2)$-slice for the $GL(2)$-space $\mathcal{B}(2)$, by Theorem 3(2) there exists an $O(2)$-equivariant retraction $\alpha : \mathcal{B}(2) \to J(2)$ such that $\alpha(A) \in GL(2)(A)$ for every $A \in \mathcal{B}(2)$. By compactness of $J(2)$ (Corollary 2), one can find $0 < \delta < \varepsilon/2$ such that $d_{\mathrm{H}}(\alpha(A), A) < \varepsilon/2$ for all A belonging to the δ-neighborhood of $J(2)$ in $\mathcal{B}(2)$, where d_{H} denotes the Hausdorff metric on $\mathcal{B}(2)$.

Fix a regular polygon $T \in J_0(2)$ with $K \subseteq O(2)_T$, circumscribing B.

For each $A \in J(2)$, let

$$\eta(A) = \frac{\operatorname{diam} A}{2} - \delta.$$

Certainly, one can assume that $\delta < 1$, so $\eta(A) > 0$ for all $A \in J(2)$.

Setting

$$h'(A) = A \cap \eta(A)T$$

we obtain a well defined K-equivariant map $h' : J(2) \to \mathcal{B}(2)$. As η depends continuously upon $A \in J(2)$, the continuity of h' follows from that of the map $\gamma : \mathcal{B}(2) \times \mathcal{B}(2) \to \mathcal{B}(2)$ defined by $\gamma(A, C) = A \cap C$. Although the continuity of γ is easy to show directly, to be more rigorous, we give yet another (analytic) argument for it. Namely, a simple computation shows that $\gamma(A, C) = h(\max\{h^{-1}(A), h^{-1}(C)\})$, where $h : \mathcal{N}(2) \to \mathcal{B}(2)$ is the $GL(2)$-homeomorphism defined in the Introduction. Because $\max\{\varphi, \psi\}$ depends continuously upon the pair $(\varphi, \psi) \in \mathcal{N}(2) \times \mathcal{N}(2)$, the continuity of γ follows.

As $d_{\mathrm{H}}(A, A \cap \eta(A)B) \le \delta$ and $A \cap \eta(A)B \subseteq h'(A) \subseteq A$, we conclude that $d_{\mathrm{H}}(A, h'(A)) \le \delta$. In particular h' is $\varepsilon/2$-close to the inclusion $J(2) \hookrightarrow \mathcal{B}(2)$.

We claim that $h'(A)$ is not an ellipse for each $A \in J(2)$. Indeed, if $A \subseteq \eta(A)T$ then $A \ne B$ and $h'(A) = A$, which clearly is not an ellipse. If $A \nsubseteq \eta(A)T$ then it is not difficult to make sure that the boundary $\partial(h'(A))$ contains a nontrivial line segment from the boundary of the polygon $\eta(A)T$, and hence, $h'(A)$ cannot be an ellipse. This proves the claim.

As $\alpha(h'(A))$ and $h'(A)$ have the same $GL(2)$-orbit, we conclude that $\alpha(h'(A)) \ne B$ for each $A \in J(2)$. As α is $O(2)$-equivariant and h' is K-equivariant, denoting by h_ε the composition $\alpha h'$, we obtain a K-equivariant map $h_\varepsilon : J(2) \to J_0(2)$, ε-close to the identity of $J(2)$. In particular $h_\varepsilon(J(2)[K]) \subseteq J_0(2)[K]$. ∎

3. More representations of $BM_0(2)$ up to homotopy. We write $BM_0(n) = J_0(n)/O(n)$. In this section we establish new homotopy representations of $BM_0(2)$.

Let G denote an arbitrary compact Lie group. For each closed subgroup $H \subseteq G$ we denote by $\mathrm{con}(G/H)$ the cone $(G/H) \times [0,1]/(G/H) \times \{0\}$ over G/H equipped with the quotient topology and with the action of G by left translations on levels. Thus $\mathrm{con}(G/H)$ naturally becomes a G-space.

LEMMA 3. *For each closed subgroup $H \subseteq G$, $\mathrm{con}(G/H)$ is a G-AR.*

Proof. It is well known that $G/H \in G$-ANR [20, p. 27]. Hence G/H is locally G-contractible (see [17]). Evidently, $\mathrm{con}(G/H)$ inherits from G/H its local G-contractibility, and since it is finite-dimensional, it then follows from a result of J. Jaworowski [17] that $\mathrm{con}(G/H) \in G$-ANR. Observing that $\mathrm{con}(G/H)$ is globally G-contractible to its vertex, we complete the proof. ∎

LEMMA 4. *Let X be a metrizable G-space with the sequence of orbit types $(H_1), \ldots, (H_k), \ldots$ Let $Q(H_i) = (\mathrm{con}(G/H_i))^\infty$. Then there exists an isovariant map $f : X \to \prod_{i=1}^\infty Q(H_i)$.*

Proof. The proof can be extracted by an easy modification from the proof of [4, Lemma 5]. Indeed, let us return to the proof of [4, Lemma 3]. Instead of an equivariant embedding of G/H in a Euclidean G-space E_n, we now consider the natural equivariant embedding $x \mapsto (x, 1)$, $x \in G/H$, into $\mathrm{con}(G/H)$. It remains to repeat the rest of the proof of [4, Lemma 5], using the above Lemma 3 instead of the fact that the unit ball of a Euclidean G-space is a G-AR, which we have used in the proof of [4, Lemma 5]. ∎

Hearafter we assume that $(H_1), (H_2), \ldots$ is the sequence of all $O(n)$-orbit types occurring in $J_0(n)$. Let $\Pi(n)$ be the product $\prod_{i=1}^\infty Q(H_i)$, equipped with the diagonal $O(n)$-action, where $Q(H_i) = (\mathrm{con}(O(n)/H_i))^\infty$. Define $\Pi_0(n) = \Pi(n) \setminus \{a\}$, where a is the unique $O(n)$-fixed point of $\Pi(n)$. As $\mathrm{con}(O(n)/H_i) \in AR$, $i \geq 1$, it follows from a result of J. West [29] that $\Pi(n)$ is a Hilbert cube.

THEOREM 5. *$J_0(2)$ and $\Pi_0(2)$ have the same $O(2)$-homotopy type.*

For the proof we need the following

LEMMA 5. *Every $O(2)$-equivariant map $f : J_0(2) \to \Pi_0(2)$ is an $O(2)$-homotopy equivalence.*

Proof. We apply the following result of I. James and G. Segal [16]: Let G be a compact Lie group and $f : T \to Z$ be a G-map of G-ANR's. Then f is a G-homotopy equivalence iff for each closed subgroup $K \subseteq G$, the restriction of f to the K-fixed point set $T[K]$ is an ordinary homotopy equivalence.

