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Embedding Cohen algebras using pcf theory

by

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Abstract. Using a theorem from pcf theory, we show that for any singular cardinal ν, the product of the Cohen forcing notions on κ, $\kappa < \nu$, adds a generic for the Cohen forcing notion on ν^+.

The following question (problem 5.1 in Miller's list [Mi91]) is attributed to René David and Sy Friedman:

Does the product of the forcing notions ${}^{\aleph_n >}2$ add a generic for the forcing ${}^{\aleph_{\omega+1}>}2$?

We show here that the answer is yes in ZFC. Previously Zapletal [Za] showed this result under the assumption $\square_{\aleph_{\omega+1}}$.

In fact, a similar theorem can be shown about other singular cardinals as well. The reader who is interested only in the original problem should read $\aleph_{\omega+1}$ for λ, $\aleph_\omega$ for μ and $\{\aleph_n : n \in (1,\omega)\}$ for $\mathfrak{a}$.

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Definition 1. (1) Let $\mathfrak{a}$ be a set of regular cardinals. $\prod \mathfrak{a}$ is the set of all functions f with domain $\mathfrak{a}$ satisfying $f(\kappa) \in \kappa$ for all $\kappa \in \mathfrak{a}$.

(2) A set $\mathfrak{b} \subseteq \mathfrak{a}$ is *bounded* if $\sup \mathfrak{b} < \sup \mathfrak{a}$, and *cobounded* if $\mathfrak{a} \setminus \mathfrak{b}$ is bounded.

(3) If J is an ideal on $\mathfrak{a}$, $f, g \in \prod \mathfrak{a}$, then $f <_J g$ means $\{\kappa \in \mathfrak{a} : f(\kappa) \not< g(\kappa)\} \in J$. We write $\prod \mathfrak{a}/J$ for the partial (quasi)order $(\prod \mathfrak{a}, <_J)$.

(4) $\lambda = \operatorname{tcf}(\prod \mathfrak{a}/J)$ (λ is the *true cofinality* of $\prod \mathfrak{a}/J$) means that there is a strictly increasing cofinal sequence of functions in the partial order $(\prod \mathfrak{a}, <_J)$.

(5) $\operatorname{pcf}(\mathfrak{a}) = \{\lambda : (\exists J)\, (\lambda = \operatorname{tcf}(\prod \mathfrak{a}/J))\}$.

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We will use the following theorem from pcf theory:

LEMMA 2. *Let μ be a singular cardinal. Then there is a set $\mathfrak{a}$ of regular cardinals below μ with $|\mathfrak{a}| = \mathrm{cf}(\mu) < \min \mathfrak{a}$ and $\mu^+ \in \mathrm{pcf}(\mathfrak{a})$. Moreover, we can even have $\mathrm{tcf}(\prod \mathfrak{a}/J^{\mathrm{bd}}) = \mu^+$, where J^{bd} is the ideal of all bounded subsets of $\mathfrak{a}$.*

Proof. See [Sh 355, Theorem 1.5].

THEOREM 3. *Let $\mathfrak{a}$ be a set of regular cardinals, $\mu = \sup \mathfrak{a} \notin \mathfrak{a}$, $2^{<\lambda} = 2^{\mu}$, $\lambda > \mu$, $\lambda \in \mathrm{pcf}(\mathfrak{a})$, and moreover:*

(∗) *There is an ideal J on $\mathfrak{a}$ containing all bounded sets such that $\lambda = \mathrm{tcf}(\prod \mathfrak{a}/J)$.*

Then the forcing notion $\prod_{\kappa \in \mathfrak{a}} {}^{\kappa>}2$ adds a generic for ${}^{\lambda>}2$.

COROLLARY 4. *If ν is a singular cardinal, and P is the product of the forcing notions ${}^{\kappa>}2$ for $\kappa < \nu$, then P adds a generic for ${}^{\nu^+>}2$.*

Proof. By Lemma 2 and Theorem 3.

REMARK 5. (1) The condition (∗) in the theorem is equivalent to:

(∗∗) For all bounded sets $\mathfrak{b} \subset \mathfrak{a}$ we have $\lambda \in \mathrm{pcf}(\mathfrak{a} \setminus \mathfrak{b})$.

(2) Clearly the assumption $2^{<\lambda} = 2^{\mu}$ is necessary, because otherwise the forcing notion $\prod_{\kappa \in \mathfrak{a}} {}^{\kappa>}2$ would be too small to add a generic for ${}^{\lambda>}2$.

Proof of Theorem 3. By our assumption we have some ideal J containing all bounded sets such that $\mathrm{tcf}(\prod \mathfrak{a}/J) = \lambda$.