In our case $G = O(2)$, $T = J_0(2)$ and $Z = \Pi_0(2)$. It follows from Corollary 2 that $J_0(2) \in O(2)$-ANR. On the other hand, as $O(2)/H_i \in O(2)$-ANR [20, p. 27], Lemma 3 yields that $\mathrm{con}(O(2)/H_i) \in O(2)$-AR. Consequently,

$Q(H_i) \in O(2)$-AR, $i \geq 1$ and $\Pi(2) \in O(2)$-AR, implying $\Pi_0(2) \in O(2)$-ANR.

Now, if K is an infinite closed subgroup of $O(2)$ then evidently $J_0(2)[K] = \Pi_0(2)[K] = \emptyset$. For $K \subseteq O(2)$ a finite subgroup we shall show that the K-fixed point sets $J_0(2)(2)[K]$ and $\Pi_0(2)[K]$ are both contractible. In fact, by Lemma 2 the $O(2)$-fixed point set $\{B\}$ is a Z-set in $J(2)[K]$. As $J(2)[K] \in \mathrm{AR}$ [2, Theorem 7], it follows from [13] that $J(2)[K]$ and $J_0(2)[K]$ have the same homotopy type, and hence $J_0(2)[K]$ is contractible. To prove the contractibility of $\Pi_0(2)[K]$, we shall show that in fact $\Pi(2)[K]$ is a Hilbert cube. As the finite subgroup K is either cyclic or dihedral, there is a regular polygon $A \in J_0(2)$ with $K \subset O(2)_A$; so there is an orbit type (H_j) such that $(O(2)/H_j)[K] \neq \emptyset$. Next we have $\Pi(2)[K] = \prod_{i=1}^{\infty}(Q(H_i)[K])$. As $O(2)/H_i \in O(2)$-ANR, it follows that $(O(2)/H_i)[K] \in \mathrm{ANR}$, implying that $(\mathrm{con}(O(2)/H_i))[K]= \mathrm{con}((O(2)/H_i)[K]) \in \mathrm{AR}$. If for an index $i \geq 1$, $(O(2)/H_i)[K]$ is nonempty then $\mathrm{con}((O(2)/H_i)[K])$ is nondegenerate. In this case, according to West's theorem [29], the countable product $Q(H_i)[K] = (\mathrm{con}(O(2)/H_i)[K])^{\infty}$ is a Hilbert cube. In particular $Q(H_j)[K]$ is a Hilbert cube. If $(O(2)/H_i)[K] = \emptyset$, then $\mathrm{con}((O(2)/H_i)[K])$ as well as $Q(H_i)[K]$ are singletons. It follows that $\prod_{i=1}^{\infty}(Q(H_i)[K])$ is a Hilbert cube, and hence, $\Pi_0(2)[K]$ is contractible. ∎

Proof of Theorem 5. By Lemma 4 there is an isovariant map $f : J_0(2) \to \Pi(2)$. By isovariance, the image of f in fact lies in $\Pi_0(2)$, so we have an $O(2)$-equivariant map $f : J_0(2) \to \Pi_0(2)$. By Lemma 5, f is an $O(2)$-homotopy equivalence. ∎

For each integer $k \geq 1$, let $\mu_k : O(2) \times \mathbb{R}^2 \to \mathbb{R}^2$ denote the natural action on the plane $\mathbb{R}^2$ whose kernel is the cyclic subgroup $\mathbb{Z}_k$ of $O(2)$. More precisely, let us identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ and let $\sigma : \mathbb{C} \to \mathbb{C}$ be the complex conjugation. Then the orthogonal group $O(2)$ is the disjoint union $SO(2) \cup SO(2) \cdot \sigma$, where $SO(2)$ is the group of all rotations of $\mathbb{R}^2$ about the origin, and $SO(2) \cdot \sigma = \{\varphi\sigma : \varphi \in SO(2)\}$. The action μ_k is then defined by

$$\begin{aligned} \mu_k(e^{it}, re^{ix}) &= re^{i(kt+x)} \qquad \text{if } e^{it} \in SO(2),\ re^{ix} \in \mathbb{C}, \\ \mu_k(e^{it} \cdot \sigma, re^{ix}) &= re^{i(kt-x)} \qquad \text{if } e^{it} \cdot \sigma \in SO(2) \cdot \sigma,\ re^{ix} \in \mathbb{C}. \end{aligned}$$

Let $\mathbb{R}^2_k$ be the $O(2)$-space $(\mathbb{R}^2, O(2), \mu_k)$ and let $D_k = (D, O(2), \mu_k)$ with D the unit disk of $\mathbb{R}^2$.

In what follows we denote by $\Delta(2)$ the $O(2)$-space $\prod_{k=1}^{\infty} D_k$ endowed with the diagonal action. Similarly, we let $\Psi(2) = \prod_{k=1}^{\infty} D_k^{\infty}$ and $\Gamma(2) = \prod_{k=1}^{\infty} D_{2k}^{\infty}$. Furthermore, we denote by $\Psi_0(2)$ (resp. $\Delta_0(2)$ and $\Gamma_0(2)$) the complement of the unique $O(2)$-fixed point in $\Psi(2)$ (resp., in $\Delta(2)$ and $\Gamma(2)$).

THEOREM 6. *The three $O(2)$-spaces $\Gamma_0(2)$, $\Psi_0(2)$ and $\Delta_0(2)$ have the same $O(2)$-homotopy type.*

Proof. 1. $\Gamma_0(2) \simeq_{O(2)} \Psi_0(2)$. The idea of the proof is the same as that used in Lemma 5. First we observe that each of $\Gamma(2)$ and $\Psi(2)$ is an $O(2)$-AR. This follows from the fact that $D_k \in O(2)$-AR, $k \geq 1$. Hence $\Gamma_0(2)$ and $\Psi_0(2)$ are $O(2)$-ANR's. Let $f : \Gamma(2) \to \Psi(2)$ be the natural inclusion map defined by $f(x)_{2k} = x_k$ and $f(x)_{2k-1} = O$ for all $x \in \Gamma(2)$, $k = 1, 2, \ldots$, where O denotes the origin of $\mathbb{R}^2$. Then the restriction $f_0 : \Gamma_0(2) \to \Psi_0(2)$ of f is an $O(2)$-equivariant map of $O(2)$-ANR's. As in the proof of Lemma 5, using the James–Segal theorem, one can establish that f_0 is an $O(2)$-homotopy equivalence. In fact, if K is an infinite subgroup of $O(2)$ then evidently $\Gamma_0(2)[K] = \Psi_0(2)[K] = \emptyset$. For $K \subseteq O(2)$ a finite subgroup we show that the K-fixed point sets $\Gamma(2)[K]$ and $\Psi(2)[K]$ are both Hilbert cubes. Indeed,

$$\Psi(2)[K] = \Big(\prod_{i=1}^{\infty} D_i^{\infty}\Big)[K] = \prod_{i=1}^{\infty}(D_i^{\infty}[K]).$$

If K is a cyclic subgroup of order k then $D_i[K] = D_i$ if i is a multiple of k, and $D_i[K] = \{O\}$ otherwise. If K is a dihedral subgroup of order $2k$ then $D_i[K]$ is the diameter of D_i lying on the x-axis if i is a multiple of k, and $D_i[K] = \{O\}$ otherwise. As $D_i^{\infty}[K] = (D_i[K])^{\infty}$, we see that $\Psi(2)[K]$, being a countable product of 1-or 2-dimensional discs, is a Hilbert cube. By the same reason $\Gamma(2)[K]$ is a Hilbert cube.

Consequently, $\Gamma_0(2)[K]$ and $\Psi_0(2)[K]$ are both contractible, and the James–Segal theorem completes the proof.

2. The $O(2)$-homotopy equivalence $\Psi_0(2) \simeq_{O(2)} \Delta_0(2)$ can be established in a similar way. Indeed, for every $k \geq 1$ there is a natural inclusion map $f_k : D_k \to D_k^{\infty}$ defined by $f_k(x) = (x, O, O, \ldots)$, $x \in D_k$. The diagonal product of these maps is an $O(2)$-equivariant embedding of $\Delta(2)$ into $\Psi(2)$, which maps the $O(2)$-ANR space $\Delta_0(2)$ into the $O(2)$-ANR space $\Psi_0(2)$. To show that this embedding is an $O(2)$-homotopy equivalence, it remains to repeat the argument used above. ■

THEOREM 7. *$\Pi_0(2)$ and $\Gamma_0(2)$ have the same $O(2)$-homotopy type.*

Proof. We claim that there is an $O(2)$-equivariant map $f : \Pi_0(2) \to \Gamma_0(2)$.

Indeed, let H_i, $i \geq 1$, be a stabilizer occurring in $J_0(2)$. Then it is either a cyclic group $\mathbb{Z}_{2k}$ or a dihedral group $\mathbb{Z}_{2k} \cup \mathbb{Z}_{2k} \cdot \sigma$, where $k \geq 1$ is an integer and σ is a reflection. In both cases we choose a point $a \in D_{2k}$ with $\|a\| = 1$ and consider the natural $O(2)$-map $h_i : O(2)/H_i \to O(2)(a)$ into the orbit $O(2)(a) \subseteq D_{2k}$. If we consider D_{2k} as a cone over the unit circle, the map h_i admits a conic extension to an $O(2)$-equivariant map $f_i : \mathrm{con}(O(2)/H_i) \to D_{2k}$.

Observe that f_i is $O(2)$-isovariant if H_i is dihedral, and it is "$O(2)$-semiisovariant" if H_i is cyclic, i.e., $O(2)_{f_i(x)} = O(2)_x \cup O(2)_x \cdot \sigma$ for a reflection $\sigma \in O(2)$. Let $F : \Pi(2) \to \Gamma(2)$ be the direct product of the maps $f_i^\infty : Q(H_i) \to D_{2k}^\infty$, $i = 1, 2, \ldots$ Then F is $O(2)$-equivariant and, due to the above mentioned property of f_i, it maps $\Pi_0(2)$ onto $\Gamma_0(2)$. Now as in the proof of Theorem 6, by applying the James–Segal theorem we infer that the restriction $f = F|_{\Pi_0(2)} : \Pi_0(2) \to \Gamma_0(2)$ is an $O(2)$-homotopy equivalence. ■

COROLLARY 5. *For each closed subgroup* $N \subseteq O(2)$, *the* N*-orbit spaces* $J_0(2)/N$, $\Pi_0(2)/N$, $\Gamma_0(2)/N$, $\Psi_0(2)/N$ *and* $\Delta_0(2)/N$ *have the same homotopy type. In particular*, $BM_0(2)$, $\Pi_0(2)/O(2)$, $\Gamma_0(2)/O(2)$, $\Psi_0(2)/O(2)$ *and* $\Delta_0(2)/O(2)$ *have the same homotopy type.*

Proof. Immediate from Theorems 5–7. ■

4. Homotopy types of $J_0(2)/SO(2)$ and $BM_0(2)$. In this section we investigate homotopy properties of $J_0(2)/SO(2)$ and $BM_0(2)$. Following the general idea in [26], we first prove the following

THEOREM 8. *The space* $J_0(2)/SO(2)$ *is an Eilenberg–MacLane space* $\mathbf{K}(\mathbb{Q}, 2)$; *consequently*, *the compactum* $J(2)/SO(2)$ *is not homeomorphic to the Hilbert cube.*

For the proof we need some lemmas.

Below we denote by $\mathcal{F}(2)$ the set of all finite subgroups of the circle group $SO(2)$.

For each $H \in \mathcal{F}(2)$ let $\Psi^H(2) = \{x \in \Psi(2) : SO(2)_x \subseteq H\}$. We observe that $\Psi^H(2) \subseteq \Psi_0(2)$.

LEMMA 6. *For each* $H \in \mathcal{F}(2)$ *the set* $\Psi(2)[H] \setminus \Psi^H(2)$ *is a* Z*-set in* $\Psi(2)[H]$.