We will write $(\forall^J \kappa \in \mathfrak{a})\,(\varphi(\kappa))$ for $\{\kappa \in \mathfrak{a} : \neg\varphi(\kappa)\} \in J$. So we have a sequence $\langle f_\alpha : \alpha < \lambda \rangle$ such that:

(a) $f_\alpha \in \prod \mathfrak{a}$.
(b) If $\alpha < \beta$, then $(\forall^J \kappa \in \mathfrak{a})\,(f_\alpha(\kappa) < f_\beta(\kappa))$.
(c) $(\forall f \in \prod \mathfrak{a})(\exists \alpha)(\forall^J \kappa \in \mathfrak{a})(f(\kappa) < f_\alpha(\kappa))$.

The next lemma shows that if we allow these functions to be defined only almost everywhere, then we can additionally assume that in each block of length μ these functions have disjoint graphs:

LEMMA 6. *Assume that $\mathfrak{a}$, λ, μ are as above. Then there is a sequence $\langle g_\alpha : \alpha < \lambda \rangle$ such that:*

(a) $\mathrm{dom}(g_\alpha) \subseteq \mathfrak{a}$ *is cobounded (so in particular* $(\forall^J \kappa \in \mathfrak{a})(\kappa \in \mathrm{dom}(g_\alpha(\kappa)))$).
(b) *If* $\alpha < \beta$, *then* $(\forall^J \kappa \in \mathfrak{a})(g_\alpha(\kappa) < g_\beta(\kappa))$.
(c) $(\forall f \in \prod \mathfrak{a})(\exists \alpha)(\forall^J \kappa \in \mathfrak{a})(f(\kappa) < g_\alpha(\kappa))$. *Moreover, we may choose α to be divisible by μ.*
(d) *If* $\alpha < \beta < \alpha + \mu$, *then* $(\forall \kappa \in \mathrm{dom}(g_\alpha) \cap \mathrm{dom}(g_\beta))(g_\alpha(\kappa) < g_\beta(\kappa))$.

Proof. Let $\langle f_\alpha : \alpha < \lambda\rangle$ be as above. Now define $\langle g_\alpha : \alpha < \lambda\rangle$ by induction as follows:

If $\alpha = \mu\cdot\zeta$, then let $g_\alpha \in \prod\mathfrak{a}$ be any function that satisfies $g_\beta <_J g_\alpha$ for all $\beta < \alpha$, and also $f_\alpha <_J g_\alpha$. Such a function can be found because the set of functions of size $< \lambda$ can be $<_J$-bounded by some f_β.

If $\alpha = \mu\cdot\zeta + i$, $0 < i < \mu$, then let

$$g_\alpha(\kappa) = \begin{cases} g_{\mu\cdot\zeta}(\kappa) + i & \text{if } i < \kappa,\\ \text{undefined} & \text{otherwise.}\end{cases}$$

It is easy to see that (a)–(d) are satisfied.

Definition 7. (1) Let P_κ be the set ${}^{\kappa>}2$, partially ordered by inclusion (= sequence extension). Let $P = \prod_{\kappa\in\mathfrak{a}} P_\kappa$. [We will show that P adds a generic for ${}^{\lambda>}2$.]

(2) Assume that $\langle g_\alpha : \alpha < \lambda\rangle$ is as in Lemma 6.

(3) Let $H : {}^{\mu}2 \to {}^{\lambda>}2$ be onto.

(4) For $\kappa \in \mathfrak{a}$, let $\underset{\sim}{\eta}_\kappa$ be the P_κ-name for the generic function from κ to 2. Define a P-name of a function $\underset{\sim}{h} : \lambda \to 2$ by

$$\underset{\sim}{h}(\alpha) = \begin{cases} 0 & \text{if } (\forall^J \kappa\in\mathfrak{a})(\underset{\sim}{\eta}_\kappa(g_\alpha(\kappa))=0),\\ 1 & \text{otherwise.}\end{cases}$$

(5) For $\xi < \lambda$ let $\underset{\sim}{\varrho}_\xi$ be a P-name for the element of ${}^{\mu}2$ that satisfies $\underset{\sim}{\varrho}_\xi \simeq \underset{\sim}{h}\restriction[\mu\cdot\xi, \mu\cdot(\xi+1))$, i.e.,

$$i < \mu \Rightarrow \Vdash_P \underset{\sim}{\varrho}_\xi(i) = \underset{\sim}{h}(\mu\cdot\xi + i).$$

Define $\underset{\sim}{\varrho} \in {}^{\lambda}2$ by

$$\underset{\sim}{\varrho} = H(\underset{\sim}{\varrho}_0)^\frown H(\underset{\sim}{\varrho}_1)^\frown\cdots{}^\frown H(\underset{\sim}{\varrho}_\xi)^\frown\cdots$$

Main Claim 8. *$\underset{\sim}{\varrho}$ is generic for ${}^{\lambda>}2$.*

Definition 9. For $\alpha < \lambda$ let $P^{(\alpha)}$ be the set of all conditions p satisfying $(\forall^J\kappa)(\mathrm{dom}(p_\kappa) = g_\alpha(\kappa))$.