Proof. As H is a finite subgroup of $SO(2)$ it must be a cyclic group $\mathbb{Z}_k$ for some $k \geq 1$. Observe that

$$\Psi(2)[H] = \Big(\prod_{i=1}^{\infty} D_i^\infty\Big)[H] = \prod_{i=1}^{\infty} (D_i^\infty[H]).$$

But $D_i^\infty[H] = D_i^\infty$ if i is a multiple of k, and $D_i^\infty[H] = \{O^*\}$ otherwise, where $O^* = (O, O, \ldots)$ with O the origin of $\mathbb{R}^2$ (see the proof of Theorem 6). Therefore

$$\Psi(2)[H] = \prod_{i=1}^{\infty} X_i \quad \text{with } X_i = D_i^\infty \text{ if } i \text{ is a multiple of } k, \text{ and } X_i = \{O^*\} \text{ otherwise.}$$

So, $\Psi(2)[H]$ is a Hilbert cube. Similarly,

$$\Psi(2)[H] \setminus \Psi^H(2) = X_1 \times \ldots \times X_{k-1} \times \{O^*\} \times \prod_{i=k+1}^{\infty} X_i.$$

As points are Z-sets in the Hilbert cube, we infer that $\{O^*\}$ is a Z-set in the Hilbert cube $X_k = D_k^\infty$, and hence $X_1 \times \ldots \times X_{k-1} \times \{O^*\} \times \prod_{i=k+1}^\infty X_i$ is a Z-set in $\prod_{i=1}^\infty X_i$. Thus $\Psi(2)[H] \setminus \Psi^H(2)$ is a Z-set in $\Psi(2)[H]$. ∎

LEMMA 7. *For each $H \in \mathcal{F}(2)$, $\Psi^H(2)$ is an H-AR.*

Proof. The slice existence theorem yields that $\Psi^H(2)$ is an open invariant subspace of $\Psi_0(2)$ [20, p. 37]. Since $\Psi(2)$ is an $SO(2)$-AR (see the proof of Theorem 6), it follows that $\Psi_0(2)$ as well as $\Psi^H(2)$ are $SO(2)$-ANR's. In order to prove the second statement we apply the following corollary of the James–Segal theorem [16], already used in the proof of Lemma 5: if G is a compact Lie group and T is a G-ANR then T is a G-AR iff for every closed subgroup $K \subseteq G$ the K-fixed point set $T[K]$ is contractible. In our case $G = H$ and $T = \Psi^H(2)$. For let K be any closed subgroup of H, and denote briefly by Y the subset of all K-fixed points of $\Psi^H(2)$, i.e., $Y = \{x \in \Psi(2) : K \subseteq SO(2)_x \subseteq H\}$.

Consider $\Psi(2)[K]$, the subspace of all K-fixed points of $\Psi(2)$, which is a Hilbert cube (see the proof of Lemma 6). Let us show that $\Psi(2)[K] \setminus Y$ is a Z-set in $\Psi(2)[K]$. Indeed, since $K \subseteq H$ one has $\Psi(2)[K] \setminus Y \subseteq \Psi(2)[K] \setminus \Psi^K(2)$. As by Lemma 6, $\Psi(2)[K] \setminus \Psi^K(2)$ is a Z-set in $\Psi(2)[K]$, we conclude that $\Psi(2)[K] \setminus Y$ is also a Z-set in $\Psi(2)[K]$.

On the other hand, the complement $\Psi(2)[K] \setminus Y$ is contractible to its unique $O(2)$-fixed point $O^* = (O, O, \ldots)$; the corresponding contraction is given by

$$F_t(x_1, x_2, \ldots) = ((1-t)x_1, (1-t)x_2, \ldots), \qquad t \in [0,1].$$

Now the complement theorem of T. Chapman [9, §25] yields that Y is homeomorphic to Q_0, the complement of the point O^* in the Hilbert cube $\Psi(2)[K]$. As Q_0 is contractible, the proof is complete. ∎

LEMMA 8. *For each $H \in \mathcal{F}(2)$ let $M(H) = \Psi^H(2)/SO(2)$. Then*

(1) *$M(H)$ is an Eilenberg–MacLane space $\mathbf{K}(\mathbb{Z}, 2)$;*

(2) *if $F \in \mathcal{F}(2)$ and $F \subseteq H$, then the inclusion $i_H^F : M(F) \to M(H)$ induces multiplication by $|H/F|$: $(i_H^F)_* : \mathbb{Z} = \pi_2(M(F)) \to \pi_2(M(H)) = \mathbb{Z}$.*

Proof. (1) By Lemma 7, $\Psi^H(2)$ is an H-AR, and by Theorem 2 the H-orbit space $\Psi^H(2)/H$ is an AR, and hence, is contractible. Observe that the group $SO(2)/H$ is topologically isomorphic to $SO(2)$, and the natural action of $SO(2)/H$ on $\Psi^H(2)/H$ (see Preliminaries) is free. According to a theorem of A. Gleason [7, II, §5], a locally trivial fibration $SO(2) \hookrightarrow$

$\Psi^H(2)/H \to M(H)$ arises. It follows from the homotopy exact sequence of this fibration that $\pi_k(M(H)) = \pi_{k-1}(SO(2))$ for all $k = 1, 2, \ldots$, so $M(H)$ is a $\mathbf{K}(\mathbb{Z}, 2)$. This proves (1).

(2) From the functoriality of the above homotopy exact sequence we obtain the commutative diagram

$$\begin{tikzcd}
\pi_2(\Psi^F(2)/F) \arrow[r] \arrow[d] & \pi_2(M(F)) \arrow[r, "\partial"] \arrow[d, "(i_H^F)_*"] & \pi_1(SO(2)/F) \arrow[r] \arrow[d, "f"] & \pi_1(\Psi^F(2)/F) \arrow[d] \\
\pi_2(\Psi^H(2)/H) \arrow[r] & \pi_2(M(H)) \arrow[r, "\partial"] & \pi_1(SO(2)/H) \arrow[r] & \pi_1(\Psi^H(2)/H)
\end{tikzcd}$$

which is

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "\partial"] \arrow[d, "(i_H^F)_*"] & \mathbb{Z} \arrow[r] \arrow[d, "f"] & 0 \arrow[d] \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\partial"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

with f being multiplication by $|H/F|$. Hence $(i_H^F)_*$ is also multiplication by $|H/F|$. ∎

Following [26] we consider the direct system

$$\mathcal{M}(SO(2)) = \{M(H), i_H^F : M(F) \to M(H) \text{ (whenever } F \subseteq H) : F, H \in \mathcal{F}(2)\},$$

formed by inclusions. Observe that the index set $\mathcal{F}(2)$ is *directed* by inclusions.

LEMMA 9. *The space $\Psi_0(2)/SO(2)$ is the topological direct limit of the direct system $\mathcal{M}(SO(2))$ of Q-manifolds $\mathbf{K}(\mathbb{Z}, 2)$; consequently, $\Psi_0(2)/SO(2)$ is a Q-manifold $\mathbf{K}(\mathbb{Q}, 2)$.*

Proof. For each $k = 1, 2, \ldots$, let $\mathbb{Z}_k$ be a copy of $\mathbb{Z}$ and let $\mathbb{Z}_m \to \mathbb{Z}_k$ be the multiplication by k/m whenever m divides k. It is well known that

$$(**) \qquad \{\mathbb{Z}_k,\ \mathbb{Z}_m \xrightarrow{k/m} \mathbb{Z}_k : k \in \mathbb{N}\}$$

is a direct system with direct limit $\mathbb{Q}$ (see, for example, [23, Ch. I]).

Since each $M(H)$ is open in $\Psi_0(2)/SO(2)$ and their union is the whole space $\Psi_0(2)/SO(2)$, we see that $\Psi_0(2)/SO(2) = \varinjlim \mathcal{M}(O(2))$, the topological direct limit. Being an open subset of the Q-manifold $\Psi_0(2)/SO(2)$ (Theorem A1 of the Appendix), $M(H)$ is itself a Q-manifold. By Lemma 8 the direct system of p-homotopy groups

$$\pi_p(\mathcal{M}(SO(2))) = \{\pi_p(M(H)),\ (i_H^F)_* : \pi_p(M(F)) \to \pi_p(M(H)) : F, H \in \mathcal{F}(2)\}$$

is the trivial direct system for each $p \neq 2$, and is the direct system $(**)$ for $p = 2$. Now, as $\pi_p(\Psi_0(2)/SO(2)) = \varinjlim \pi_p(\mathcal{M}(SO(2)))$, $p = 1, 2, \ldots$, the result follows. ∎

Proof of Theorem 8. By Corollary 5, $J_0(2)$ has the same $SO(2)$-homotopy type as $\Psi_0(2)$, and therefore, $J_0(2)/SO(2)$ is homotopically equivalent to $\Psi_0(2)/SO(2)$. Now the result follows from Lemma 9. ∎

Corollary 6. *The space $BM_0(2)$ is not contractible*; *consequently*, *the Banach–Mazur compactum $BM(2)$ is not a Hilbert cube.*

Proof. We show that $H^{4k}(BM_0(2)) \approx \mathbb{Q}$ and $H^{2k+1}(BM_0(2)) \approx 0$ for all $k \geq 0$, where singular cohomology with rational coefficients is considered (as $J_0(2)/SO(2)$ and $BM_0(2)$ are ANR's, their singular and Čech cohomologies coincide). Consider the induced action of $\mathbb{Z}_2 = O(2)/SO(2)$ on $J_0(2)/SO(2)$. For convenience we redenote $X = J_0(2)/SO(2)$ and $G = \mathbb{Z}_2$. According to [7, p. 142, Theorem 7.2] (see also [8, p. 139]), $H^k(X/G) \approx H^k(X)^G$, where $H^k(X)^G$ is the submodule of $H^k(X)$ consisting of all elements fixed under the induced action of G on $H^k(X)$.

By Theorem 8, X is an Eilenberg–MacLane space $\mathbf{K}(\mathbb{Q}, 2)$. The cohomology algebras of Eilenberg–MacLane spaces of arbitrary type were computed by D. Sullivan [24, p. 91, Theorem]: $H^*(\mathbf{K}(\mathbb{Q}, 2), \mathbb{Q}) = \mathbb{Q}[x, 2]$, the graded polynomial algebra over $\mathbb{Q}$ in one indeterminate x of degree 2. This already implies that $H^{2k+1}(X) \approx 0$.

Let $\varphi \in G$ be the generator, i.e., $\varphi : X \to X$ is a homeomorphism with $\varphi^2 = 1_X$, and let $\varphi_k^* : H^k(X) \to H^k(X)$ be the induced isomorphism. As $H^{2k}(X)$ is a 1-dimensional free module over $\mathbb{Q}$ and $\varphi_{2k}^* : H^{2k}(X) \to H^{2k}(X)$ is a linear isomorphism with $(\varphi_{2k}^*)^2 = 1_{H^{2k}(X)}$, there are only two possibilities: either $\varphi_{2k}^* = 1_{H^{2k}(X)}$ or $\varphi_{2k}^* = -1_{H^{2k}(X)}$ (i.e., $\varphi_{2k}^*(a) = -a$ for all $a \in H^{2k}(X)$). In both cases $\varphi_{2k}^*(x^k) \cdot \varphi_{2k}^*(x^k) = x^{2k}$ (recall that x^k is the generator of the free module $H^{2k}(X)$). On the other hand, as $\varphi^* : H^*(X) \to H^*(X)$ is also an algebra homomorphism, it must preserve the product, so we have $\varphi_{2k}^*(x^k) \cdot \varphi_{2k}^*(x^k) = \varphi_{4k}^*(x^{2k})$. Hence $\varphi_{4k}^*(x^{2k}) = x^{2k}$, which implies that $\varphi_{4k}^* = 1_{H^{4k}(X)}$. Consequently, $H^{4k}(X)^G = H^{4k}(X) \approx \mathbb{Q}$, and hence $H^{4k}(BM_0(2)) = H^{4k}(X/G) \approx \mathbb{Q}$. In particular, $BM_0(2)$ is not contractible. As the Hilbert cube with a point removed is contractible, we conclude that $BM(2)$ is not a Hilbert cube. ∎

5. Further relevant results. In this section we establish further properties of the Banach–Mazur compactum $BM(2)$ and of related spaces. We begin with the following

Corollary 7. *$BM(2)$ is nonhomogeneous.*

Proof. Because of Corollary 6 it suffices to show that there is a point $O(2)(A) \in BM(2)$, different from the singular point $B \in BM(2)$, such that the complement $BM(2) \setminus \{O(2)(A)\}$ is contractible. Let $A \in J(2)$ be the square $\{(x,y) \in \mathbb{R}^2 : |x| \le 1,\ |y| \le 1\}$ and let X denote the complement of the orbit $O(2)(A)$ in $J(2)$. As X is an open invariant subset of the $O(2)$-AR space $J(2)$ (Corollary 2), it is an $O(2)$-ANR. We assert that X is in fact an $O(2)$-AR. It suffices to check that X is $O(2)$-contractible. Consider the $O(2)$-contraction $f_t : J(2) \to J(2)$ to the point $B \in J(2)$ defined in Lemma 1. We claim that $f_t(C) \in X$ for every $C \in X$ and every $0 \le t \le 1$. Indeed, $f_0(C) = B \in X$. Now suppose the contrary, that $f_t(C)$ is a square for some $t > 0$ and for some $C \in X$. Then there is a $g \in O(2)$ such that $f_t(gC) = A$. Since the point $x = (1,0)$ belongs to $\partial(f_t(gC)) \cap \partial B$, it must belong to $\partial(gC)$. As $(1,1)$ and $(1,-1)$ belong to $f_t(gC)$, there exists $u > 1$ such that $a = (u,u)$ and $b = (u,-u)$ belong to gC. By convexity $(u,0) = \frac{1}{2}(a+b) \in gC$. Since $u > 1$, this contradicts the fact that $x \in \partial(gC)$.