Remark 10. $\bigcup_{\zeta<\lambda} P^{(\mu\cdot\zeta)}$ is dense in P.

Proof. By Lemma 6(c).

Fact 11. *Let $\alpha = \mu\cdot\zeta$, $p \in P^{(\alpha)}$, $\sigma \in {}^{\mu}2$. Then there is a condition $q \in P^{(\alpha+\mu)}$, $q \ge p$ and*

$$(\forall j < \mu)(\forall^J\kappa)(q_\kappa(g_{\alpha+j}(\kappa)) = \sigma(j)).$$

Proof. Let $p = (p_\kappa : \kappa \in \mathfrak{a})$. There is a set $\mathfrak{b} \in J$ such that for all $\kappa \in \mathfrak{a}\setminus\mathfrak{b}$ we have $\mathrm{dom}(p_\kappa) = g_\alpha(\kappa)$. Define $q \in P^{(\alpha+\mu)}$, $q = (q_\kappa : \kappa\in\mathfrak{a})$, as follows:

$$q_\kappa(\gamma) = \begin{cases} p_\kappa(\gamma) & \text{if } \gamma\in\mathrm{dom}(p_\kappa),\\ \sigma(j) & \text{if } \gamma = g_{\alpha+j}(\kappa),\ \kappa\in\mathfrak{a}\setminus\mathfrak{b},\\ 0 & \text{otherwise.}\end{cases}$$

We have to explain why q is well defined: First note that the first and the second case are mutually exclusive. Indeed, if $\gamma = g_{\alpha+j}(\kappa)$, then $\gamma > g_\alpha(\kappa)$, whereas $\kappa \notin \mathfrak{b}$ implies that $\mathrm{dom}(p_\kappa) = g_\alpha(\kappa)$, so $\gamma \notin \mathrm{dom}(p_\kappa)$.

Next, by the property (d) from Lemma 6 there is no contradiction between various instances of the second case. Also the third case causes no contradiction. Now obviously $q_\kappa \in P_\kappa$ and $p_\kappa \le q_\kappa$. So $p \le q \in P_\kappa$.

Hence we find that for all $j < \mu$, whenever $\kappa \in \mathfrak{a} \setminus \mathfrak{b}$ and $\kappa > j$, then $q_\kappa(g_{\alpha+j}(\kappa)) = \sigma(j)$. Since J contains all bounded sets, this means that $(\forall^J \kappa)(q_\kappa(g_{\alpha+j}(\kappa)) = \sigma(j))$.

Remark 12. Assume that $\alpha = \mu \cdot \zeta$, and p, q, σ are as above. Then $q \Vdash \underset{\sim}{\varrho}_\zeta = \sigma$.

Proof of the main claim. Let $p \in P$, and $D \subseteq {}^{\lambda>}2$ be a dense open set. We may assume that for some $\alpha^* < \lambda$, $\zeta^* < \lambda$ we have $\alpha^* = \mu \cdot \zeta^*$ and $p \in P^{(\alpha^*)}$, i.e., for some $\mathfrak{b} \in J$ we have $(\forall \kappa \notin \mathfrak{b})(\mathrm{dom}(p_\kappa) = g_{\alpha^*}(\kappa))$, by Remark 10.

So p decides the values of $h \restriction \alpha^*$, and hence also the values of $\underset{\sim}{\varrho}_\zeta$ for $\zeta < \zeta^*$. Specifically, for each $\zeta < \zeta^*$ we can define $\sigma_\zeta \in {}^\mu 2$ by

$$\sigma_\zeta(i) = \begin{cases} 0 & \text{if } (\forall^J \kappa)(p_\kappa(g_{\mu\cdot\zeta+i}(\kappa)) = 0), \\ 1 & \text{otherwise.} \end{cases}$$

(Note that for all $\zeta < \zeta^*$ and all $i < \mu$, and almost all κ the value of $p_\kappa(g_{\mu\cdot\zeta+i}(\kappa))$ is defined.)

Clearly $p \Vdash \underset{\sim}{\varrho}_\zeta = \sigma_\zeta$. Since D is dense and H is onto, we can now find $\sigma_{\zeta^*} \in {}^\mu 2$ such that $H(\sigma_0)^\frown \cdots {}^\frown H(\sigma^*_\zeta) \in D$. Using 11 and 12, we can now find $q \ge p$ such that $q \Vdash \underset{\sim}{\varrho}_{\zeta^*} = \sigma_{\zeta^*}$.

Hence $q \Vdash \underset{\sim}{\varrho} \in D$, and we are done.

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