Thus the restriction of (f_t) to X realizes an equivariant contraction of X to its point B. Passing to the orbit space we obtain a contraction of $BM(2) \setminus \{O(2)(A)\}$ to its point $\{B\}$. This completes the proof. ∎

Corollary 8. *The space $\mathcal{N}(n) = \mathcal{B}(n)$ is homeomorphic to the product $\mathbb{R}^k \times J(n)$ with $k = n(n+1)/2$. Moreover, this homeomorphism can be made $O(n)$-equivariant under a suitable linear $O(n)$-action on $\mathbb{R}^k$.*

Proof. First we note that $\mathcal{B}(n) = \mathcal{N}(n)$ is a proper $GL(n)$-space [6]. As $O(n)$ is a maximal compact subgroup of $GL(n)$, and $J(n)$ is a global $O(n)$-slice of $\mathcal{B}(n)$ (Theorem 4), Theorem 3(1) implies the equality of $O(n)$-spaces:

$$\mathcal{B}(n) = (GL(n)/O(n)) \times J(n).$$

In general, when G and K are as in Theorem 1, G/K is K-homeomorphic to a linear K-space $\mathbb{R}^p$, with $p = \dim(G/K)$ [1, Corollary A.6]. For G a Lie group this is a classical result [14, Ch. XV, Theorem 3.1]. In fact $\mathbb{R}^p$ is the complementary subalgebra of the Lie algebra $\mathcal{L}(K)$ in the Lie algebra $\mathcal{L}(G)$. In our case $\dim(GL(n)/O(n)) = n(n+1)/2$ (this comes from the standard result of linear algebra to the effect that each invertible matrix can be uniquely represented as the product of an orthogonal matrix and a diagonal matrix with positive diagonal elements). Thus $GL(n)/O(n)$ is $O(n)$-homeomorphic to $\mathbb{R}^k$. This completes the proof. ∎

Let $\exp S^1$ denote the hyperspace of all nonvoid closed subsets of the circle S^1 equipped with the Hausdorff metric and with the natural $O(2)$-action. We denote by $\exp_0 S^1$ the subspace $(\exp S^1) \setminus \{S^1\}$. In [26, Corollary 6] it is proved that the orbit space $(\exp_0 S^1)/SO(2)$ is a Q-manifold Eilenberg–MacLane space $\mathbf{K}(\mathbb{Q}, 2)$. On the other hand, according to Corollary 5 and

Theorem 8, $\Delta_0(2)/SO(2)$ is a $\mathbf{K}(\mathbb{Q},2)$ as well. By Theorem A1 from the Appendix, $\Delta_0(2)/SO(2)$ is a Q-manifold. Hence the two Q-manifolds have the same homotopy type. Further, $(\exp_0 S^1)/SO(2)$ is homeomorphic to its product with the half-open interval $[0,1)$ (see [26, Theorem 1]). The space $\Delta_0(2)/SO(2)$ has an obvious *proper* (preimage of each compact set is compact) deformation to infinity $\big(\Delta_0(2)/SO(2)\big)\times[0,1)\to\Delta_0(2)/SO(2)$ (using the conic structure). So, by a result of R. Y. T. Wong [31] (see also [26, p. 445]), $\Delta_0(2)/SO(2)$ is homeomorphic to its product with $[0,1)$. Therefore, applying [9, Theorem 21.2], we get the following

COROLLARY 9. *$\Delta_0(2)/SO(2)$ is homeomorphic to $(\exp_0 S^1)/SO(2)$; consequently, $\Delta(2)/SO(2)$ is homeomorphic to $(\exp S^1)/SO(2)$.*

COROLLARY 10. *For each closed subgroup $N\subseteq O(2)$, the N-orbit spaces $\Pi_0(2)/N$, $\Gamma_0(2)/N$, $\Psi_0(2)/N$ and $\Delta_0(2)/N$ are mutually homeomorphic. In particular, $\Pi_0(2)/O(2)\approx\Gamma_0(2)/O(2)\approx\Psi_0(2)/O(2)\approx\Delta_0(2)/O(2)$, and hence, $\Pi(2)/O(2)\approx\Gamma(2)/O(2)\approx\Psi(2)/O(2)\approx\Delta(2)/O(2)$.*

Proof. By Corollary 5, the N-orbit spaces $\Pi_0(2)/N$, $\Gamma_0(2)/N$, $\Psi_0(2)/N$ and $\Delta_0(2)/N$ have the same homotopy type. Further, $\Pi(2)$, $\Gamma(2)$, $\Psi(2)$ and $\Delta(2)$ all satisfy the hypothesis of Theorem A1 from the Appendix, so all the orbit spaces $\Pi_0(2)/N$, $\Gamma_0(2)/N$, $\Psi_0(2)/N$ and $\Delta_0(2)/N$ are Q-manifolds. As the relevant spaces have obvious proper deformations to infinity, by applying the results of R. Y. T. Wong [31] and T. A. Chapman [9] quoted above, we get the desired homeomorphisms. ∎

6. Appendix. Orbit spaces as Q-manifolds. Here we prove a theorem announced in [5], whose particular cases were used in Section 5. Perhaps it is also of independent interest.

THEOREM A1. *Let G be a compact Lie group, X_i, $i=1,2,\ldots$, be a sequence of metrizable compact G-AR's and $a_i\in X_i$ be a G-fixed point. Assume that if (N) is an orbit type occurring in some X_i then there are infinitely many indices j such that $X_j\setminus\{a_j\}$ contains an orbit type $\succeq(N)$. Let $a=(a_i)$ and $X=(\prod_{i=1}^{\infty}X_i)\setminus\{a\}$. Then for each closed subgroup $H\subseteq G$ the H-orbit space X/H is a Q-manifold.*

The proof depends upon the three lemmas stated below.

For spaces T and Z we denote by $C(T,Z)$ the space of all continuous maps $T\to Z$ equipped with the compact-open topology. If T and Z are G-spaces then $E(T,Z)$ denotes the subspace of $C(T,Z)$ consisting of all equivariant maps.

DEFINITION A1. A closed invariant subset A of a G-space Y is called a *GZ-set* in Y if for every compact G-space K the set $\{f\in E(K,Y): f(K)\cap A=\emptyset\}$ is dense in $E(K,Y)$.

LEMMA A1. *Let $k \geq 1$ be a fixed integer. Then the set $A = \{x \in X : x_i = a_i \text{ for all } i \geq k\}$ is a GZ-set in X.*

Proof. Let $f : N \to X$ be a G-map with N a compact G-space, and let $\varepsilon > 0$. Choose an integer $r > k$ such that $\sum_{i=r+1}^{\infty} 1/2^i < \varepsilon$. Let ϱ_i be a metric on X_i with diameter ≤ 1. Consider the metric $\varrho(x, z) = \sum_{i=1}^{\infty} \varrho_i(x_i, z_i)/2^i$ on $\prod_{i=1}^{\infty} X_i$, where $x = \{x_i\}$, $z = \{z_i\} \in \prod_{i=1}^{\infty} X_i$. Let $f(y) = \{f_i(y)\}_{i=1}^{\infty}$, $y \in N$.

CLAIM 2. *There exists a sequence $\{\varphi_i : N \to X_i : i \geq r+1\}$ of G-maps such that for each $y \in N$ there is an index $j \geq r+1$ with $\varphi_j(y) \neq a_j$.*

Proof. By the slice existence theorem there is a tubular neighborhood about each orbit $G(y) \subseteq N$, i.e., there is a G-neighborhood U_y having a G-retraction $p_y : U_y \to G(y)$ [7, Ch. II, §5]. Clearly U_y can be assumed to be a closed neighborhood. By compactness of N one can choose a finite number of neighborhoods $U_{y_1}, \ldots, U_{y_m}$ which cover N. Now for each index $1 \leq i \leq m$ we define a G-map $\psi_{q(i)} : U_{y_i} \to X_{q(i)}$ with $q(i) \geq r+1$ as follows. Since $G_{y_i} \subseteq G_{f(y_i)}$ and $f(y_i) \neq a$ there is an index $s(i) \geq 1$ such that $f_{s(i)}(y_i) \neq a_{s(i)}$. Then $G_{f(y_i)} \subseteq G_{f_{s(i)}(y_i)}$ and hence, $G_{y_i} \subseteq G_{f_{s(i)}(y_i)}$. According to the hypothesis of Theorem A1 there are infinitely many indices $j \geq 1$ with $X_j \setminus \{a_j\}$ containing points whose stabilizers are larger than $G_{f_{s(i)}(y_i)}$. Choose an index $q(i) \geq r+1$ and a point $b_{q(i)} \in X_{q(i)} \setminus \{a_{q(i)}\}$ with $G_{f_{s(i)}(y_i)} \subseteq G_{b_{q(i)}}$. Now, as $G_{y_i} \subseteq G_{b_{q(i)}}$, there is an obvious G-map $h_{q(i)} : G(y_i) \to G(b_{q(i)})$. Put $\psi_{q(i)} = h_{q(i)} p_{y_i}$. We emphasize that $\psi_{q(i)}(y) \neq a_{q(i)}$ for all $y \in U_{y_i}$. Now, as U_{y_i} is a closed invariant subset of N and $X_{q(i)} \in G$-AR, we can extend each $\psi_{q(i)}$ to a G-map $\varphi_{q(i)} : N \to X_{q(i)}$, $1 \leq i \leq m$. If an index $l \geq r+1$ does not belong to the set $\{q(1), \ldots, q(m)\}$, then we define the corresponding map $\varphi_l : N \to X_l$ by putting $\varphi_l(y) = a_l$, $y \in N$. The family $\{\varphi_j : j \geq r+1\}$ of G-maps is as desired. Indeed, let $y \in N$. Then $y \in U_{y_i}$ for some $1 \leq i \leq m$. By construction $\varphi_{q(i)}(y) = \psi_{q(i)}(y) \neq a_{q(i)}$, and since $q(i) \geq r+1$, the proof of Claim 2 is complete.

Now we define a new G-map $f' : N \to X$ by

$$f'(y) = (f_1(y), \ldots, f_r(y), \varphi_{r+1}(y), \varphi_{r+2}(y), \ldots), \qquad y \in N.$$

Since $r > k$ and for each $y \in N$ there is an index $j \geq r+1$ with $\varphi_j(y) \neq a_j$, we conclude that f' is well defined, i.e., $f'(N) \subseteq X$ and $f'(N) \cap A = \emptyset$. To complete the proof observe that by the choice of r,

$$\varrho(f(y), f'(y)) = \sum_{i=r+1}^{\infty} \varrho_i(f_i(y), \varphi_i(y))/2^i \leq \sum_{i=r+1}^{\infty} 1/2^i < \varepsilon \quad \text{for all } y \in N. \ \blacksquare$$

LEMMA A2. *Let A be a GZ-set in a metric G-space Y. Then A/G is a Z-set in the orbit space Y/G.*

Proof. Let M be an arbitrary compact space. One should prove that the set $\{\varphi \in C(M, Y/G) : \varphi(M) \cap A = \emptyset\}$ is dense in $C(M, Y/G)$. Fix a G-invariant metric ϱ on Y. It is well known [20, Proposition 1.1.12] that the quotient topology on Y/G is generated by the metric

$$\varrho^*(G(x), G(y)) = \inf\{\varrho(x, gy) : g \in G\}, \quad G(x), G(y) \in Y/G.$$

Evidently,

$$(A_1) \qquad \varrho^*(G(x), G(y)) \le \varrho(x, y), \quad x, y \in Y.$$

Let $\varphi \in C(M, Y/G)$ and $\varepsilon > 0$. Denote by p the orbit map $Y \to Y/G$. It is well known [15, Ch. IV, Proposition 4.1] that we have the following commutative (pull-back) diagram:

$$\begin{tikzcd} N \arrow[r, "f"] \arrow[d, "\pi"] & Y \arrow[d, "p"] \\ M \arrow[r, "\varphi"] & Y/G \end{tikzcd}$$

where N is a compact G-space with $N/G = M$, $\pi : N \to M$ the orbit map and f an equivariant map inducing φ. Since A is a GZ-subset of Y, there is an equivariant map $l : N \to Y$ such that $l(N) \cap A = \emptyset$, and $\varrho(l(x), f(x)) < \varepsilon$ for all $x \in N$. Then the map $\psi : M = N/G \to Y/G$ induced by l is ε-close to φ (by (A_1)), and $\psi(M) \cap A/G = \emptyset$. This completes the proof. ■

Lemma A3. *Let $H \subseteq G$ be a closed subgroup of a compact Lie group G and A be a GZ-set in a metric G-space Y. Then A is an HZ-set in Y.*

Proof. Let $f : N \to Y$ be an H-map with N a compact H-space and let $\varepsilon > 0$. Consider the twisted product $G \times_H N$ and the induced G-map $f' : G \times_H N \to Y$ defined by $f'([g, x]) = gf(x)$, where $[g, x] \in G \times_H N$ (see [7, II, §4]). As A is a GZ-subset of Y, there is a G-map $\varphi' : G \times_H N \to Y$ with $A \cap \mathrm{Im}(\varphi') = \emptyset$, ε-close to f'. Now $\varphi = \varphi'|_N$ is an H-map of N to Y with $A \cap \mathrm{Im}(\varphi) = \emptyset$, ε-close to f. ■

Proof of Theorem A1. Since each $X_i \in G$-AR, also $X_i \in H$-AR (see [27]). Hence $\prod_{i=1}^{\infty} X_i \in H$-AR, implying $X \in H$-ANR. Now by Theorem 2 we have $X/H \in$ ANR. Being an open continuous image of the locally compact space X, the orbit space X/H is itself locally compact. So, according to Toruńczyk's characterization criterion [25], it remains to verify that for each compactum M the set of all Z-maps $M \to X/H$ is dense in $C(M, X/H)$. To this end take $\varphi \in C(M, X/H)$ and let $p : X \to X/H$ denote the orbit map. Consider the diagram from the proof of Lemma A2 with $Y = X$ and $G = H$. As $f(N)$ is compact and lies in X, there is an integer $m \ge 1$ satisfying the following condition:

(A_2) for each $y \in N$ there exists $1 \le i \le m$ such that $f_i(y) \neq a_i$, where f_j, $j = 1, 2, \ldots$, are the coordinate maps of f.

(Otherwise there exists a sequence $\{x_k\} \subseteq N$ with $f(x_k) \to a$, implying that $a \in f(N)$, a contradiction.)

Let $\varepsilon > 0$ and let $k > m$ be such that $\sum_{i=k}^{\infty} 1/2^i < \varepsilon$. Define an equivariant map $s : N \to X$ by

$$s(y) = (f_1(y), \ldots, f_{k-1}(y), a_k, a_{k+1}, \ldots), \qquad y \in N.$$

By the choice of k it follows from condition (A_2) that $s(y) \neq a$ for all $y \in N$, i.e., the map s is well defined. Next we have

$$(A_3) \qquad \varrho(s(y), f(y)) = \sum_{i=k}^{\infty} \varrho_i(a_i, f_i(y))/2^i \le \sum_{i=k}^{\infty} 1/2^i < \varepsilon.$$

Thus the equivariant maps f and s are ε-close, and $s(N)$ lies in a "finite-dimensional face" of the product $\prod_{i=1}^{\infty} X_i$, namely in $A = \{x \in X : x_i = a_i$ for all $i \ge k\}$. By Lemmas A1–A3, A/H is a Z-subset of X/H. Let $q : M = N/H \to X/H$ be induced by s. As $q(M) \subseteq A/H$, $q(M)$ is a Z-set in X/H. Now let $u \in M$. Then $u = \pi(t)$ for some $t \in N$, implying $q(u) = p(s(t))$ and $\varphi(u) = p(f(t))$. By (A_1) and (A_3) it follows that

$$\varrho^*(q(u), \varphi(u)) \le \varrho(s(t), f(t)) < \varepsilon \quad \text{ for all } u \in M.$$

So, q is a Z-map ε-close to φ. The proof is complete. ∎

7. Concluding remarks. I. As $J_0(2)/SO(2)$ and $(\exp_0 S^1)/SO(2)$ are both Eilenberg–MacLane spaces $\mathbf{K}(\mathbb{Q}, 2)$, they have the same homotopy type. Moreover, they are homeomorphic. This can be proved exactly in the same way as Corollaries 9 and 10, after having established that $J_0(2)/SO(2)$ is a Q-manifold. In fact $J_0(n)/H$ is a Q-manifold for every closed subgroup $H \subseteq O(n)$ and each $n \ge 2$. Whenever H is a subgroup occurring in $J_0(n)$ as a stabilizer then both $J(n)/H$ and $J(n)[H]$ are Hilbert cubes. These and related results will be presented in our next paper.

The picture seems to be more complicated in the case of the $O(n)$-space $\exp_0 S^{n-1}$ of all closed proper nonempty subsets of the $(n-1)$-dimensional sphere S^{n-1}. Here we have the following

CONJECTURE 1. *For each closed subgroup $H \subseteq O(n)$, the orbit spaces $J_0(n)/H$ and $(\exp_0 S^{n-1})/H$ are homeomorphic Q-manifolds for all $n \ge 2$.*

For this, one needs first to establish that $(\exp_0 S^{n-1})/H \in$ ANR. This would follow from Theorem 2 if we knew that $\exp_0 S^{n-1} \in O(n)$-ANR. Let us formulate the following more general assertion, which can be regarded as a case of the equivariant Wojdysławski Theorem:

Conjecture 2. *Let G be a compact Lie group and X be a connected G-ANR. Then* $\exp X$ *equipped with the induced G-action is a G-AR.*

II. In connection with Theorem 5 we have the following

Conjecture 3. $J_0(n)$ *and* $\Pi_0(n)$ (*and hence* $J(n)$ *and* $\Pi(n)$) *are* $O(n)$-*equivariantly homeomorphic for all* $n \geq 2$.

The desired equivariant homeomorphism could be obtained just as in the nonequivariant case [9, §21], by first establishing equivariant versions of the results of R. Y. T. Wong and T. A. Chapman quoted in Section 5.

III. Yet another $O(n)$-space related to the Banach–Mazur compactum $BM(n)$ is provided by the hyperspace $J'(n)$ of *all* (not only centrally symmetric) compact convex bodies in $\mathbb{R}^n$ having the unit ball B as their John ellipsoid. In the same way as in Section 2 one can prove that $J'(n) \in O(n)$-AR. An easy modification of the method used in Section 3 (Corollary 5) allows one to prove that $J'_0(2)/N$ and $\Pi_0(2)/N$ have the same homotopy type for every closed subgroup $N \subseteq O(2)$, where $J'_0(n) = J'(n) \setminus \{B\}$. In particular this implies that $BM_0(2) \simeq J'_0(2)/O(2)$.

Conjecture 4. $J'_0(n)$ *and* $\exp_0 S^{n-1}$ *are* $O(n)$-*equivariantly homeomorphic for all* $n \geq 2$.

IV. Finally we have the following

Conjecture 5. $BM_0(n)$ *is not contractible for any* $n \geq 3$.

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